Map enumeration and bijective proofs

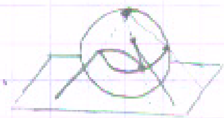
In this part of the course we will make a more detailed study of a combinatorial family of great importance not only in discrete mathematics, but also in probability theory, algebraic geometry and statistical mechanics.

Def/ (Topological planar map) A planar map M is a multigraph Γ embedded into a surface homeomorphic to $\mathbb{S}^2$ in such a way that:

i) Vertices are represented as distinct points of the surface.
ii) Edges are represented as curves on the surface that intersect only at the vertices.

Under these situation, each connected component of $\mathbb{S}^2 - \Gamma$ is homeomorphic to an open disk. Each one of these objects is called a face.

Obs/ a) Taking out the north pole, we can represent maps in the plane.

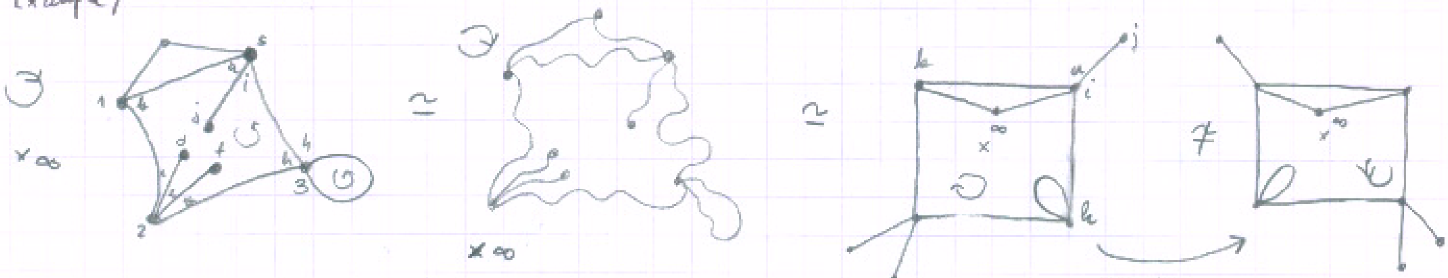


b) Planar maps satisfies Euler relation: # faces + # vertices = # edges + 2.

Now the next point is to define when two planar maps are "equal":

Def/ Two maps M_1 and M_2 are isomorphic if there exists an orientation preserving homeomorphism $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ such that the restriction of φ to Γ_1 is a graph isomorphism between Γ_1 and Γ_2.

Example/



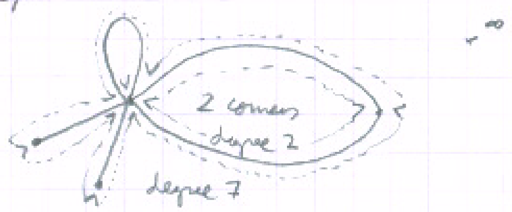
We have changed the orientation!

We need some extra definitions:

Def/ The dual map M^* of a given map M is the map built by duality.
Def/ The degree of a vertex v of a map M is the vertex degree in the associated multigraph.
Def/ The degree of a face f of a map M is the degree of the vertex that arise by duality.

Def/ A corner in a map is each ocurrence of a vertex when doing a walk contour around a face.

Example/



In particular, the degree of a face is the number of corners it defines.
⇓
A map is obtained by "pasting" disks (ribbon graphs)

So now we can try to count. The first maps are the following (in terms of edges):

Embedded trees are maps with 1 face
⇓
Unicellular maps!

Rooted planar maps and enumeration.

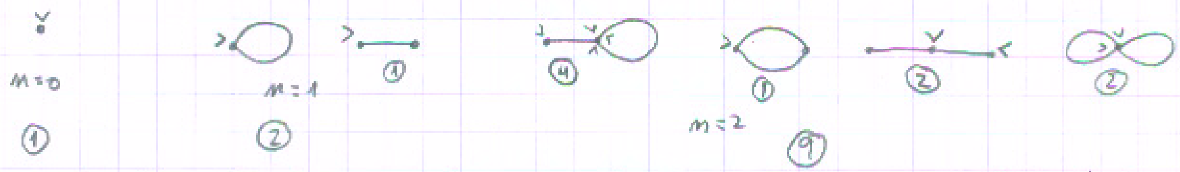
As it happens in tree enumeration, counting maps is a difficult matter. Instead of doing this, we will count rooted planar maps.

Def / A <u>rooted planar map</u> is a planar map where a vertex v, an edge e incident with v and the face on the right of the edge are distinguished.

Usually we represent the root using a corner or an oriented edge over the unbounded face:

⇒ Now the isomorphism must preserve the root.

Ex / The first rooted planar maps are the following:



So let us try to count them. For this purpose, let us use Tutte's strategy by deleting the root. However, we will need an auxiliar variable (the catalytic variable) which encode the degree of the root face. Write $M(x,u)$ the GF, where x marks edges and u marks root face degree. 3 situations may happen:

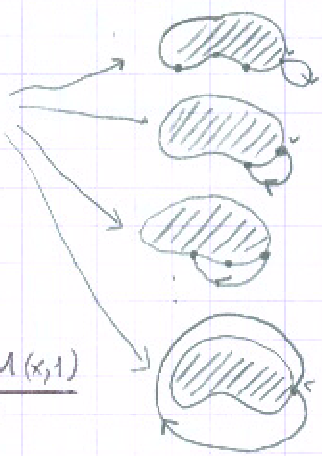
i) my map does not have edges $\Rightarrow 1$

ii) The root of the map is a bridge:

$\Rightarrow \quad x u^2 M(x,u)^2$

pair of rooted maps

iii) The root of the map is NOT a bridge: such a map is obtained from an existing map by adding conveniently the root edge:



From a map encoded by $x^r u^k$, we construct maps

$x^{r+1}(u^{k+1}+\dots+u)$

$\Downarrow$

$$M(x,u)=\underbrace{\sum_{r,k\geq 0} M_{r,k}\, x^r u^k}_{\Downarrow}$$

$$xu\frac{uM(x,u)-M(x,1)}{u-1} = u x^{r+1}\frac{1-u^{k+1}}{1-u}$$

Consequently, we have the following equation:

$$M(x,u)=1+xu^2M(x,u)^2+xu\frac{uM(x,u)-M(x,1)}{u-1}$$

So now we have an equation and two indeterminates. Now the strategy (due to Tutte) is to exploit the fact that we have a bit of freedom for choosing u as a function of x. This is what is called the <u>quadratic Method</u>.