The Symbolic Method: OGF

Let $\mathcal{A}$ be a combinatorial family (possibly infinite), joint with a function $|\cdot|_\mathcal{A}: \mathcal{A} \to \mathbb{N}$ (called the size). We write $\mathcal{A}_n = \{a \in \mathcal{A} : |a| = n\}$, and $|\mathcal{A}_n| = a_n$. Assuming that $|\mathcal{A}_n| < \infty$, we can define the GF

$(\mathcal{A}, |\cdot|)$ ||| combinatorial class

$$A(x) = \sum_{n \geq 0} a_n x^n = \sum_{a \in \mathcal{A}} x^{|a|}$$

This is the so-called ordinary generating function (OGF). We also write (as we did for f.p.s.) $[x^n] A(x) = a_n$.

Def/ Let Φ be an m-ary construction that associate to any collection of classes $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(m)}$ a new class

$$\mathcal{A} = \Phi(\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(m)})$$

We say that Φ is admissible if the counting series of $\mathcal{A}$ only depends on the counting sequences of $\mathcal{B}^{(1)}, \dots, \mathcal{B}^{(m)}$.

In other words, admissible constructions just need of the enumerative information!

Examples/ Admissible constructions:

0) Atomic class and neutral class: $\mathcal{X}, \mathcal{E}$, $X(x) = x$, $E(x) = 1$; $\mathcal{E} = \{\varepsilon\}$

a) Disjoint union: given two combinatorial classes $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \cap \mathcal{B} = \emptyset$, the disjoint union of $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{A} \cup \mathcal{B}$, whose counting formula is $A(x) + B(x)$.

b) (Cartesian) product: given two classes $\mathcal{A}$ and $\mathcal{B}$, the cartesian product of $\mathcal{A}$ and $\mathcal{B}$ is $\mathcal{A} \times \mathcal{B} = \{(a, b) : a \in \mathcal{A}, b \in \mathcal{B}\}$. The size of an element is $|(a,b)| = |a|_\mathcal{A} + |b|_\mathcal{B}$, and the related OGF is $A(x) \cdot B(x)$.

c) Sequence: is simply $\mathcal{E} \cup \mathcal{A} \cup \mathcal{A} \times \mathcal{A} \cup \dots$, whose generating function is $(1 - A(x))^{-1}$. We can also have restricted sequences: $\Delta \subseteq \mathbb{N}$, then $\mathrm{Seq}_\Delta(\mathcal{A})$ is $\bigcup_{\delta \in \Delta} \mathcal{A}^{\times \delta}\!\times \mathcal{A}$, and the OGF is:

$$\sum_{\delta \in \Delta} A(x)^\delta \; ; \text{ if } \Delta = \{k\}, \text{ then } \mathrm{Seq}_\Delta \equiv \mathrm{Seq}_k .$$

d) Multiset: multisets are like sets but repetitions of elements are allowed. Hence,

$$\mathrm{MSet}(\mathcal{A}) = \mathrm{Seq}_+(\mathcal{A}) / M, \text{ where } (a_1, \dots, a_r) M (b_1, \dots, b_s) \text{ iff } r = s \text{ and } a_i = b_{\sigma(i)} \text{ for some } \sigma \in S_r$$

Then, $\mathrm{MSet}(\mathcal{A}) = \prod_{a \in \mathcal{A}} \mathrm{Seq}(\{a\})$; hence, the generating function is the following:

$$\prod_{a \in \mathcal{A}} (1 - x^{|a|})^{-1} = \prod_{n=1}^{\infty} (1 - x^n)^{-a_n} = \exp\left(\sum_{n=1}^{\infty} a_n \log(1 - x^n)^{-1} \right)$$

$$= \exp\left(\sum_{r=1}^{\infty} \frac{1}{r} A(x^r) \right) = \overline{\exp}(A(x))$$

||| Pólya operator

e) Powerset: is the subset of the multisets with NO repetition! Then, $\mathrm{PSet}(\mathcal{A}) = \prod_{a \in \mathcal{A}} (\{\varepsilon\} \cup \{a\})$, so:

$$\prod_{a \in \mathcal{A}} (1 + x^{|a|}) = \prod_{n} (1 + x^n)^{a_n} = \exp\left(\sum_{n=1}^{\infty} a_n \log(1 + x^n) \right) = \exp\left(\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} A(x^r) \right)$$

f) Cyclic construction: $Cyc(A)$ is defined in terms of $Seg(A)$: $Cyc(A) = Seg(A)/c$, where $(a_1, \dots, a_r) c (b_1, \dots, b_s)$ iff $r = s$ and the first sequence is obtained from the second one by a cyclic permutation.

Then, the corresponding OGF is $-\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log(1 - A(x^k))$

g) Pointing operator: given a combinatorial class A, and a fixed collection of neutral objects $\{\varepsilon_1, \varepsilon_2, \dots\}$ of size 0, the pointed class $A^\bullet$ is $A^\bullet = \bigcup_{n \geq 0} A_n \times \{\varepsilon_1, \dots, \varepsilon_n\}$.
This is translated into GFs as $\sum_{n \geq 0} n a_n x^n = x A'(x)$

h) The substitution: given two combinatorial class A and B, the composition of B into A is

$$A \circ B := \bigcup_{\alpha \in A} Seq_{|\alpha|}(B) \Rightarrow A(B(x))$$

So we have a table with all these constructions

Name	Construction	OGF
Atomic class	X	x
Neutral class	$\mathcal{E}$	1
Disjoint union	$A \cup B$	$A(x) + B(x)$
Cartesian product	$A \times B$	$A(x)\, B(x)$
Sequence	$Seq(A)$	$(1 - A(x))^{-1}$
Multiset	$MSet(A)$	$\exp\left(\sum_{r=1}^{\infty} \frac{1}{r} A(x^r)\right) = \overline{\exp}(A(x))$
Powerset	$PSet(A)$	$\exp\left(\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} A(x^r)\right)$
Cyclic	$Cyc(A)$	$-\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log(1 - A(x^k))$
Pointing	$A^\bullet$	$x A'(x)$
Substitution	$A \circ B$	$A(B(x))$

In all cases, the GF associated to A and B is $A(x)$ and $B(x)$, respectively

Multivariate OGF

Many times we will also be interested in another parameters apart from the counting formulas. We will then need extra variables to encode such information.

Def/ A parameter in a combinatorial class is a function $\chi: A \to \mathbb{N}$.

Writing now $A_{n,k} = \{a \in A : |a| = n, \chi(a) = k\}$, we can define a multivariate GF by including this new parameter:

$$A(x,u) = \sum_{n,k} |A_{n,k}| x^n u^k = \sum_{a \in A} x^{|a|} u^{\chi(a)}$$