GRAPH CLASSES WITH GIVEN 3-CONNECTED COMPONENTS: ASYMPTOTIC ENUMERATION AND RANDOM GRAPHS

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Abstract. Consider a family $\mathcal{T}$ of 3-connected graphs of moderate growth, and let $\mathcal{G}$ be the class of graphs whose 3-connected components are graphs in $\mathcal{T}$. We present a general framework for analyzing such graphs classes based on singularity analysis of generating functions, which generalizes previously studied cases such as planar graphs and series-parallel graphs. We provide a general result for the asymptotic number of graphs in $\mathcal{G}$, based on the singularities of the exponential generating function associated to $\mathcal{T}$. We derive limit laws, which are either normal or Poisson, for several basic parameters, including the number of edges, number of blocks and number of components. For the size of the largest block we find a fundamental dichotomy: classes similar to planar graphs have almost surely a unique block of linear size, while classes similar to series-parallel graphs have only sublinear blocks. This dichotomy also applies to the size of the largest 3-connected component. For some classes under study both regimes occur, because of a critical phenomenon as the edge density in the class varies.

1. Introduction

Several enumeration problems on planar graphs have been studied in the last years. It has been shown [15] that the number of labelled planar graphs with n vertices is asymptotically equal to

$$c \cdot n^{-7/2} \cdot \gamma^n n!, \tag{1}$$

for suitable constants c and γ. For series-parallel graphs [4], the asymptotic estimate is of the form, again for suitable constants d and δ,

$$d \cdot n^{-5/2} \cdot \delta^n n!. \tag{2}$$

As can be seen from the proofs in [4, 15], the difference in the subexponential term comes from a different behaviour of the counting generating functions near their dominant singularities. Related families of labelled graphs have been studied, like outerplanar graphs [4], graphs not containing $K_{3,3}$ as a minor [14] and, more generally, classes of graphs closed under minors [3]. In most cases where asymptotic estimates have been obtained, the subexponential term is systematically either $n^{-7/2}$ or $n^{-5/2}$. The present paper grew up as an attempt to understand this dichotomy.

A *class* of graphs is a family of labelled graphs which is closed under isomorphism. A class $\mathcal{G}$ is *closed* if the following condition holds: a graph is in $\mathcal{G}$ if and only if its connected, 2-connected and 3-connected components are in $\mathcal{G}$. A closed class is completely determined by its 3-connected members. The basic example is the class of planar graphs, but there are others, particularly minor-closed classes whose excluded minors are 3-connected. Let us observe that there are natural classes of graphs which are not closed, like the class of cubic planar graphs [5]. Another important example is the class of graphs embeddable in a fixed surface; the enumeration problem in this case has been solved recently [6].

In this paper we present a general framework for enumerating closed classes of graphs. Let $T(x,z)$ be the generating function associated to the family of 3-connected graphs in a closed class $\mathcal{G}$, where x marks vertices and z marks edges, and let g_n be the number of graphs in $\mathcal{G}$ with n vertices. Our first result shows that the asymptotic estimates for g_n depend crucially on the singular behaviour of $T(x,z)$. For a fixed value of x, let $r(x)$ be the dominant singularity of $T(x,z)$. If $T(x,z)$ has an expansion at $r(x)$ in powers of $Z = \sqrt{1 - z/r(x)}$ with dominant

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term Z^5, then the estimate for g_n is as in Equation (1); if $T(x,z)$ is either analytic everywhere or the dominant term is Z^3, then the pattern is that of Equation (2). Our analysis gives a clear explanation of these facts in term of the vanishing of certain coefficients in singular expansions (Propositions 3.5, 3.8, and 3.10).

There also mixed cases, where 2-connected and connected graphs in $\mathcal{G}$ get different exponents. And there are *critical* cases too, due to the confluence of two sources for the dominant singularity, where a subexponential term $n^{-8/3}$ appears. This is the content of Theorem 3.1, whose proof is based on a detailed analysis of singularities.

Section 2 presents technical preliminaries needed in the paper, and Section 3 contains the main results. In Section 4, extending the analytic techniques developed for asymptotic enumeration, we analyze random graphs from closed classes of graphs. We show that several basic parameters converge in distribution either to a normal law or to a Poisson law. In particular, the number of edges, number of blocks and number of cut vertices are asymptotically normal with linear mean and variance. This is also the case for the number of special copies of a fixed graph or a fixed block in the class. On the other hand, the number of connected components converges to a discrete Poisson law.

In Section 5 we study a key extremal parameter: the size of the largest block (2-connected component). And in this case we find a striking difference depending on the class of graphs. For planar graphs there is asymptotically almost surely a block of linear size, and the remaining blocks are of order $O(n^{2/3})$. For series-parallel graphs there is no block of linear size. This applies more generally to the classes considered in Theorem 3.1. The same dichotomy arises when considering the size of the largest 3-connected component. This is proved using the techniques developed by Banderier et al. [1] for analyzing largest components in random maps.

In particular, for planar graphs we prove the following precise result in Theorem 5.4. If X_n is the size of the largest block in a random planar graph with n vertices, then

$$\mathbf{P}\left(X_n = \alpha n + x n^{2/3}\right) \sim n^{-2/3} c g(cx),$$

where $\alpha \approx 0.95982$ and $c \approx 128.35169$ are well-defined analytic constants, and $g(x)$ is the so called Airy distribution of the map type, which is a particular instance of a stable law of index $3/2$. Moreover, the size of the second largest block is $O(n^{2/3})$. The giant block is uniformly distributed among the planar 2-connected graphs with the same number of vertices, hence according to the results in [2] it has about $2.2629 \cdot 0.95982\, n = 2.172\, n$ edges, again with deviations of order $O(n^{2/3})$ (the deviations for the normal law are of order $n^{1/2}$, but the $n^{2/3}$ term coming from the Airy distribution dominates). We remark that the size of the largest block has been analyzed too in [22] using different techniques. The main improvement with respect to [22] is that we are able to obtain the limiting distribution.

With respect to the largest 3-connected component in a random planar graph, we show that it follows an Airy distribution and has ηn vertices and ζn edges, where $\eta \approx 0.7346$ and $\zeta \approx 1.7921$ are again well-defined constants. This is technically more involved since we have to analyze the concatenation of two Airy laws and different probability distributions in 2-connected graphs.

The picture that emerges for random planar graphs is the following. Start with a large 3-connected planar graph M (or the skeleton of a polytope in the space if one prefers a more geometric view), and perform the following operations. First edges of M are substituted by small blocks with a distinguished oriented edge, giving rise to the giant block L; then small connected graphs are attached to some of the vertices of L, which become cut vertices, giving rise to the largest connected component C. The component C constitutes almost the whole graph since, as we show later, it has $n - O(1)$ vertices. As a consequence, we obtain a very precise qualitative description of how a random planar graph looks like. Moreover, we have explicit continuous and discrete limit laws with computable parameters for many parameters of interest.

An interesting open question is whether there are other parameters besides the size of the largest block (or largest 3-connected component) for which planar graphs and series-parallel graphs differ in a qualitative way. We remark that with respect to the largest component there is no qualitative difference. This is also true for the degree distribution: if d_k is the probability that a given vertex has degree $k > 0$, then in both cases it can be shown that the d_k decay as $c \cdot n^{\alpha} q^k$, where c, α and q depend on the class under consideration (see [9, 10]).

In Section 7 we apply the previous machinery to the analysis of several classes of graphs closed under minors, including planar graphs and series-parallel graphs. Whenever the generating function $T(x, z)$ can be computed explicitly, we obtain precise asymptotic estimates for the number of graphs g_n, and limit laws for the main parameters. In particular we determine the asymptotic probability of a random graph being connected, the constant κ such that the expected number of edges is asymptotically κn, and other fundamental constants.

Our techniques allow also to study graphs with a given density, or average degree. To fix ideas, let $g_{n,\lfloor \mu n \rfloor}$ be the number of planar graphs with n vertices and $\lfloor \mu n \rfloor$ edges: μ is the edge density and 2μ is the average degree. For $\mu \in (1, 3)$, a precise estimate for $g_{n,\lfloor \mu n \rfloor}$ can be obtained using a local limit theorem [15]. And parameters like the number of components or the number of blocks can be analyzed too when the edge density varies. It turns out that the family of planar graphs with density $\mu \in (1, 3)$ shares the main characteristics of planar graphs, no matter the value of μ. This is also the case for series-parallel graphs, where $\mu \in (1, 2)$ since maximal graphs in this class have only $2n - 3$ edges. In Section 8 we show examples of *critical phenomena* by a suitable choice of the family $\mathcal{T}$ of 3-connected graphs. In the associated closed class $\mathcal{G}$, graphs below a critical density μ_0 behave like series-parallel graphs, and above μ_0 they behave like planar graphs, or conversely. We even have examples with more than one critical value.

