Rational formal power series and linear recurrences

Def/ A formal power series $A(x)$ is <u>rational</u> if it can be written in the form:

$$A(x) = \frac{P(x)}{Q(x)} \quad , \text{where } [x^0]\,Q(x) \neq 0 \;,\; P(x), Q(x) \in \mathbb{C}[x] \;,$$

in fact, a subring of $\mathbb{C}[[x]]$!

It is clear then that the set of rational f. p. series is a ring on $\mathbb{C}(x)$. Additionally, a rational formal power series has inverse iff $[x^0]\,P(x) \neq 0$.

Obs/ The functional inverse of a rational formal power series does not need to be rational! Take for instance $A(x) = x/(1-x^2)$.

Rational formal power series encode the basic sequences:

Def/ The sequence $(a_n)_{n\geq 0}$ satisfy a <u>linear and homogeneous recurrence</u> if, $\forall n$, $\exists\, c_1, \dots c_k$ s.t.

$$a_{n+k} + c_1 a_{n+k-1} + \dots + c_k a_n = 0$$

The main theorem here relates both worlds:

Theorem/ Let $(a_n)_{n\geq 0}$ be a sequence. Then, the following properties are equivalent:

1. The numbers satisfy a recurrence of the form $a_{n+k} + c_1 a_{n+k-1} + \dots + c_k a_n = 0$ (c_i are $\in \mathbb{C}$)
2. The corresponding GF is rational, with denominator $1 - c_1 x - \dots - c_k x^k$ and the degree of the numerator is $< k$.
3. The numbers a_n satisfy an expression of the form $a_n = P_1(n)\lambda_1^n + \dots + P_r(n)\lambda_r^n$, where $1 - c_1 x - \dots - c_k x^k = (1-\lambda_1 x)^{e_1} \dots (1-\lambda_r x)^{e_r}$, and $P_i(n)$ is a polynomial of degree $\leq e_i$.

To prove the theorem we need the usual technique used to decompose a rational function:

$$\frac{P(x)}{(1-\lambda_1 x)^{e_1} \dots (1-\lambda_r x)^{e_r}} = \sum_{i=1}^{r} \sum_{j=1}^{e_i} \frac{A_{ij}}{(1-\lambda_i x)^j} \qquad (A_{ij} \text{ are constants})$$

Obs/ If we have a recurrence of the form $a_{n+k} + c_1 a_{n+k-1} + \dots + c_k a_n = f(n)$, and $f(n)$ is also written in the form $\sum_{i=1}^{r} Q_i(n)\,\gamma_i^n$, then the resulting GF is also rational.
↳ polynomial

Algebraic formal power series. D-finite formal power series.

Def/ A formal power series $A(x)$ is <u>algebraic</u> if it satisfies an equation of the form:

$$p_n(x)A^n(x) + p_{n-1}(x)A^{n-1}(x) + \dots + p_1(x)A(x) + p_0(x) = 0,$$

where the $p_i(x)$ are polynomials over $\mathbb{C}$ (not all of them 0!) The <u>degree</u> of an algebraic formal power series is the smallest n such an equation exist.

Obs/ 1) Rational functions are algebraic functions of degree 1: $A(x) = \frac{P(x)}{Q(x)} \Rightarrow P(x) - Q(x)A(x) = 0$.

2) Catalan numbers: Catalan numbers satisfy the non-linear recurrence relation

$$C_{n+1} = C_0 C_n + \dots + C_n C_0 \;;\; C_0 = 1 \qquad (\Rightarrow C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14, \dots)$$

3) If the degree of A is d then the degree of the extension $\mathbb{C}(x, A)$ over $\mathbb{C}(x)$ is d

Then it is not difficult to check that the corresponding GF $C(x)$ satisfies the equation $x\,C(x)^2 - C(x) + 1 = 0$.

4) The sum and product of algebraic formal power series is also algebraic (as it happens with algebraic numbers). Also the inverse. From here we derive that the derivative of an algebraic FPS is also algebraic.

5) The functional inverse of an algebraic FPS (if it exist!) is also algebraic: for the formal power series $A(x)$, with inverse $B(x)$ $(A(B(x))=x)$, we have that:

$$p_m(x)A^n(x) + \dots + p_1(x)A(x) + p_0(x) = 0 \Rightarrow p_m(B(x))A^n(B(x)) + \dots + p_1(B(x))A(B(x)) + p_0(B(x)) = 0$$

$$\Rightarrow q_m(x)B^m(x) + \dots + q_1(x)B(x) + q_0(x) = 0 \quad \text{for some polynomials } q_i(x).$$

We can generalize even more this notion:

Def / A formal power series $A(x)$ is D-finite if satisfies an equation of the form:

$$p_n(x)A^{n)}(x) + \dots + p_1(x)A^{1)}(x) + p_0(x)A(x) = q(x)$$

Example / $E(x) = \sum_{n\geq 0} \frac{1}{n!}x^n$ is NOT algebraic, but it is D-finite because $E'(x) - E(x) = 0$.
$R(x) = \sum_{n\geq 0} n!\,x^n$ satisfies that $x^2R'(x) + (x-1)R(x) = 1$.

Def / A sequence $(a_n)_{n\geq 0}$ it is called P-recursive if it satisfies a relation of the form:

$$P_e(n)a_{n+e} + P_{e-1}(n)a_{n+e-1} + \dots + P_0(n)a_n = 0 \qquad n \geq n_0$$

It is then easy to see that the sequence $(a_n)_{n\geq 0}$ is P-recursive iff the corresponding GF is D-finite. Indeed, the language of D-finite FPS covers, in particular, the one of algebraic FPS:

Theorem / Let $A(x)$ be an algebraic formal power series of degree d. Then it is also D-finite.

Proof / If $A(x)$ is algebraic, then its derivatives are rational functions of $A(x)$ and x. As $\mathbb{C}(x,A)$ is finite dimensional over $\mathbb{C}(x)$, necessarily all the derivatives of A are NOT independent. Hence we have an equation of the form:

$$r_d(x)A^{d)}(x) + r_{d-1}(x)A^{d-1)}(x) + \dots + r_1(x)A'(x) + r_0(x) = 0$$

⇓ Multiply by the l.c.m. of the denominators.

In particular, the coefficients of an algebraic f.p.s are P-recursive.

Example / Catalan numbers. One can check that Catalan numbers satisfy the following relation:

$$(n+1)C_{n+1} - 2(2n+1)C_n = 0.$$

Obs / 1) Many combinatorial families we will study are algebraic, but there are natural families which are NOT algebraic, and even NOT D-finite.

↓ $C_n \cdot C_{n+1}$ ↓ Walks in the first quadrant.

Rational f.p.s. $\subseteq$ Algebraic f.p.s $\subseteq$ D-finite f.p.s

Linear recurrences $\subseteq$ P-recursiveness