Analytic Techniques in Enumerative Combinatorics

Why Complex Analysis?

In the previous lectures we have encounter the following scheme:

Combinatorial Problem $\xrightarrow{\text{Symbolic Method}}$ Generating Function $\xrightarrow{\text{Lagrange Inversion}}$ Coefficient ≡ Counting

$C(x) = 1 + xC(x)^2$

$C_n = \frac{1}{n+1}\binom{2n}{n} \nearrow C_n \simeq \frac{1}{n}\frac{(2n)!}{n!\,n!} \simeq \frac{1}{n}\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{\sqrt{2\pi n}^{2}\left(\frac{n}{e}\right)^{2n}} \simeq \frac{1}{\sqrt{\pi}}\, n^{-3/2}\, 4^n\,(1+o(1))$

Question: Can we deduce asymptotic estimates without knowing the coefficient $\Rightarrow$ YES!

A crash course on Complex Analysis and Singularities

The main notion needed is the one of analyticity:

Def / Let $f: U \subseteq \mathbb{C} \to \mathbb{C}$ be a function defined over an open subset U of $\mathbb{C}$. We say that f is (complex)-differentiable at $z_0 \in U$ iff the limit

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \quad \text{exists}$$

If f is differentiable at all $z_0 \in U$, we say that f is holomorphic in U.

The strength of this notion will arise later, but the main idea is that this derivative is defined in terms of a quotient of 2-dimensional objects. Compare this with the definition of differentiability in $\mathbb{R}^2$.

Ex / $f(z) = z^k$ is holomorphic in $\mathbb{C}$, but $g(z) = \bar{z}$ is not differentiable at $z=0$.

Differentiability would provide a way to write functions in a very easy way: as power series. We need the following result by Abel:

Theorem / Let $f(z) = \sum_{n\geq 0} a_n (z-a)^n$ with radius of convergence $R>0$. Then,

i) For all $k \geq 1$ the series $\sum_{n\geq k} n(n-1)\dots(n-k+1)\, a_n (z-a)^{n-k}$ converges in $B(a,R)$.
ii) $f(z)$ and its derivatives are holomorphic in $B(a,R)$.
iii) $a_n = f^{(n)}(a)/n!$.

This tells us that every power series is holomorphic in a certain region. In fact, the opposite result is also true, in a local way:

Theorem / (Cauchy) Let $f: U \subseteq \mathbb{C} \to \mathbb{C}$ be holomorphic at z_0. Then

$$f(z) = \sum_{n\geq 0} a_n (z-z_0)^n \quad \text{in a } B(z_0,R), \text{ for some } R>0.$$

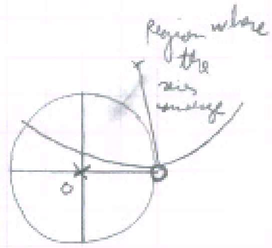
From this theorem we conclude something stranger: if f is holomorphic at $z=z_0$, then all its derivatives exist (at $z=z_0$)

Def / $f: U \subseteq \mathbb{C} \to \mathbb{C}$ is analytic at $z=z_0$ iff in an open set $z_0 \in V \subseteq U$, $f(z) = \sum_{n\geq 0} a_n (z-z_0)^n$.

Theorem / f is holomorphic at $z=z_0 \iff f$ is analytic at $z=z_0$

Def/ A function which is analytic in $\mathbb{C}$ is called entire.

Examples/ i) Polynomials on z are entire functions.



ii) $f(z) = \frac{1}{1-z}$. This function is defined at every point minus at $z=1$.

a) $z=0 \Rightarrow$ Analytic $\Rightarrow f(z) = \sum_{n\geq 0} z^n$. This is valid for $|z|<1$.

b) $z=z_0 \neq 1 \Rightarrow$ Write

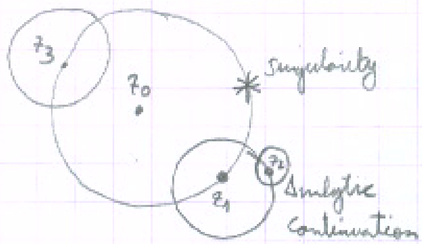
$$\frac{1}{1-z} = \frac{1}{(1-z_0)-(z-z_0)} = \frac{1}{1-z_0}\,\frac{1}{1-\frac{z-z_0}{1-z_0}} = \sum_{n\geq 0}\left(\frac{1}{1-z_0}\right)^{n+1}(z-z_0)^n$$

We need finally another definition:

Def/ A function $h(z)$ is meromorphic at $z=z_0$ iff in a neighbourhood of $z=z_0$, $h(z)$ can be written as $f(z)/g(z)$, with f,g analytic at $z=z_0$

Singularities.