We remark that graph classes with given 3-connected components are analyzed also in [7] and [12], where the emphasis is on combinatorial decompositions rather than asymptotic analysis.

2. Preliminaries

Generating functions (GF for short) are of the exponential type, unless we say explicitly the contrary. The partial derivatives of $A(x, y)$ are written $A_x(x, y)$ and $A_y(x, y)$. In some cases the derivative with respect to x is written $A'(x, y)$. The second derivatives are written $A_{xx}(x, y)$, and so on.

The decomposition of a graph into connected components, and of a connected graph into blocks (2-connected components) are well known. We also need the decomposition of a 2-connected graph decomposes into 3-connected components [27]. A 2-connected graph is built by series and parallel compositions and 3-connected graphs in which each edge has been substituted by a block; see below the definition of networks.

A class of labelled graphs $\mathcal{G}$ is *closed* if a graph G is in $\mathcal{G}$ if and only if the connected, 2-connected and 3-connected components of G are in $\mathcal{G}$. A closed class is completely determined by the family $\mathcal{T}$ of its 3-connected members. Let g_n be the number of graphs in $\mathcal{G}$ with n vertices, and let $g_{n,k}$ be the number of graphs with n vertices and k edges. We define similarly c_n, b_n, t_n for the number of connected, 2-connected and 3-connected graphs, respectively, as well as the corresponding $c_{n,k}, b_{n,k}, t_{n,k}$. We introduce the EGFs

$$G(x, y) = \sum_{n,k} g_{n,k}\, y^k \frac{x^n}{n!},$$

and similarly for $C(x, y)$ and $B(x, y)$. When $y = 1$ we recover the univariate EGFs

$$B(x) = \sum b_n \frac{x^n}{n!}, \qquad C(x) = \sum c_n \frac{x^n}{n!}, \qquad G(x) = \sum g_n \frac{x^n}{n!}.$$

The following equations reflect the decomposition into connected components and 2-connected components:

$$G(x, y) = \exp(C(x, y)), \qquad xC'(x, y) = x \exp\left(B'(xC'(x, y), y)\right), \tag{3}$$

In the first decomposition, one must notice that a general graph is simply a *set* of labelled connected graphs, hence the equation $G(x, y) = \exp(C(x, y))$. The second decomposition is a bit more involved. The EGF $xC'(x, y)$ is associated to the family of connected graphs with rooted at a vertex. Then, the second equation in (3) says that a connected graph with a rooted vertex is obtained from a set of rooted 2-connected graphs (where the root bears no label), in which we substitute each vertex by a connected graph with a rooted vertex (the roots allow us to recover the graph from its constituents). We also define

$$T(x,z) = \sum_{n,k} t_{n,k}\, z^n \frac{x^n}{n!},$$

where the only difference is that the variable for marking edges is now z. This convention is useful and will be maintained throughout the paper.

A *network* is a graph with two distinguished vertices, called *poles*, such that the graph obtained by adding an edge between the two poles is 2-connected. Moreover, the two poles are not labelled. Networks are the key technical device for encoding the decomposition of 2-connected graphs into 3-connected components. We distinguish between three kinds of networks. A network is *series* if it is obtained from a cycle C with a distinguished edge e, whose endpoints become the poles, and every edge different from e is replaced by a network. Equivalently, when removing the root edge if present, the resulting graph is not 2-connected. A network is *parallel* if it is obtained by gluing two or more networks, none of them containing the root edge, along the common poles. Equivalently, when the two poles are a 2-cut of the network. Finally, an *h-network* is obtained from a 3-connected graph H rooted at an oriented edge, by replacing every edge of H (other than the root) by an arbitrary network. Trakhtenbrot [24] showed that a network is either series, parallel or an h-network, and Walsh [28] translated this fact into generating functions as we show next.

Let $D(x,y)$ be the GF associated to networks, where again x and y mark vertices and edges. Then $D = D(x,y)$ satisfies (see [2], who draws on [24, 28])

$$\frac{2}{x^2}T_z(x,D) - \log\left(\frac{1+D}{1+y}\right) + \frac{xD^2}{1+xD} = 0, \tag{4}$$

and $B(x,y)$ is related to $D(x,y)$ through

$$B_y(x,y) = \frac{x^2}{2}\left(\frac{1+D(x,y)}{1+y}\right). \tag{5}$$

For future reference, we set

$$\Phi(x,z) = \frac{2}{x^2}\,T_z(x,z) - \log\left(\frac{1+z}{1+y}\right) + \frac{xz^2}{1+xz}, \tag{6}$$

so that Equation (4) is written in the form $\Phi(x,D) = 0$, for a given value of y. By integrating (5) using the techniques developed in [15], we obtain an explicit expression for $B(x,y)$ in terms of $D(x,y)$ and $T(x,z)$ (see the first part of the proof of Lemma 5 in [15]):

$$\begin{aligned} B(x,y) \quad = \quad & T(x,D(x,y)) - \frac{1}{2}xD(x,y) + \frac{1}{2}\log(1+xD(x,y)) + \\ & \frac{x^2}{2}\left(D(x,y) + \frac{1}{2}D(x,y)^2 + (1+D(x,y))\log\left(\frac{1+y}{1+D(x,y)}\right)\right). \end{aligned} \tag{7}$$

This relation is valid for every closed class defined in terms of 3-connected graphs, and can be proved in a more combinatorial way [7] (see also [12]).

We use singularity analysis for obtaining asymptotic estimates; the main reference here is [11]. The singular expansions we encounter in this paper are always of the form

$$f(x) = f_0 + f_2X^2 + f_4X^4 + \cdots + f_{2k}X^{2k} + f_{2k+1}X_{2k+1} + O(X_{2k+2}),$$

where $X = \sqrt{1-x/\rho}$. That is, $2k+1$ is the smallest odd integer i such that $f_i \neq 0$. The even powers of X are analytic functions and do not contribute to the asymptotics of $[x^n]f(x)$. The number $\alpha = (2k+1)/2$ is called the *singular exponent*, and by the Transfer Theorem [11] we obtain the estimate

$$[x^n]f(x) \sim c \cdot n^{\alpha-1}\rho^{-n},$$

where $c = f_{2k+1}/\Gamma(-\alpha)$.

We assume that, for a fixed value of x, $T(x,z)$ has a unique dominant singularity $r(x)$, and that there is a singular expansion near $r(x)$ of the form

$$T(x,z) = \sum_{n\geq n_0} T_n(x)\left(1-\frac{z}{r(x)}\right)^{n/\kappa}, \tag{8}$$

where n_0 is an integer, possibly negative, and the functions $t_n(x)$ and $r(x)$ are analytic. This is a rather general assumption, as it includes singularities coming from algebraic and meromorphic functions.

The case when $\mathcal{T}$ is empty (there are no 3-connected graphs) gives rise to the class of series-parallel graphs. It is shown in [4] that, for a fixed value $y=y_0$, $D(x,y_0)$ has a unique dominant singularity $R(y_0)$. This is also true for arbitrary $\mathcal{T}$, since adding 3-connected graphs can only increase the number of networks.

3. Asymptotic enumeration

Throughout the rest of the FS we assume that $\mathcal{T}$ is a family of 3-connected graphs whose GF $T(x,z)$ satisfies the requirements described in Section 2. We assume that a singular expansion like (8) holds, and we let $r(x)$ be the dominant singularity of $T(x,z)$, and α the singular exponent.

Our main result gives precise asymptotic estimates for g_n, c_n, b_n depending on the singularities of $T(x,z)$. Cases (1) and (2) in the next statement can be considered as generic, whereas (1) and (2.1) are those encountered in 'natural' classes of graphs. The two situations in case (3) come from critical conditions, when two possible sources of singularities coincide. This is the reason for the unusual exponent $-8/3$, which comes from a singularity of cubic-root type instead of the familiar square-root type.

