Formal power series and generating functions

The basics on formal power series

We assume that in the following all the definitions are done over $\mathbb{C}$. However, a similar theory could be done over an arbitrary field.

Def / A <u>formal power series</u> over $\mathbb{C}$ is an expression of the form $A(x) = \sum_{n\geq 0} a_n x^n$, where $a_i \in \mathbb{C}$ for all i.

In this case we will write that $[x^n]A(x) = a_n$. We will denote by $\mathbb{C}[[x]]$ the set of formal power series over $\mathbb{C}$.

Obs / Obviously, $\mathbb{C}[x] \hookrightarrow \mathbb{C}[[x]]$, because each polynomial over $\mathbb{C}$ is a formal power series whose coefficients are 0 from a point on.

We can define the operations "+" and "·" over $\mathbb{C}[[x]]$:

a) $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$, then $A(x) + B(x) := \sum_{n\geq 0} s_n x^n$, where $s_n = a_n + b_n$

b) $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$, then $A(x)B(x) := \sum_{n\geq 0} p_n x^n$, where $p_n = \sum_{m=0}^{m} a_m b_{n-m}$.

$\mathbb{C}(x)$
$\mathbb{C}((x))$

These operations extend naturally the ones over $\mathbb{C}[x]$. Hence we can prove easily the following theorem:

Theorem / $(\mathbb{C}[[x]], +, \cdot)$ is a commutative ring. Furthermore, $\mathbb{C}[[x]]$ contains a subring which is isomorphic to $\mathbb{C}[x]$.

$\begin{cases} O(x) = 0 + \dots \\ 1(x) = 1 + 0x + \dots \end{cases}$

Let us look now for the elements in $\mathbb{C}[[x]]$ which have a multiplicative inverse:

Theorem / $A(x) = \sum_{n\geq 0} a_n x^n$ has a multiplicative inverse iff $a_0 \neq 0$.

Proof / $\Rightarrow$] If $B(x)$ is the multiplicative inverse of $A(x)$, then $A(x)B(x) = 1$, and $a_0 \neq 0$.

$\Leftarrow$] We determine the inverse of $A(x)$. We write $b_0 = a_0^{-1}$, $b_n = -a_0^{-1} \sum_{i=1}^{n} a_i b_{n-i}$, which could be solved recursively.

We move to another operation. Observe that if $B(x) = \sum_{n\geq 0} b_n x^n$, $b_0 = 0$, then the operation of substitution of "x" into another generating function $A(x)$ has sense:

Def / If $b_0 = 0$, the <u>composition</u> of $A(x) = \sum_{n\geq 0} a_n x^n$ and $B(x) = \sum_{m\geq 0} b_m x^m$ is the generating function

$$A \circ B(x) = A(B(x)) = \sum_{n\geq 0} a_n B(x)^n$$

Observe that $A(B(x))$ is well defined because in order to compute its coefficients we only need a finite number of computations.

Def / We say that $A(x)$ has a <u>functional inverse</u> $B(x)$ iff $A(B(x)) = B(A(x)) = x$.

Theorem / $A(x)$ has a functional inverse iff $[x^0]A(x) = 0$, $[x^1]A(x) \neq 0$. In this case, the functional inverse is unique.

Proof/ Exercise for the student!

We finish this introduction with the properties of the derivative.

Def/ If $A(x)=\sum_{n\geq 0} a_n x^n$ is a formal power series, then the derivative of $A(x)$ is:

$$\frac{d}{dx}A(x) := \sum_{n\geq 0} n\, a_n x^{n-1}.$$

We will also write $A'(x)$ for the derivative of $A(x)$.

Lemma/ If $A(x)=\sum_{n\geq 0} a_n x^n$ and $B(x)=\sum_{n\geq 0} b_n x^n$, then

0) $\frac{d}{dx}A(x)=0 \Rightarrow A(x)=c$

I) $\frac{d}{dx}\left(A(x)+B(x)\right) = A'(x)+B'(x)$

II) $\frac{d}{dx}\left(A(x)B(x)\right) = A'(x)B(x)+A(x)B'(x) \rightsquigarrow \frac{d}{dx}A(x)^n = n A(x)^{n-1}A'(x)$.

III) If $a_0=0$, $\frac{d}{dx}\left(B(A(x))\right) = B'(A(x))\cdot A'(x)$

IV) $\frac{d}{dx}A(x)^{-1} = -A(x)^{-2}\cdot A'(x) \longrightarrow \frac{d}{dx}A(x)^{-n} = -n A(x)^{-n-1} A(x)^{-2}$.

Proof/ We just prove II). Observe that $A(x)B(x)=\sum_{n\geq 0}\left(\sum_{m=0}^{n} a_m b_{n-m}\right)x^n$, hence its derivative is equal to:

$$\sum_{n\geq 0}\left(\sum_{m=0}^{n} n\, a_m b_{n-m}\right)x^n = \sum_{n\geq 0}\Big(\sum_{m=0}^{n} \underbrace{m\, a_m b_{n-m}}_{A'(x)B(x)} + \underbrace{(n-m)\, a_m b_{n-m}}_{A(x)B'(x)}\Big)x^n$$

In the opposite direction, we can define the integral of a formal power series in the following way:

Def/ Let $A(x)=\sum_{n\geq 0} a_n x^n$ be a formal power series over $\mathbb{C}$. Then $\int A(s)\,ds$ is a formal power series of the form:

$$\int A(s)\,ds = c + \sum_{n\geq 0} a_n \frac{x^{n+1}}{n+1}, \text{ where } c\in\mathbb{C}.$$

Generating functions and sequences.

Def/ Given a sequence of complex numbers $(a_n)_{n\geq 0}$, the ordinary generating function (GF) associated to $(a_n)_{n\geq 0}$ is the formal power series $\sum_{n\geq 0} a_n x^n$.

We can translate operations between sequences to equations involving the corresponding GF's:

Lemma/ Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be to sequences with GF's $A(x)$ and $B(x)$, respectively. Then,

1) $(a_n+b_n)_{n\geq 0} \rightarrow A(x)+B(x)$
2) $(a_{n+h})_{n\geq 0} \rightarrow (A(x)-a_0-a_1x-\ldots-a_{h-1}x^{h-1})/x^h$
3) $\left(\sum_{m=0}^{n} a_m b_{n-m}\right)_{n\geq 0} \rightarrow A(x)B(x)$
4) $(n a_n)_{n\geq 0} \rightarrow x\frac{d}{dx}A(x)$
5) $(a_0+\ldots+a_n)_{n\geq 0} \rightarrow \frac{1}{1-x}A(x)$

We will usually use the following sequence: $\left(\binom{\alpha}{n}\right)_{n\geq 0}$, where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}$,

and $\alpha\in\mathbb{R}$ …