A singularity can be informally defined as a point $z_0 \in \mathbb{C}$ where a certain function ceases to be analytic (poles are the simplest type of singularity)



Def/ Let f be analytic at $z=z_0$, and consider the radius of convergence of its expansion as a power series. Then a point in the boundary ζ_0 a singularity if f is not analytically continuable at ζ_0.

Something to have in mind: it would be possible that $f(\zeta_0) \neq \infty$! (We will see some examples).

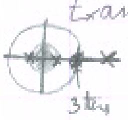
We have now all the language in order to go back to power series whose coefficients count combinatorial objects. Let $A(z) = \sum_{n\geq 0} a_n z^n$ be a formal power series whose coefficient growth $\leq R^n$, for some $R>0$. Then necessarily $A(z)$ defines an analytic function around $z=0$. We also assume that $A(z)$ is NOT entire (otherwise we need some technical integral ...)

We have then the following Theorem:

Theorem/ (Pringsheim's Theorem) If $A(z) = \sum_{n\geq 0} a_n z^n$ has positive coefficients and and radius of convergence R, then $A(z)$ has a singularity at $z=R$.

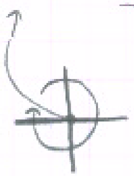
Observe that this result does not say that R is the unique singularity of $A(z)$ at $|z|=R$.

Examples/ i) $A(z) = \frac{1}{(1-4x^2)(1-x)^3(1-\frac{1}{2}x)}$. All the singularities are isolated (poles); the diagram of singularities is the following:



ii) The logarithm function: the sum $\sum_{n=0}^{\infty}\frac{z^n}{n!}$ defines an entire function. We write it as e^z. Observe that $f'(z)=f(z)$, hence if $f(z_0)=0$, then $f(z)\equiv 0$, which we know that it is not true.

Recall now that if $\alpha \in \mathbb{R}$, $e^{i\alpha} = \cos\alpha + i\sin\alpha$. Hence, for each choice of $k \in$ we have that $e^{i(\alpha+2\pi k)} = e^{i\alpha} = \cos\alpha + i\sin\alpha$. Writing now $w=e^z$, we write $\log w = z$. In particular, $\log(0)$ is not defined. For a fixed w, the solutions of



$$w = e^z \Rightarrow z = \log|w| + i(\arg w + 2\pi k)$$

We need to pick 1 value of $k \Rightarrow$ logarithm branch

This is the reason why we cannot define $\log z$ over a closed curve around 0.

iii) Square roots: this is a slightly different example, because here singularities are not infinite. Consider the entire function $f(z) = 1 - z^2$, and let $g(z)$ such that $f(g(z)) = z$. Of course both $\pm\sqrt{1-z}$ define an inverse for $f(z)$.

But, in both cases, such an inverse ceases to be analytic because $f'(1) \to \infty$. This example shows an example of a singularity which is finite.

The previous singularities can be explained in terms of the following theorem:

Theorem / (Inverse function Theorem, simplified version) Let $f: \mathbb{C} \to \mathbb{C}$ be an entire function such that there exists $g: U \subseteq \mathbb{C} \to \mathbb{C}$ with $f(g(z)) = z$ $\forall z \in U$. If $f'(g(z)) \neq 0$, then g is holomorphic at z and $g'(z) = f'(g(z))^{-1}$.

In particular, the inverse function of f ceases to be analytic at the points where $f'(a) = 0$.

The Main 2 principles of asymptotic estimates.

We are now almost ready to start studying how the coefficients of a GF growths. Write $A(z) = \sum_{n \geq 0} a_n z^n$, and assume that $\exists R > 0$, such that $(a_n / R)^n \to 0$ (coefficients do not growth too fast). We know that if $A(z)$ has a finite singularity, then the smallest in modulo must be real and positive.

(A) Exponential growth: given a sequence $\{a_n\}$, we say that it is of exponential order K^n iff

$$\limsup |a_n|^{1/n} = K$$

Considering now the corresponding generating function $A(z) = \sum_{n \geq 0} a_n z^n$ whose radius of convergence is $R < \infty$; then we have that:

a) taking $z = R - \varepsilon$, for all $\varepsilon > 0$, we have that the sum $\sum_{n \geq 0} a_n (R-\varepsilon)^n$ is convergent. Hence, $\lim a_n (R-\varepsilon)^n \to 0$.

b) taking $z = R + \varepsilon$, for all $\varepsilon > 0$, we have that $\sum_{n \geq 0} a_n (R+\varepsilon)^n$ does NOT converge.

