Laurent series and Lagrange inversion formula

We have seen that that rational power series is a ring of $\mathbb{C}((x))$. However, we would be interested in studying the structure of the whole field $\mathbb{C}((x))$:

Def/ A Laurent series is an expression of the form $A(x) = \sum_k a_n x^n$, where $k \in \mathbb{Z}$. If $a_k \neq 0$ we say that k is the order of $A(x)$ (namely, the smallest k, $n \geq k$ such that $a_k \neq 0$)

Theorem/ $\mathbb{C}((x)) = \{$ Laurent series over $\mathbb{C} \}$

Proof/ It is clear that $\{$Laurent series over $\mathbb{C}\} \subseteq \mathbb{C}((x))$: if $A(x)$ has order k, then $A(x) = x^{-k} \cdot B(x)$, where $B(x) \in \mathbb{C}[[x]]$.

On the other side, if $A(x), B(x) \in \mathbb{C}[[x]]$ with $B(x) \neq 0$, and the order of $B(x)$ is k, then $B(x) = x^k C(x)$, and

$$\frac{A(x)}{B(x)} = x^{-k} A(x)\, C(x)^{-1}$$

which is a Laurent series.

In Laurent series we have the same operations as we had for formal power series. In particular, the derivative of a quotient is the following:

$$\frac{d}{dx}\frac{A(x)}{B(x)} = \frac{A'(x)B(x) - A(x)B'(x)}{B^2(x)}$$

Def/ Let $A(x) = \sum_{n \geq k} a_k x^k$ be a Laurent series. The residue of $A(x)$ is $\operatorname{Res}(A(x)) = a_{-1}$

Lemma/ If $A(x)$ is a Laurent series, then

0) $\operatorname{Res}(A(x) + B(x)) = \operatorname{Res}(A(x)) + \operatorname{Res}(B(x))$

a) $\operatorname{Res}(A'(x)) = 0$

b) $\operatorname{Res}(A'(x)/A(x)) = m$, where m is the order of $A(x)$.

Proof/ Exercise for the reader.

Theorem/ Let $A(x) = \sum_{k \geq 1} a_k x^k$, $a_1 \neq 0$, and let $B(x) = \sum_{k \geq 1} b_k x^k$ the functional inverse of $A(x)$. Then,

$$b_n = \operatorname{Res}\left(\frac{1}{n A^n(x)}\right)$$

Proof/ We have that $B(A(x)) = x$, hence $x = \sum_{k=1}^{\infty} b_k A^k(x)$. Hence,

$$1 = \sum_{k=1}^{\infty} b_k k\, A(x)^{k-1} A'(x) \Rightarrow A(x)^{-n} = \sum_{k=1}^{\infty} b_k \cdot k\, A(x)^{k-n-1} \cdot A'(x)$$

$$\Rightarrow b_1 A'(x) + 2b_2 A(x)A'(x) + \dots + n b_n A^{n-1}(x) A'(x) + \dots = 1$$ (and the order of $A(x)$ is 1)

are derivatives!

$$\Rightarrow b_1 \frac{A'(x)}{A^n(x)} + 2b_2 \frac{A'(x)}{A^{n-1}(x)} + \dots + n b_n \frac{A'(x)}{A(x)} + \dots = A(x)^{-n} \Rightarrow n \cdot b_n = \operatorname{Res}\left(\frac{1}{A(x)^n}\right)$$

We can apply then this fact to the following result

Theorem/ (Lagrange inversion Formula) Let $U(x) = \sum_{n \geq 0} u_n x^n$, where $U(x) = x\,\phi(U(x))$, where ϕ is a function with $[t^0]\phi(t) \neq 0$. Then, Proof/ We have that $U(x)/\phi(U(x)) = x$, hence $U(x)$ is the functional inverse of $t/\phi(t)$. Consequently,

$$u_n = [x^n]\,U(x) = \operatorname{Res}\left(\frac{1}{n\,t^n}\phi^n(t)\right) = \frac{1}{n}\,[t^{-1}]\left(\frac{\phi^n(t)}{t^n}\right) = \frac{1}{n}\,[t^{n-1}]\,\phi^n(t)$$

In fact, by applying similar arguments we have the following generalization:

Theorem/ (Lagrange inversion formula, v2) Let $U(x) = \sum_{n\geq 0} u_n x^n$, where $U(x) = x\,\phi(U(x))$ for $\phi(x)$ satisfying that $[t^0]\,\phi(t) \neq 0$. Then

$$[x^n]\,U(x)^k = \frac{k}{n}\,[t^{n-k}]\,\phi(t)^n$$

Example/ Catalan numbers: we know that its GF is $C(x) = 1 + x\,C(x)^2$. Writing $C(x) = 1 + U(x)$, we have that $U(x) = x\,(U(x)+1)^2$. Then $\phi(t) = (1+t)^2$, and $[t^0]\,\phi(t) \neq 0$; hence,

$$C_n = [x^n]\,U(x) = \frac{1}{n}\,[t^{n-1}]\,(1+t)^{2n} = \frac{1}{n}\binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$$

Let us also see some property about $C'(x)$. Recall that $C(x)$ is algebraic of degree 2, hence:

$$C'(x) = C(x)^2 + 2x\,C(x)\,C'(x) \Rightarrow C'(x) = \frac{C(x)^2}{1 - 2x\,C(x)} \in \mathbb{C}(x, C)$$

From this is difficult to get an expression of $C'(x)$ in the form $\frac{p_1(x)}{q_1(x)} + \frac{p_2(x)}{q_2(x)}\,C(x)$.

Then, we use the property $(n+2)\,C_{n+1} = (4n+2)\,C_n$:

$$\begin{array}{rlcl}
 & 0\cdot C_1 + 2\cdot C_1 & = & 4\cdot 0\cdot C_0 + 2C_0 \\
 & 1\cdot C_2\,x + 2C_2\,x & = & 4\cdot 1\,C_1\,x + 2C_1\,x \\
+ & 2\cdot C_3\,x^2 + 2C_3\,x^2 & = & 4\cdot 2\,C_2\,x^2 + 2C_2\,x^2 \\
 & \vdots & & \vdots \\
 & n\,C_{n+1}\,x^n + 2C_{n+1}\,x^n & = & 4n\,C_n\,x^n + 2C_n\,x^n \\
\hline
 & x\left(\frac{C(x)-1}{x}\right)' + 2\,\frac{C(x)-1}{x} & = & 4x\,C'(x) + 2C(x)
\end{array}$$

$$\Rightarrow x\left(\frac{C'(x)}{x} - \frac{C(x)-1}{x^2}\right) + 2\,\frac{C(x)-1}{x} = 4x\,C'(x) + 2C(x)$$

$$\Rightarrow C'(x) + \frac{C(x)-1}{x} = 4x\,C'(x) + 2C(x)$$

$$\Rightarrow C'(x) = \frac{-1}{1-4x}\,\overbrace{\frac{C(x)-1}{x}}^{\in\,\mathbb{C}[[x]]} + \frac{2}{1-4x}\,C(x)$$

$$= \underbrace{\frac{1}{x(1-4x)}}_{\notin\,\mathbb{C}[[x]]} + \frac{2x+1}{x(1-4x)}\,C(x)$$

If $V(x) = \frac{C(x)-1}{x} \Rightarrow C(x) = 1 + x\,V(x) \Rightarrow$

$$\Rightarrow \cancel{1} + x\,V(x) = \cancel{1} + x\,(1 + x\,V(x))^2 \Rightarrow V(x) = x + 2x^2\,V(x) + x^3\,V(x)^2 \Rightarrow$$ is also algebraic