Theorem 3.1. *Let $\mathcal{G}$ be a closed family of graphs, and let $T(x,z)$ be the GF of the family of 3-connected graphs in $\mathcal{G}$. In all cases b,c,g,R,ρ are explicit positive constants and $\rho<R$.*

(1) *If $T_z(x,z)$ is either analytic or has singular exponent $\alpha<1$, then*

$$b_n \sim b\,n^{-5/2}R^{-n}n!, \qquad c_n \sim c\,n^{-5/2}\rho^{-n}n!, \qquad g_n \sim g\,n^{-5/2}\rho^{-n}n!$$

(2) *If $T_z(x,z)$ has singular exponent $\alpha=3/2$, then one of the following holds:*
- (2.1) $b_n \sim b\,n^{-7/2}R^{-n}n!$, $\quad c_n \sim c\,n^{-7/2}\rho^{-n}n!$, $\quad g_n \sim g\,n^{-7/2}\rho^{-n}n!$
- (2.2) $b_n \sim b\,n^{-7/2}R^{-n}n!$, $\quad c_n \sim c\,n^{-5/2}\rho^{-n}n!$, $\quad g_n \sim g\,n^{-5/2}\rho^{-n}n!$
- (2.3) $b_n \sim b\,n^{-5/2}R^{-n}n!$, $\quad c_n \sim c\,n^{-5/2}\rho^{-n}n!$, $\quad g_n \sim g\,n^{-5/2}\rho^{-n}n!$

(3) *If $T_z(x,z)$ has singular exponent $\alpha=3/2$, and in addition a critical condition is satisfied, one of the following holds:*
- (3.1) $b_n \sim b\,n^{-8/3}R^{-n}n!$, $\quad c_n \sim c\,n^{-5/2}\rho^{-n}n!$, $\quad g_n \sim g\,n^{-5/2}\rho^{-n}n!$
- (3.2) $b_n \sim b\,n^{-7/2}R^{-n}n!$, $\quad c_n \sim c\,n^{-8/3}\rho^{-n}n!$, $\quad g_n \sim g\,n^{-8/3}\rho^{-n}n!$

Using Transfer Theorems [11], the previous result is a direct application of the following analytic result for $y=1$. We prove it for arbitrary values of $y=y_0$, since this has important consequences later on.

Theorem 3.2. *Let $\mathcal{G}$ be a closed family of graphs, and let $T(x,z)$ be the GF of the family of 3-connected graphs in $\mathcal{G}$.*

For a fixed value $y=y_0$, let $R=R(y_0)$ be the dominant singularity of $D(x,y_0)$, and let $D_0=D(R,y_0)$.

(1) *If $T_z(x,z)$ is either analytic or has singular exponent $\alpha<1$ at (R,D_0), then $B(x,y_0)$, $C(x,y_0)$ and $G(x,y_0)$ have singular exponent $3/2$.*

(2) *If $T_z(x,z)$ has singular exponent $\alpha=3/2$ at (R,D_0), then one of the following holds:*
- (2.1) *$B(x,y_0), C(x,y_0)$ and $G(x,y_0)$ have singular exponent $5/2$.*
- (2.2) *$B(x,y_0)$ has singular exponent $5/2$, and $C(x,y_0), G(x,y_0)$ have singular exponent $3/2$.*
- (2.3) *$B(x,y_0), C(x,y_0)$ and $G(x,y_0)$ have singular exponent $3/2$.*

(3) *If $T_z(x,z)$ has singular exponent $\alpha = 3/2$ at (R, D_0), and in addition a critical condition is satisfied for the singularities of either $B(x,y)$ or $C(x,y)$, then one of the following holds:*
   (3.1) *$B(x,y_0)$ has singular exponent $5/3$, and $C(x,y_0), G(x,y_0)$ have singular exponent $3/2$.*
   (3.2) *$B(x,y_0)$ has singular exponent $5/2$, and $C(x,y_0), G(x,y_0)$ have singular exponent $5/3$.*

The rest of the section is devoted to the proof of Theorem 3.2, which implies Theorem 3.1. First we study the singularities of $B(x,y)$, which is the most technical part. Then we study the singularities of $C(x,y)$ and $G(x,y)$, which are always of the same type since $G(x,y) = \exp(C(x,y))$.

3.1. **Singularity analysis of $B(x,y)$.** From now on, we assume that $y = y_0$ is a fixed value, and let $D(x) = D(x,y_0)$. Recall from Equation (6) that $D(x)$ satisfies $\Phi(x, D(x)) = 0$, where

$$\Phi(x,z) = \frac{2}{x^2} T_z(x,z) - \log\left(\frac{1+z}{1+y_0}\right) + \frac{xz^2}{1+xz}.$$

Since a 3-connected graph has at least four vertices, $T(x,z)$ is $O(x^4)$. If follows that $D(0) = y_0$ and $\Phi_z(0, D(0)) = -1/(1+y_0) < 0$. It follows from the implicit function theorem that $D(x)$ is analytic at $x = 0$.

The next result shows that $D(x)$ has a positive singularity R and that $D(x)$ is finite at R.

Lemma 3.3. *With the previous assumptions, $D(x)$ has a positive singularity $R = R(y)$, and $D(R)$ is also finite.*

Proof. We first show that $D(x)$ has a finite singularity. Consider the family of networks without 3-connected components, which corresponds to series-parallel networks, and let $D_\emptyset(x,y)$ be the associated GF. It is shown in [4] that the radius of convergence $R_\emptyset(y_0)$ of $D_\emptyset(x,y_0)$ is finite for all $y > 0$. Since the set of networks enumerated by $D(x,y_0)$ contains the networks without 3-connected components, it follows that $D_\emptyset(x,y) \leq D(x,y)$ and $D(x)$ has a finite singularity $R(y_0) \leq R_\emptyset(y_0)$.

Next we show that $D(x)$ is finite at its dominant singularity $R = R(y_0)$. Since R is the smallest singularity and $\Phi_z(0, D(0)) < 0$, we have $\Phi_z(x, D(x)) < 0$ for $0 \leq x < R$. We also have $\Phi_{zz}(x,z) > 0$ for $x, z > 0$. Indeed, the first summand in Φ is a series with positive coefficients, and all its derivatives are positive; the other two terms have with second derivatives $1/(1+z)^2$ and $2x/(1+xz)^3$, which are also positive. As a consequence, $\Phi_z(x, D(x))$ is an increasing function and $\lim_{x\to R^-} \Phi(x, D(x))$ exists and is finite. It follows that $D(R)$ cannot go to infinity, as claimed. □

Since R is the smallest singularity of $D(x)$, $\Phi(x,z)$ is analytic for all $x < R$ along the curve defined by $\Phi(x, D(x)) = 0$. For $x, z > 0$ it is clear that Φ is analytic at (x,z) if and only if $T(x,z)$ is also analytic. Thus $T(x,z)$ is also analytic along the curve $\Phi(x, D(x)) = 0$ for $x < R$. As a consequence, the singularity R can only have two possible sources:

(a) A branch-point (R, D_0) when solving $\Phi(x,z) = 0$, that is, Φ and Φ_z vanish at (R, D_0).
(b) $T(x,z)$ becomes singular at (R, D_0), so that $\Phi(x,z)$ is also singular.

Case (a) corresponds to case (1) in Theorem 3.2. For case (b) we assume that the singular exponent of $T(x,z)$ at the dominant singularity is $5/2$, which corresponds to families of 3-connected graphs coming for 3-connected planar maps, and related families of graphs. We could allow more general types of singular exponents but they do not appear in the main examples we have analyzed.

The typical situation is case (2.1) in Theorem 3.2, but (2.2) and (2.3) are also possible. It is also possible to have a critical situation, where (a) and (b) both hold, and this leads to case (3.1): this is treated at the end of this subsection. Finally, a confluence of singularities may also arise when solving equation

$$xC'(x,y) = x\exp\left(B'(xC'(x,y), y)\right).$$

Indeed the singularity may come from: (a) a branch point when solving the previous equation; or (b) $B(x,y)$ becomes singular at $\rho(y)C'(\rho(y), y)$, where $\rho(y)$ is the singularity of $C(x,y)$. When the two sources (a) and (b) for the singularity coincide, we are in case (3.2). This is treated at the end of Section 3.2.

3.1.1. Φ *has a branch-point at* (R, D_0). We assume that $\Phi_z(R, D_0) = 0$ and that Φ is analytic at (R, D_0). We have seen that $\Phi_{zz}(x, z) > 0$ for $x, z > 0$. Under these conditions, $D(x)$ admits a singular expansion near R of the form

$$D(x) = D_0 + D_1 X + D_2 X^2 + D_3 X^3 + O(X^4), \tag{9}$$

where $X = \sqrt{1 - x/R}$, and $D_1 = -\sqrt{2R\Phi_x(R, D_0)/\Phi_{zz}(R, D_0)}$ (see [11]). We remark that R and the D_i's depend implicitly on y_0.

In the next result we find an explicit expression for D_1, which is the dominant term in (9). This puts into perspective the result found in [4] for series-parallel graphs, where it was shown that $D_1 < 0$ for that class.