These two facts suggest the following theorem:

Theorem / (Exponential growth): if $A(z)$ is analytic at 0 and R is the smallest (real) singularity, then a_n is of exponential order R^{-n}.

In a more precise way, we can say that $a_n = \Theta(n) R^{-n} (1 + o(1))$, where $\limsup \Theta(n)^{1/n} = 1$. So, this gives us the first principle in asymptotic enumeration:

"The location (or position) of a function singularity dictates the exponential growth of its coefficients".

Example / Surjections: a surjection is a function $f: [n] \to [r]$, where all the elements in the image have antiimage. Then, $S = \mathrm{Seq}(\mathrm{Set}_{\geq 1}(\circ))$. Hence, the corresponding GF is:

$$S(z) = (1 - (e^z - 1))^{-1} = (2 - e^z)^{-1}$$

Let us study how these coefficients grow. The singularities of this function are located at the points where $2 = e^z$. This means that the singularities are located at the points

$$\rho_k = \log 2 + 2\pi i k$$

The dominant singularity is then when $k = 0$. Hence, $s_n = [z^n] S(z)$ grows like $\Big(\underbrace{\frac{1}{\log(2)}}_{3.321928}\Big)^n$.

Example/ Derangement: recall that a derangement is a permutation with Nº fixed point. Hence, $D(z) = e^{-z}/(1-z)$. Hence, the unique singularity is located at $z=1$. For this reason, we have that $d_n = [z^n]\, D(z)$ grows like 1^n. So in this case we would need something more precise.

A very important example is the following one, which connects with all the theory of Lagrangian schemes. Assume that $A(z)$ satisfies an equation of the form $A(z) = z\,\phi(A(z))$, where $\phi(u)$ satisfies the following condition:

(H1) $\phi(u)$ is analytic at $u=0$ and satisfies that $\phi(0)\neq 0$, $[u^n]\,\phi(u) \geq 0$, $\phi \neq \phi_0 + \phi_1 u$ (the inverse function is well defined)

(H2) If R is the radius of convergence of ϕ, then there is a unique $\tau \in \mathbb{R}^+$, $0<\tau<R$ such that $\phi(\tau) = \tau\phi'(\tau)$.

Under these conditions we can state the following theorem:

Theorem/ (Exponential growth for inverse functions) Assume that $A(z)$ satisfies an equation of the form $A(z) = z\,\phi(A(z))$, where ϕ satisfies (H1) and (H2). Then, if τ is the solution to the equation $\phi(\tau) = \tau\phi'(\tau)$. Then

$$a_n = [z^n]\, A(z) \text{ grows like } \rho^{-n} = \left(\frac{\tau}{\phi(\tau)}\right)^{-n}.$$

Proof/ The inverse function of $A(z)$ is $\frac{u}{\phi(u)}$. Hence the singularities of $A(z)$ arise when $\left(\frac{u}{\phi(u)}\right)' = 0$. This happens at $u=\tau$. Consequently, if writing $A(\rho) = \tau$, we have that $\rho = \tau/\phi(\tau)$, and this must be the singularity of $A(z)$.

Example/ Catalan numbers. We have the equation $C(z) = 1 + z\,C(z)^2$, which is transformed into $U(z) = z(1+U(z))^2$ by writing $U(z) = C(z) - 1$. Then $\phi(u) = (1+u)^2$, which satisfies (H1). Concerning (H2):

$$(1+\tau)^2 = 2\tau(1+\tau) \Rightarrow 1 + 2\tau + \tau^2 = 2\tau + 2\tau^2 \Rightarrow \tau^2 = 1 \Rightarrow \tau = \pm 1 \Rightarrow \underline{\tau = 1}$$

$$\Rightarrow \rho = 1/(1+1)^2 = 1/4 \Rightarrow [z^n]\, C(z) = c_n \text{ grows like } 4^n.$$

Example/ Cayley trees: we now have the equation $T(z) = z\,e^{T(z)}$, hence here $\phi(u) = e^u$. Again, $\phi(u)$ satisfies (H1), and concerning (H2) we have the equation $e^\tau = \tau e^\tau \Rightarrow \tau = 1$. Consequently, $\tau = 1 \Rightarrow \rho = e^{-1}$, and we know that

$$\frac{t_n}{n!} = [z^n]\, T(z) \text{ grows like } e^n \Rightarrow t_n \text{ grows like } e^n\, n!$$

(B) Subexponential growth: once studied where the smallest singularity is, which provides us the information concerning the exponential growth of the coefficients, the next step is based on studying the subexponential growth $\theta(n)$ of the coefficients. As we will see, the principle here is the following:

> "The nature of the singularities with the smallest modulo defines the subexponential growth of the coefficient".

We start studying this principle in the context of rational functions. Those are the ones related to linear recurrence relations for the coefficients. Assume that $A(z) = N(z)/D(z)$, where $N(z)$ and $D(z)$ are polynomials. Then $A(z)$ ceases to be analytic at the points a such that $D(a)=0$. These types of singularities are called poles. If $D(z) = (1-z\alpha_1)^{r_1} \cdots (1-z\alpha_s)^{r_s}$, we say that $A(z)$ has a pole at $z = \alpha_i$ of order r.

we have then the following theorem:

Theorem/ If $A(z)$ is a rational function of the form $N(z)/D(z)$, analytic at zero and with poles at points $\alpha_1, \dots, \alpha_d$, then

$$a_n = [z^n]A(z) = \sum_{j=1}^{d} \Pi_j(n)\, \alpha_j^{-n},$$

where $\Pi_j(n)$ is a polynomial in n of degree the order of the pole at α_j minus one.

Proof/ Since $A(z)$ is rational it can be written in the form:

$$A(z) = Q(z) + \sum_{(\alpha,r)} \frac{c_{\alpha,r}}{(z-\alpha)^r}$$

where $Q(z)$ is a polynomial of degree $n_0 = \deg(N) - \deg(D)$, and the sum is over all poles, and r is bounded by the degree of the pole α. Then we can extract the coefficients one by one.

It is important to notice that, asymptotically speaking, the main contributions will arise from the α's whose modulo is the largest.

Example/ Fibonacci numbers have the GF: $\frac{z}{1-z-z^2}$. Hence, as the roots of $1-z-z^2$ are $\frac{-1\pm\sqrt{5}}{2}$ ($\varphi_1 < \varphi_2$), we have that

$$f_n = [z^n]F(z) = \frac{1}{\sqrt{5}}\varphi_1^{-n} + \frac{-1}{\sqrt{5}}\varphi_2^{-n} \approx \frac{1}{\sqrt{5}}\varphi_1^{-n} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

② Meromorphic functions: these objects are the natural generalization of rational functions. If $F(z)$ is a meromorphic function, analytic at the origin and with singularities $\alpha_1, \alpha_2, \dots \alpha_m$ of the same (smallest) modulo. Then a similar result can be stated as in the case of rational functions, but now the order of the poles (same terminology) must be computed different:

Ex/ The EGF for surjections is $(2-e^z)^{-1}$; we have already determined the poles, with values $f_k = \log 2 + 2\pi i k$ $(k \in \mathbb{Z})$. The smallest singularity is located at $f_0 = \log(2)$. Let us study its order: we try with order 1:

$$\lim_{\substack{z\to\log(2)\\ z\in\mathbb{R}}} \frac{z-\log(2)}{(2-e^z)} = \lim_{\substack{z\to\log(2)\\ z\in\mathbb{R}}} \frac{1}{-e^z} = -\frac{1}{2} \Rightarrow \frac{1}{2-e^z} = -\frac{1}{2}\frac{1}{(z-\log(2))} + \underbrace{f(z)}$$

($f(z)$: analytic in a neighbourhood of $z = \log(2)$)

Hence, we can apply the same estimates as in the case of rational functions.

③ Lagrangian schemes: we go back to equations of the form $A(z) = z\phi(A(z))$, where we also assume that hypothesis (H1) and (H2) are satisfied. Write for convenience $\psi(u) = u/\phi(u)$, and hence $\psi(A(z)) = z$.

We saw that if $\psi'(u) \neq 0$, then $A(z)$ is analytic at $z = z_0$ such that $A(z_0) = u$, and the obstruction to the analyticity of $A(z)$ is given by $\psi'(u_0) = 0$.