Proposition 3.4. *Consider the singular expansion (9). Then* $D_1 < 0$ *is given by*

$$D_1 = -\left(\frac{2RT_{xz} - 4T_z + \dfrac{R^3 D_0^2}{(1 + RD_0)^2}}{\dfrac{R^2}{2(1 + D_0)^2} + \dfrac{R^3}{(1 + RD_0)^3} + T_{zzz}} \right)^{1/2},$$

where the partial derivatives of T *are evaluated at* (R, D_0).

Proof. We plug the expansion (9) inside (6) and extract work out the undetermined coefficients D_i. The expression for D_1 follows from a direct computation of Φ_x and Φ_{zz}, and evaluating at (R, D_0). To show that D_1 does not vanish, notice that

$$2RT_{xz} - 4T_z = R^3 \frac{\partial}{\partial x} \left(\frac{2}{x^2} T_z(x, z) \right).$$

This is positive since $2/x^2 T_z$ is a series with positive coefficients. Since $R, D_0 > 0$, the remaining term in the numerator inside the square root is clearly positive, and so is the denominator. Hence $D_1 < 0$. $\square$

From the singular expansion of $D(x)$ and the explicit expression (7) of $B(x, y_0)$ in terms of $D(x, y_0)$, it is clear that $B(x) = B(x, y_0)$ also admits a singular expansion at the same singularity R of the form

$$B(x) = B_0 + B_1 X + B_2 X^2 + B_3 X^3 + O(X^4). \tag{10}$$

The next result shows that the singular exponent of $B(x)$ is 3/2, as claimed. Again, the fact that $B_1 = 0$ and $B_3 > 0$ explains the results found in [4] for series-parallel graphs.

Proposition 3.5. *Consider the singular expansion (10). Then* $B_1 = 0$ *and* $B_3 > 0$ *is given by*

$$B_3 = \frac{1}{3} \left(4T_z - 2RT_{xz} - \frac{R^3 D_0^2}{(1 + RD_0)^2} \right) D_1, \tag{11}$$

where the partial derivatives of T *are evaluated in* (R, D_0).

Proof. We plug the singular expansion (9) of $D(x)$ into Equation (7) and work out the undetermined coefficients B_i. One can check that $B_1 = 2R^2\Phi(R, D_0)D_1$, which vanishes because $\Phi(x, D(x)) = 0$.

When computing B_3, it turns out that the values D_2 and D_3 are irrelevant because they appear in a term which contains a factor Φ_z, which by definition vanishes at (R, D_0). This observation gives directly Equation (11). The fact $B_3 \neq 0$ follows from applying the same argument as in the proof of Proposition 3.4, that is, $4T_z - 2RT_{xz} < 0$. Then $B_3 > 0$ since it is the product of two negative numbers. $\square$

3.1.2. Φ *is singular at* (R, D_0). In this case we assume that $T(x,z)$ is singular at (R, D_0) and that $\Phi_z(R, D_0) < 0$. The situation where both $T(x,z)$ is singular and $\Phi_z(R, D_0) = 0$ is treated in the next subsection. We start with a technical lemma.

Lemma 3.6. *The function* T_{zz} *is bounded at the singular point* (R, D_0).

Proof. By differentiating Equation (6) with respect to z we obtain

$$\Phi_z(x,z) = \frac{2}{x^2}T_{zz}(x,z) - \frac{1}{1+z} - \frac{1}{(1+xz)^2} + 1.$$

Since $\Phi_z(R, D_0) < 0$, we have

$$\frac{2}{R^2}T_{zz}(R,D_0) < \frac{1}{1+D_0} + \frac{1}{(1+RD_0)^2} - 1 < 1.$$

Hence $T_{zz}(R, D_0) < R^2/2$. □

Let us consider now the singular expansions of Φ and T in terms of $Z = \sqrt{1 - z/r(x)}$, where $r(x)$ is the dominant singularity. Note that, by Equation (6), Φ and T_z have the same singular behaviour. By Lemma 3.6, the singular exponent α of the dominant singular term Z^α of T_{zz} must be greater than 0 and, consequently, the singular exponent of T_z and Φ is greater than 1. As discussed above, we only study the case where the singular exponent of $T(x,z)$ is $5/2$ (equivalently, the singular exponent of $\Phi(x,z)$ is $3/2$), which corresponds to several families of 3-connected graphs arising from maps. That is, we assume that T has a singular expansion of the form

$$T(x,z) = T_0(x) + T_2(x)Z^2 + T_4(x)Z^4 + T_5(x)Z^5 + O(Z^6),$$

where $Z = \sqrt{1 - z/r(x)}$, and the functions $r(x)$ and $T_i(x)$ are analytic in a neighborhood of R. Notice that $r(R) = D_0$. Since we are assuming that the singular exponent is $5/2$, we have that $T_5(R) \neq 0$.

We introduce now the Taylor expansion of the coefficients $T_i(x)$ at R. However, since we aim at computing the singular expansions of $D(x)$ and $B(x)$ at R, we expand in even powers of $X = \sqrt{1 - x/R}$:

$$\begin{aligned} T(x,z) &= T_{0,0} + T_{0,2}X^2 + O(X^4) \\ &\quad + \left(T_{2,0} + T_{2,2}X^2 + O(X^4)\right) \cdot Z^2 \\ &\quad + \left(T_{4,0} + T_{4,2}X^2 + O(X^4)\right) \cdot Z^4 \\ &\quad + \left(T_{5,0} + T_{5,2}X^2 + O(X^4)\right) \cdot Z^5 + O(Z^6). \end{aligned} \tag{12}$$

Notice that $T_{5,0} = T_5(R) \neq 0$. Similarly, we also consider the expansion of Φ given by

$$\begin{aligned} \Phi(x,z) &= \Phi_{0,0} + \Phi_{0,2}X^2 + O(X^4) \\ &\quad + \left(\Phi_{2,0} + \Phi_{2,2}X^2 + O(X^4)\right) \cdot Z^2 \\ &\quad + \left(\Phi_{3,0} + \Phi_{3,2}X^2 + O(X^4)\right) \cdot Z^3 + O(Z^4), \end{aligned} \tag{13}$$

where $\Phi_{2,0} \neq 0$ because $\Phi_z(R, D_0) < 0$.

The next result shows that $D_1 = 0$ and $D_3 > 0$. This was proved in [2] for the class of planar graphs, but there was no obvious reason explaining this fact. Now we see it follows directly from our general assumptions on $T(x,z)$, which are satisfied when $T(x,z)$ is the GF of 3-connected planar graphs.

Proposition 3.7. *The function* $D(x)$ *admits the following singular expansion*

$$D(x) = D_0 + D_2X^2 + D_3X^3 + O(X^4),$$

where $X = \sqrt{1 - x/R}$. *Moreover,*

$$D_2 = D_0\frac{P}{Q} - Rr', \qquad D_3 = -\frac{5T_{5,0}(-P)^{3/2}}{R^2\,Q^{5/2}} > 0,$$

where r' is the evaluation of the derivative $r'(x)$ at $x = R$, and $P < 0$ and $Q > 0$ are given by

$$P = \Phi_{0,2} = -\frac{4T_{2,0} + 2T_{2,2}}{R^2 D_0} - \frac{2T_{2,0} r'}{R D_0^2} + \frac{R r'}{1 + D_0} - \frac{R D_0 (D_0 + (2 + R D_0) R r')}{(1 + R D_0)^2},$$

$$Q = \Phi_{2,0} = -\frac{4T_{4,0}}{R^2 D_0} + \frac{D_0}{1 + D_0} - \frac{2 R D_0^2}{1 + R D_0} + \frac{R^2 D_0^3}{(1 + R D_0)^2}.$$

Proof. We consider Equation (13) as a power series $\Phi(X, Z)$, where $X = \sqrt{1 - x/R}$ and $Z = \sqrt{1 - z/D_0}$. We look for a solution $Z(X)$ such that $\Phi(X, Z(X)) = 0$; we also impose $Z(0) = 0$, since $\Phi_{0,0} = \Phi(R, D_0) = 0$. Define $D(x)$ as

$$D(x) = r(x)(1 - Z(X)^2),$$

which satisfies $\Phi(x, D(x)) = 0$ and $D(R) = D_0$. By indeterminate coefficients we obtain

$$Z(X) = \pm \sqrt{\frac{-\Phi_{0,2}}{\Phi_{2,0}}} X + \frac{\Phi_{3,0}\, \Phi_{0,2}}{2\, {\Phi_{2,0}}^2} X^2 + O(X^3),$$

where the sign of the coefficient in X is determined later. Now we use this expression and the Taylor series of the analytic function $r(x)$ at $x = R$ to obtain the following singular expansion for $D(x)$:

$$D(x) = D_0 + \left(D_0 \frac{\Phi_{0,2}}{\Phi_{2,0}} - R r' \right) X^2 \pm D_0 \frac{(-\Phi_{0,2})^{3/2}\, \Phi_{3,0}}{{\Phi_{2,0}}^{5/2}} X^3 + O(X^4).$$

Observe in particular that the coefficient of X vanishes. We define $P = \Phi_{0,2}$ and $Q = \Phi_{2,0}$. The fact that $P < 0$ and $Q > 0$ follows from the relations

$$\Phi_z = \frac{-1}{D_0} \Phi_{2,0}, \qquad \Phi_x = \frac{-1}{R} \Phi_{0,2} + \frac{r'}{D_0^2} \Phi_{2,0},$$

that are obtained by differentiating Equation (13). We have $\Phi_z < 0$ by assumption, and $\Phi_x > 0$ following the proof of Proposition 3.4.

The coefficient D_3 must have positive sign, since $D''(x)$ is a positive function and its singular expansion is $D_{xx}(x) = 3 D_3 (4R^2)^{-1} X^{-1} + O(1)$. The coefficients $\Phi_{i,j}$ in Equation (13) are easily expressed in term of the $T_{i,j}$, and a simple computation gives the result as claimed. □

Proposition 3.8. *The function $B(x)$ admits the following singular expansion*

$$B(x) = B_0 + B_2 X^2 + B_4 X^4 + B_5 X^5 + O(X^6),$$

where $X = \sqrt{1 - x/R}$. Moreover,

$$\begin{aligned}
B_0 &= \frac{R^2}{2}\left(D_0 + \frac{1}{2} D_0^2 \right) - \frac{1}{2} R D_0 + \frac{1}{2} \log(1 + R D_0) - \frac{1}{2}(1 + D_0) \frac{R^3 D_0^2}{1 + R D_0} \\
&\quad + T_{0,0} + \frac{1 + D_0}{D_0} T_{2,0}, \\
B_2 &= \frac{R^2 D_0 (D_0^2 R - 2)}{2(1 + R D_0)} + T_{0,2} - \left(2 \frac{1 + D_0}{D_0} + \frac{R r'}{D_0} \right) T_{2,0}, \\
B_4 &= \left(T_{0,4} + \frac{2 R^3 D_0^2 - R^4 D_0^4 + 2 R^2 D_0}{4(1 + R D_0)^2} \right) + \left(\frac{1 + D_0 + r''}{D_0} \right) T_{2,0} + \frac{P^2}{Q} \frac{R^2 D_0}{4} \\
&\quad + \left(\frac{2R}{D_0} T_{2,0} + \frac{R^4 D_0^2}{2(1 + R D_0)^2} \right) r' + \frac{R^4}{4} \left(\frac{D_0}{1 + D_0} - \frac{1}{(1 + R D_0)^2} \right) (r')^2, \\
B_5 &= T_{5,0} \left(-\frac{P}{Q} \right)^{5/2} < 0,
\end{aligned}$$

where P and Q are as in Proposition 3.7, and r' and r'' are the derivatives of $r(x)$ evaluated at $x = R$.

Proof. Our starting point is Equation (7) relating functions D, B and T. We replace T by the singular expansion in Equation (12), D by the singular expansion given in Proposition 3.7, and we set $x = X^2(1-R)$. The expressions for B_i follow by indeterminate coefficients.

When performing these computations we observe that the coefficients B_1 and B_3 vanish identically, and that several simplifications occur in the remaining expressions. □

3.1.3. *Φ has a branch-point and $T(x,z)$ is singular at (R, D_0).* This is the first critical situation, and corresponds to case (3.1) in Theorem 3.1. To study this case we proceed exactly as in the case where Φ is singular at (R, D_0) (Section 3.1.2), except that now $\Phi_z(R, D_0) = 0$. It is easy to check that Lemma 3.6 still applies (with the bound $T_{zz}(R, D_0) \leq R^2/2$). As done in the previous section, we only take into consideration families of graphs where the singular exponent of $T(x,z)$ is 5/2 (equivalently, the singular exponent of $\Phi(x,z)$ is 3/2). Equations (12) and (13) still hold, except that now $\Phi_{2,0} = 0$ because of the branch point at (R, D_0). This missing term is crucial, as we make clear in the following analogous of Proposition 3.7. Notice that $\Phi_{3,0} \neq 0$ because of our assumptions on $T(x,z)$.

Proposition 3.9. *The function $D(x)$ admits the following singular expansion*

$$D(x) = D_0 + D_{4/3}X^{4/3} + O(X^2),$$

where $X = \sqrt{1 - x/R}$ and

$$D_{4/3} = -D_0 \left(\frac{-\Phi_{0,2}}{\Phi_{3,0}} \right)^{2/3}.$$

Proof. As in the proof of Proposition 3.7, we consider a solution $Z(X)$ of the functional equation $\Phi(X, Z(X)) = 0$, and define $D(x)$ as $r(x)(1 - Z(X)^2)$. However, the singular development of $\Phi(x,z)$ is now

$$\begin{aligned} \Phi(x,z) &= \Phi_{0,2}X^2 + O(X^4) \\ &\quad + \left(\Phi_{2,2}X^2 + O(X^4)\right) \cdot Z^2 \\ &\quad + \left(\Phi_{3,0} + \Phi_{3,2}X^2 + O(X^4)\right) \cdot Z^3 + O(Z^4). \end{aligned}$$

since $\Phi_{2,0} \neq 0$, the only way to get the necessary cancellations in $\Phi(X, Z(X)) = 0$, is that the expansion of $Z(X)$ starts with a $Z^{2/3}$ term. By indeterminate coefficients we get

$$Z(X) = \left(\frac{-\Phi_{0,2}}{\Phi_{3,0}} \right)^{2/3} X^{2/3} + O(X^{4/3}).$$

To obtain the actual development of $D(x)$ we use the equalities $D(x) = r(x)(1 - Z(X)^2)$ and $r(R) = D_0$. □

Note that $X = \sqrt{1 - x/R}$. Consequently the previous result implies that the singular exponent of $D(x)$ is 2/3. By using the explicit integration of $B_y(x,y)$ of Equation (7), one can check that the singular exponent of $B(x)$ is 5/3 (the first non-analytic term of $B(x)$ that does not vanish is $X^{10/3}$). This implies that the subexponential term in the asymptotic of b_n is $n^{-8/3}$, as claimed.

3.2. **Singularity analysis of $C(x,y)$ and $G(x,y)$.** The results in this section follow the same lines as those in the previous section. They are technically simpler, since the analysis applies to functions of one variable, whereas the second variable y behaves only as a parameter. It generalizes the analysis in Section 4 of [15] and Section 3 of [4].

Let $F(x) = xC'(x)$, which is the GF of rooted connected graphs. We know that $F(x) = x\exp(B'(F(x))$. Then $\psi(u) = u\exp(-B'(u))$ is the functional inverse of $F(x)$. Denote by ρ the dominant singularity of F. As for 2-connected graphs, there are two possible sources for the singularity:

(1) There exists $\tau \in (0, R)$ (necessarily unique) such that $\psi'(\tau) = 0$. We have a branch point and by the inverse function theorem ψ ceases to be invertible at τ. We have $\rho = \psi(\tau)$.
(2) We have $\psi'(u) \neq 0$ for all $u \in (0, R)$, and there is no obstacle to the analyticity of the inverse function. Then $\rho = \psi(R)$.

The critical case where both sources for singularity coincide is discussed at the end of this subsection. Notice that this happens precisely when $\psi'(R) = 0$.

Condition $\psi'(\tau) = 0$ is equivalent to $B''(\tau) = 1/\tau$. Since $B''(u)$ is increasing (the series $B(u)$ has positive coefficients) and $1/u$ is decreasing, we are in case (1) if $B''(R) > 1/R$, and in case (2) if $B''(R) < 1/R$. As we have already discussed, series-parallel graphs correspond to case (1) and planar graphs to case (2). In particular, if B has singular exponent $3/2$, like for series-parallel graphs, the function $B''(u)$ goes to infinity when u tends to R, so there is always a solution $\tau < R$ satisfying $B''(\tau) = 1/\tau$. This explains why in Theorem 3.1 there is no case where b_n has sub-exponential growth $n^{-5/2}$ and c_n has $n^{-7/2}$.