In such a point, roughly speaking, we have that

$$\begin{cases} \psi'(u_0) \neq 0 \Rightarrow \psi(u) \simeq \psi(u_0) + \psi'(u_0)(u-u_0) \overset{\psi(A(z))=z,\ A(z_0)=u_0}{\Rightarrow} A(z) \simeq A(z_0) + \frac{1}{\psi'(u_0)}(z-z_0) \\ \dots \end{cases}$$

This informal argument suggests that the singularity type is of square-root type.

Intermezzo: Assume that $y(z)$ is an algebraic function (namely, defined by a polynomial equation), and let ρ be its smallest singularity.

Theorem / (Newton–Puiseux expansion) On its singularity ρ, $y(z)$ admits a representation of the form:

$$y(z) = \sum_{k \geq p} c_k \cdot (1 - z/\rho)^{k/q}, \quad p \in \mathbb{Z}$$

Indeed, Puiseux series (over $\mathbb{C}$) are a field extension of Laurent series $\mathbb{C}((z))$ (in fact, an algebraic closure of $\mathbb{C}((z))$).

Now the point is that we can state something similar for functions defined by a Lagrangian scheme, and in our situation we have that

$$A(z) \underset{z \sim \rho}{=} \sum_{k \geq 0} c_k (1 - z/\rho)^{k/2} \quad \text{(Singular expansion)}$$

We would like now to extract coefficients from this expression, and something we would like to apply is the following:

$$(1 - z/\rho)^{-\alpha} \Rightarrow [z^n](1 - z/\rho)^{-\alpha} = \binom{-\alpha}{n} = \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-n+1)}{n!}\rho^{-n}$$

$$\alpha \notin \{0, -1, \dots\} \qquad \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)} = \binom{n+\alpha-1}{n}\rho^{-n} \simeq \frac{n^{\alpha-1}}{\Gamma(\alpha)}\rho^{-n}(1+o(1)) \qquad \Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt$$

Knowing this, one would like to say the following: If $A(z)$ has singular expansion of the form

$$A(z) \underset{z=\rho}{\simeq} c_0 + c_1 \underbrace{(1 - z/\rho)^{1/2}}_{X} + O(X^2) \Rightarrow [z^n]A(z) = c_1 [z^n] X + O([z^n]X^2)$$

$$= c_1 \frac{n^{-3/2}}{\Gamma(-\frac{1}{2})}\rho^{-n}(1+o(1)), \qquad \Gamma(-\tfrac{1}{2}) \to \frac{-1}{2\sqrt{\pi}}$$

This is indeed what happen:

Theorem / (Transfer theorem for singularity analysis) If $A(z)$ has a singular expansion at $z=\rho$ of the form

$$A(z) \underset{z=\rho}{=} c_0 + c_1(1 - z/\rho)^{-\alpha} + O\left((1 - z/\rho)^{-\alpha+1}\right) \Rightarrow [z^n]A(z) = c_1 \frac{n^{\alpha-1}}{\Gamma(\alpha)}\rho^{-n}(1+o(1))$$

So, in particular, for Lagrangian schemes we have estimates of the form $\frac{-c\, n^{-3/2}}{2\sqrt{\pi}}\rho^{-n}$

Example / Catalan numbers: we saw that the singularity is located at $\rho = 1/4$. Around this value we have that:

$$C(x) = 1 + xC(x)^2 \Rightarrow U(x) = C(x) - 1 \Rightarrow U(x) = x(1+U(x))^2 \overset{\rho = 1/4}{\Rightarrow} U(x) = u_0 + u_1 X + u_2 X^2 + \dots$$

$$\Rightarrow u_0 + u_1 X + u_2 X^2 + \dots = \rho(1 - X^2)(1 + u_0 + u_1 X + \dots)^2 \Rightarrow \begin{cases} u_0 = \rho(1+u_0)^2 \Rightarrow u_0 = 1 \\ u_1 = \rho \cdot 2u_1(u_0+1) \Rightarrow u_1 = u_1 \text{ (Nothing!)} \\ \cancel{u_2} = \rho\left(u_1^2 + \cancel{2(1+u_0)u_2} - (1+u_0)^2\right) \end{cases}$$

$$\Rightarrow u_1 = \pm 2 \Rightarrow \text{(We take the negative!)}$$

Hence, we conclude that $[z^n]C(z) = u_1 \frac{n^{-3/2}}{\Gamma(-1/2)} 4^n (1+o(1)) = \frac{n^{-3/2}}{\sqrt{\pi}} 4^n (1+o(1))$, which is what we found directly.