Proposition 3.10. *The value $S = RB''(R)$ determines the singular exponent of $C(x)$ and $G(x)$ as follows:*

(1) *If $S > 1$, then $C(x)$ and $G(x)$ admit the singular expansions*

$$\begin{aligned} C(x) &= C_0 + C_2X^2 + C_3X^3 + O(X^4), \\ G(x) &= G_0 + G_2X^2 + G_3X^3 + O(X^4), \end{aligned}$$

where $X = \sqrt{1 - x/\rho}$, $\rho = \psi(\tau)$, and τ is the unique solution to $\tau B''(\tau) = 1$. We have

$$C_0 = \tau(1 + \log\rho - \log\tau) + B(\tau), \qquad C_2 = -\tau,$$

$$C_3 = \frac{3}{2}\sqrt{\frac{2\rho\exp\left(B'(\rho)\right)}{\tau B'''(\tau) - \tau B''(\tau)^2 + 2B''(\tau)}},$$

$$G_0 = e^{C_0}, \qquad G_2 = C_2e^{C_0}, \qquad G_3 = C_3e^{C_0}.$$

(2) *If $S < 1$, then $C(x)$ and $G(x)$ admit the singular expansions*

$$\begin{aligned} C(x) &= C_0 + C_2X^2 + C_4X^4 + C_5X^5 + O(X^6), \\ G(x) &= G_0 + G_2X^2 + G_4X^4 + G_5X^5 + O(X^6), \end{aligned}$$

where $X = \sqrt{1 - x/\rho}$, $\rho = \psi(R)$. We have

$$C_0 = \tau(1 + \log\rho - \log R) + B_0, \qquad C_2 = -R,$$

$$C_4 = -\frac{RB_4}{2B_4 - R}, \qquad C_5 = B_5\left(1 - \frac{2B_4}{R}\right)^{-5/2},$$

$$G_0 = e^{C_0}, \qquad G_2 = C_2e^{C_0},$$

$$G_4 = \left(C_4 + \frac{1}{2}{C_2}^2\right)e^{C_0}, \qquad G_5 = C_5e^{C_0},$$

where B_0, B_4 and B_5 are as in Proposition 3.8.

Proof. The two cases $S > 1$ and $S < 1$ arise from the previous discussion. In case (1) we follow the proof of Theorem 3.6 from [4], and in case (2) the proof of Theorem 1 from [15].

First, we obtain the singular expansion of $F(x) = xC'(x)$ near $x = \rho$. This can be done by indeterminate coefficients in the equality $\psi(F(x)) = x = \rho(1 - X^2)$, with $X = \sqrt{1 - x/\rho}$. The expansion of ψ can be either at $\tau = F(\rho)$ where it is analytic, or at $R = F(\rho)$ where it is singular.

From the singular expansion of $F(x)$ we obtain C_2 and C_3 in case (1), and C_2, C_4 and C_5 in case (2) by direct computation. To obtain C_0, however, it is necessary to compute

$$C(x) = \int_0^x \frac{F(t)}{t}\,dt,$$

and this is done using the integration techniques developed in [4] and [15]. Finally, the coefficients for $G(x)$ are obtained directly from the general relation $G(x) = \exp(C(x))$. □

To conclude this section we consider the critical case where both sources of the dominant singularity ρ coincide, that is, when $\psi'(R) = 0$. In this case ψ is singular at R because R is the singularity of $B(x)$, and at the same time the inverse $F(x)$ is singular at $\rho = \psi(R)$ because of the inverse function theorem. As we have shown before, this can only happen if $B(x)$ has singular exponent $5/2$.

The argument is now as in the proof of Proposition 3.9. The singular development of $\psi(z)$ in terms of $Z = \sqrt{1 - z/R}$ must be of the form

$$\psi(z) = \psi_0 + \psi_2 Z^2 + \psi_3 Z^3 + O(Z^4),$$

where in addition ψ_2 vanishes due to $\psi'(R) = 0$. A similar analysis as that in Proposition 3.9 shows that the singular exponent of $C(x)$ is $5/3$. Indeed, since $\psi(F(x)) = x = \rho(1 - X^2)$, we deduce that the development of $F(x)$ in terms of $X = \sqrt{1 - x/\rho}$ is

$$F(x) = \rho + \left(\frac{-\rho^{5/3}}{\psi_3^{2/3}} \right) X^{4/3} + O(X^2).$$

Thus we obtain, by integration of $F(x) = xC'(x)$, that the singular exponent of $C(x)$ is $5/3$, so that the subexponential term in the asymptotic of c_n is $n^{-8/3}$. Since $G(x) = \exp(C(x))$, the same exponents hold for $G(x)$ and g_n.

4. Limit laws

In this section we discuss parameters of random graphs from a closed family whose limit laws do not depend on the singular behaviour of the GFs involved. As we are going to see, only the constants associated to the first two moments depend on the singular exponents.

The parameters we consider are asymptotically either normal or Poisson distributed. The number of edges, number of blocks, number of cut vertices, number of copies of a fixed block, and number of special copies of a fixed subgraph are all normal. On the other hand, the number of connected components is Poisson. The size of the largest connected component (rather, the number of vertices not in the largest component) also follows a discrete limit law. A fundamental extremal parameter, the size of the largest block, is treated in the next section, where it is shown that the asymptotic limit law depends very strongly on the family under consideration.

As in the previous section, let $\mathcal{G}$ be a closed family of graphs. For a fixed value of y, let $\rho(y)$ be the dominant singularity of $C(x, y)$, and let $R(y)$ be that of $B(x, y)$. We write $\rho = \rho(1)$ and $R = R(1)$. Recall that $B'(x, y)$ denotes the derivative with respect to x.

When we speak of cases (1) and (2), we refer to the statement of Proposition 3.10, which are exemplified, respectively, by series-parallel and planar graphs. That is, in case (1) the singular dominant term in $C(x)$ and $G(x)$ is $(1 - x/\rho)^{3/2}$, whereas in case (2) it is $(1 - x/\rho)^{5/2}$. Recall from the previous section that in case (1) we have $\rho(y) = \tau(y) \exp(-B'(\tau(y), y))$, where $\tau(y) B''(\tau(y)) = 1$. In case (2) we have $\rho(y) = R(y) \exp(-B'(R(y), y))$.

4.1. **Number of edges.** The number of edges obeys a limit normal law, and the asymptotic expression for the first two moments is always given in terms of the function $\rho(y)$ for connected graphs, and in terms of $R(y)$ for 2-connected graphs.

Theorem 4.1. *The number of edges in a random graph from $\mathcal{G}$ with n vertices is asymptotically normal, and the mean μ_n and variance σ_n^2 satisfy*

$$\mu_n \sim \kappa n, \qquad \sigma_n^2 \sim \lambda n,$$

where

$$\kappa = -\frac{\rho'(1)}{\rho(1)}, \qquad \lambda = -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)} \right)^2.$$

The same is true, with the same constants, for connected *random graphs from $\mathcal{G}$.*

The number of edges in a random 2-connected graph from $\mathcal{G}$ with n vertices is asymptotically normal, and the mean μ_n and variance σ_n^2 satisfy

$$\mu_n \sim \kappa_2 n, \qquad \sigma_n^2 \sim \lambda_2 n,$$

where

$$\kappa_2 = -\frac{R'(1)}{R(1)}, \qquad \lambda_2 = -\frac{R''(1)}{R(1)} - \frac{R'(1)}{R(1)} + \left(\frac{R'(1)}{R(1)} \right)^2.$$

Proof. The proof is as in [15] and [4]. In all cases the derivatives of $\rho(y)$ and $R(y)$ are readily computed, and for a given family of graphs we can compute the constants exactly. □

4.2. **Number of blocks and cut vertices.** Again we have normal limit laws but the asymptotic for the first two moments depends on which case we are. In the next statements we set $\tau = \tau(1)$.

Theorem 4.2. *The number of blocks in a random connected graph from $\mathcal{G}$ with n vertices is asymptotically normal, and the mean μ_n and variance σ_n^2 are linear in n. In case (1) we have*

$$\mu_n \sim \log(\tau/\rho)\, n, \qquad \sigma_n^2 \sim \left(\log(\tau/\rho) - \frac{1}{1+\tau^2 B'''(\tau)}\right) n.$$

In case (2) we have

$$\mu_n \sim \log(R/\rho)\, n, \qquad \sigma_n^2 \sim \log(R/\rho)\, n.$$

The same is true, with the same constants, for arbitrary random graphs from $\mathcal{G}$.

Proof. The proof for case (2) is as in [15], and is based in an application of the Quasi-Powers Theorem [11]. If $C(x,u)$ is the generating function of connected graphs where now u marks blocks, then we have

$$xC'(x,u) = x\exp\left(uB'(xC'(x,u))\right), \tag{14}$$

where derivatives are as usual with respect to x. For fixed u, $\psi(t) = t\exp(-uB'(t))$ is the functional inverse of $xC'(x,u)$. We know that for $u = 1$, $\psi'(t)$ does not vanish, and the same is true for u close to 1 by continuity. The dominant singularity of $C(x,u)$ is at $\sigma(u) = \psi(R) = R\exp(-uB'(R))$, and it is easy to compute the derivatives $\sigma'(1)$ and $\sigma''(1)$ (see [15] for details).

In case (1), Equation (14) holds as well, but now the dominant singularity is at $\psi(\tau)$. A routine (but longer) computation gives the constants as claimed. □

Theorem 4.3. *The number of cut vertices in a random connected graph from $\mathcal{G}$ with n vertices is asymptotically normal, and the mean μ_n and variance σ_n^2 are linear in n. In case (1) we have*

$$\mu_n \sim \left(1 - \frac{\rho}{\tau}\right) n, \qquad \sigma_n^2 \sim \left(\frac{\rho}{\tau} - \left(\frac{\rho}{\tau}\right)2 - \left(\frac{\rho}{\tau}\right)2\frac{1}{1+\tau 2B'''(\tau)}\right) n$$

In case (2) we have

$$\mu_n \sim \left(1 - \frac{\rho}{R}\right) n, \qquad \sigma_n^2 \sim \frac{\rho}{R}\left(1 - \frac{\rho}{R}\right) n.$$

The same is true, with the same constants, for arbitrary random graphs from $\mathcal{G}$.

Proof. If u marks cut vertices in $C(x,u)$, then we have

$$xC'(x,u) = xu(\exp\left(B'(xC'(x,u))\right) - 1) + x.$$

It follows that, for given u,

$$\psi(t) = \frac{t}{u(\exp\left(B'(t)\right) - 1) + 1}$$

is the inverse function of $xC'(x,u)$. In case (2) the dominant singularity $\sigma(u)$ is at $\psi(R)$. Taking into account that $\rho = R\exp(B'(R))$, the derivatives of σ are easily computed. In case (1) the singularity is at $\psi(\tau(u))$, where $\tau(u)$ is given by $\psi'(\tau(u)) = 0$. In order to compute derivatives, we differentiate $\psi(\tau(u)) = 0$ with respect to u and solve for $\tau'(u)$, and once more in order to get $\tau''(u)$. After several computations and simplifications using `Maple`, we get the values as claimed. □

4.3. **Number of copies of a subgraph.** Let H be a fixed rooted graph from the class $\mathcal{G}$, with vertex set $\{1,\ldots,h\}$ and root r. Following [18], we say that H *appears* in G at $W \subset V(G)$ if (a) there is an increasing bijection from $\{1,\ldots,h\}$ to W giving an isomorphism between H and the induced subgraph $G[W]$ of G; and (b) there is exactly one edge in G between W and the rest of G, and this edge is incident with the root r.

Thus an appearance of H gives a copy of H in G of a very particular type, since the copy is joined to the rest of the graph through a unique pendant edge. We do not know how to count the number of subgraphs isomorphic to H in a random graph, but we can count very precisely the number of appearances.

Theorem 4.4. *Let H be a fixed rooted connected graph in $\mathcal{G}$ with h vertices. Let X_n denote the number of appearances of H in a random rooted connected graph from $\mathcal{G}$ with n vertices. Then X_n is asymptotically normal and the mean μ_n and variance σ_n^2 satisfy*

$$\mu_n \sim \frac{\rho^h}{h!}\, n, \qquad \sigma_n^2 \sim \rho\, n,$$

Proof. The proof is as in [15], and is based on the Quasi-Powers Theorem. If $f(x,u)$ is the generating function of rooted connected graphs and u marks appearances of H then, up to a simple term that does not affect the asymptotic estimates, we have

$$f(x,u) = x \exp\left(B'(f(x,u)) + (u-1)\frac{x^h}{h!}\right).$$

The dominant singularity is computed through a change of variable, and the rest of the computation is standard; see the proof of Theorem 5 in [15] for details. For this parameter is no difference between cases (1) and (2). □

Now we study appearances of a fixed 2-connected subgraph L from $\mathcal{G}$ in rooted connected graphs. An appearance of L in this case corresponds to a block with a labelling order isomorphic to L. Notice in this case that an appearance can be anywhere in the tree of blocks, not only as a terminal block.

Theorem 4.5. *Let L be a fixed rooted 2-connected graph in $\mathcal{G}$ with $\ell+1$ vertices. Let X_n denote the number of appearances of L in a random connected graph from $\mathcal{G}$ with n vertices. Then X_n is asymptotically normal and the mean μ_n and variance σ_n^2 satisfy*

$$\mu_n \sim \frac{R^\ell}{\ell!}\, n, \qquad \sigma_n^2 \sim \frac{R^\ell}{\ell!}\, n,$$

Proof. If $f(x,u)$ is the generating function of rooted connected graphs and u marks appearances of L, then we have

$$f(x,u) = x \exp\left(B'(f(x,u)) + (u-1)\frac{f(x,u)^\ell}{\ell!}\right).$$

The reason as that each occurrence of L is single out by multiplying by u. Notice that L has $\ell+1$ vertices by the root bears no label. It follows that the inverse of $f(x,u)$ is given by (for a given value of u) is

$$\phi(t) = t \exp\left(-B'(t) - (u-1)t^\ell/\ell!\right).$$

The singularity of $\phi(t)$ is equal to R, independently of t. Since for $u=1$ we know that $\phi'(t)$ does not vanish, the same is true for u close to 1. Then the dominant singularity of $f(x,u)$ is given by

$$\sigma(u) = \phi(R) = \rho \cdot \exp(-(u-1)R^\ell/\ell!),$$

since $\rho = R\exp(-B'(R))$. A simple calculation gives

$$\sigma'(1) = -\rho\frac{R^\ell}{\ell!}, \qquad \sigma''(1) = \rho\frac{R^{2\ell}}{\ell!^2}$$

and the results follows easily as in the proof of Theorem 4.1. Again, for this parameter there is no difference between cases (1) and (2). □

4.4. **Number of connected components.** Our next parameter, as opposed to the previous one, follows a discrete limit law.

Theorem 4.6. *Let X_n denote the number of connected components in a random graph $\mathcal{G}$ with n vertices. Then $X_n - 1$ is distributed asymptotically as a Poisson law of parameter ν, where $\nu = C(\rho) = C_0$.*

As a consequence, the probability that a random graph $\mathcal{G}$ is connected is asymptotically equal to $e^{-\nu}$.

Proof. The proof is as in [15]. The generating function of graphs with exactly k connected components is $C(x)^k/k!$. Taking the k-th power of the singular expansion of $C(x)$, we have $[x^n]C(x)^k \sim k\nu^{k-1}[x^n]C(x)$. Hence the probability that a random graphs has exactly k components is asymptotically

$$\frac{[x^n]C(x)^k/k!}{[x^n]G(x)} \sim \frac{k\nu^{k-1}}{k!}\, e^{-\nu} = \frac{\nu^{k-1}}{(k-1)!}\, e^{-\nu}$$

as was to be proved. □

4.5. **Size of the largest connected component.** Extremal parameters are treated in the next two sections. However, the size of the largest component is easy to analyze and we include it here. The notation M_n in the next statement, suggesting vertices *missed* by the largest component, is borrowed from [17]. Recall that g_n, c_n are the numbers of graphs and connected graphs, respectively, R is the radius of convergence of $B(x)$, and C_i are the singular coefficients of $C(x)$.

Theorem 4.7. *Let L_n denote the size of the largest connected component in a random graph $\mathcal{G}$ with n vertices, and let $M_n = L_n - n$. Then*

$$\mathbf{P}\left(M_n = k\right) \sim p_k = p \cdot g_k \frac{\rho^k}{k!},$$

where p is the probability of a random graph being connected. Asymptotically, either $p_k \sim c\, k^{-5/2}$ or $p_k \sim c\, k^{-7/2}$ as $k \to \infty$, depending on the subexponential term in the estimate of g_k.

In addition, we have $\sum p_k = 1$ and $\mathbb{E}\left[M_n\right] \sim \tau$ in case (1) and $\mathbb{E}\left[M_n\right] \sim R$ in case (2). In case (1) the variance $\sigma^2(M_n)$ does not exist and in case (2) we have $\sigma^2(M_n) \sim R + 2C_4$.

Proof. The proof is essentially the same as in [17]. For fixed k we have

$$\binom{n}{k} \frac{c_{n-k} g_k}{g_n},$$

since there are $\binom{n}{k}$ ways of choosing the labels of the vertices not in the largest component, c_{n-k} ways of choosing the largest component, and g_k ways of choosing the complement. In case (1), given the estimates

$$g_n \sim g \cdot n^{-5/2} \rho^{-n} n!, \qquad c_n \sim c \cdot n^{-5/2} \rho^{-n} n!,$$

the estimate for p_k follows at once (we argue similarly in each subcase of (2)). Observe that $p = \lim c_n/g_n = c/g$.

For the second part of the statement notice that,

$$\sum p_k = p \sum g_k \frac{\rho^k}{k!} = p\, G(\rho) = 1,$$

since from Theorem 4.6 it follows that $p = e^{-C(\rho)} = 1/G(\rho)$. To compute the moments notice that the probability GF is $f(u) = \sum p_k u^k = pG(\rho u)$. Then the expectation is estimated as

$$f'(1) = p\, \rho G'(\rho) = p\, G(\rho)\rho C'(\rho),$$

which correspond to τ in case (1) and R in case (2), since $G(x) = \exp(C(x))$. For the variance we compute

$$f''(1) + f'(1) - f'(1)^2 = \rho C'(\rho) + \rho^2 C''(\rho).$$

In case (1) $\lim_{x\to\rho} C''(x) = \infty$, so that the variance does not exist. In case (2) we have $\rho C'(\rho) = R$ and $\rho^2 C''(\rho) = 2C_4$. □

5. Largest block and 2-connected core

The problem of estimating the largest block in random maps has been well studied. We recall that a map is a connected planar graph together with a specific embedding in the plane. Moreover, an edge has been oriented and marked as the root edge. Gao and Wormald [13] proved that the largest block in a random map with n edges has almost surely $n/3$ edges, with deviations of order $n^{2/3}$. More precisely, if X_n is the size of the largest block, then

$$\mathbf{P}\left(|X_n - n/3| < \lambda(n)n^{2/3}\right) \to 1, \qquad \text{as } n \to \infty,$$

where $\lambda(n)$ is any function going to infinity with n. The picture was further clarified by Banderier et al. [1]. They found that the largest block in random maps obeys a continuous limit law, which is called by the authors the 'Airy distribution of the map type', and is closely related to a stable law of index 3/2. As we will see shortly, the Airy distribution also appears in random planar graphs.

A useful technical device is to work with the 2-connected core, which in the case of maps is the (unique) block containing the root edge. For graphs it is a bit more delicate. Consider a connected graph R rooted at a vertex v. We would like to say that the core of R is the block containing the root, but if v is a cut vertex then there are several blocks containing v and there is no clear way to single out one of them. Another possibility is to say that the 2-connected core is the union of the blocks containing the root, but then the core is not in general a 2-connected graph.

The definition we adopt is the following. If the root is not a cut vertex, then the *core* is the unique block containing the root. Otherwise, we say that the rooted graph is *coreless.* Let $C^\bullet(x,u)$ be the generating function of rooted connected graphs, where the root bears no label, and u marks the size of the 2-connected core. Then we have

$$C^\bullet(x,u) = B'(uxC'(x)) + \exp(B'(xC'(x)) - B'(xC'(x)),$$

where $C(x)$ and $B(x)$ are the GFs for connected and 2-connected graphs, respectively. The first summand corresponds to graphs which have a core, whose size is recorded through variable u, and the second one to coreless graphs. We rewrite the former equation as

$$C^\bullet(x,u) = Q(uH(x)) + Q_L(x),$$

where

$$H(x) = xC'(x), \quad Q(x) = B'(x), \quad Q_L(x) = \exp(B'(xC'(x)) - B'(xC'(x)).$$

With this notation, $Q(uH(x))$ enumerates graphs with core and $Q_L(x)$ enumerates coreless graphs. The asymptotic probability that a graph is coreless is

$$p_L = \lim_{n\to\infty} \frac{[x^n]Q_L(x)}{[x^n]C'(x)} = 1 - \lim_{n\to\infty} \frac{[x^n]Q(H(x))}{[x^n]C'(x)}.$$

The key point is that graphs with core fit into a composition scheme

$$Q(uH(x)).$$

This has to be understood as follows. A rooted connected graph whose root is not a cut vertex is obtained from a 2-connected graph (the core), replacing each vertex of the core by a rooted connected graph. It is shown in [1] that such a composition scheme leads either to a discrete law or to a continuous law, depending on the nature of the singularities of $Q(x)$ and $H(x)$.

Our analysis for a closed class $\mathcal{G}$ is divided into two cases. If we are in case (1) of Proposition 3.10, we say that the class $\mathcal{G}$ is *series-parallel-like*; in this situation the size of the core follows invariably a discrete law which can be determined precisely in terms of $Q(x)$ and $H(x)$. If we are in case (2) we say that the class $\mathcal{G}$ is *planar-like.* In this situation the size of the core has two modes, a discrete law when the core is small, and a continuous Airy distribution when the core has linear size. Moreover, for planar-like classes, the size of the largest block follows the same Airy distribution and is concentrated around αn for a computable constant α. The critical case, discussed at the end of Section 3, is not treated here.

5.1. **Core of series-parallel-like classes.** Recall that in case (1) of Proposition 3.10 we have $H(\rho) = \rho C'(\rho) = \tau$, where τ is the solution to the equation $\tau B''(\tau) = 1$. Since $RB''(R) > 1$ and $uB''(u)$ is an increasing function, we conclude that $H(\rho) < R$. This gives rise to the so called *subcritical* composition scheme. We refer to the exposition in section IX.3 of [11]. The main result we use is Proposition IX.1 from [11], which is the following.

Proposition 5.1. *Consider the composition scheme $Q(uH(x))$. Let R, ρ be the radius of convergence of Q and H, respectively. Assume that Q and H satisfy the subcritical condition $\tau = H(\rho) \leq R$, and that $H(x)$ has a unique singularity at ρ on its disk of convergence with a singular expansion*

$$H(x) = \tau - c_\lambda (1 - z/\rho)^\lambda + o((1 - z/\rho)^\lambda),$$

where $\tau, c_\lambda > 0$ and $0 < \lambda < 1$. Then the size of the Q-core follows a discrete limit law,

$$\lim_{n\to\infty} \frac{[x^n u^k]Q(uH(x))}{[x^n]Q(H(x))} = q_k.$$

The probability generating function $q(u) = \sum q_k u^k$ of the limit distribution is

$$q(u) = \frac{uQ'(\tau u)}{Q'(\tau)}.$$

The previous result applies to our composition scheme $Q(uH(x))$, that is, to the family of rooted connected graphs that have core.

Theorem 5.2. *Let $\mathcal{G}$ be a series-parallel-like class, and let Y_n be the size of the 2-connected core in a random rooted connected graph $\mathcal{G}$ with core and n vertices. Then $\mathbf{P}(Y_n = k)$ tends to a limit q_k as n goes to infinity. The probability generating function $q(u) = \sum q_k u^k$ is given by*

$$q(u) = \tau u B''(\tau u).$$

The estimates of q_k for large k depend on the singular behaviour of $B(x)$ near R as follows, where $X = \sqrt{1 - x/R}$:

(a) *If $B(x) = B_0 + B_2X^2 + B_3X^3 + O(X^4)$, then $q_k \sim \dfrac{3B_3}{4R\sqrt{\pi}} k^{-1/2} \left(\dfrac{\tau}{R}\right)^k$.*

(b) *If $B(x) = B_0 + B_2X^2 + B_4X^4 + B_5X^5 + O(X^6)$, then $q_k \sim -\dfrac{5B_5}{2R\sqrt{\pi}} k^{-3/2} \left(\dfrac{\tau}{R}\right)^k$.*

Finally, the probability of a graph being coreless is asymptotically equal to $1 - \rho/\tau$.

Proof. We apply Proposition 5.1 with $Q(x) = B'(x)$ and $H(x) = xC'(x)$. Since $\tau B''(\tau) = 1$, we have

$$q(u) = \frac{uB''(\tau u)}{B''(\tau)} = \tau u B''(\tau u),$$

as claimed. The dominant singularity of $q(u)$ is at $u = R/\tau$. The asymptotic for the tail of the distribution follows by the corresponding singular expansions. In case (a) we have

$$B''(X) = \frac{3B_3}{4R^2} X^{-1} + O(1).$$

In case (b) we have

$$B''(X) = \frac{2B_4}{R^2} + \frac{5B_5}{R^2} X + O(X^2).$$

By applying singularity analysis to $q(u)$, the result follows. We remark that $B_3 > 0$ and $B_5 < 0$, so that the multiplicative constants are in each case positive. □

It is shown in [22] that the largest block in series-parallel classes is of order $O(\log n)$. This is to be expected given the exponential tails of the distributions in the previous theorem.

5.2. **Largest block of planar-like classes.** In order to state our main result, we need to introduce the Airy distribution. Its density is given by

$$g(x) = 2e^{-2x^3/3}(x\mathrm{Ai}(x^2) - \mathrm{Ai}'(x^2)), \tag{15}$$

where $\mathrm{Ai}(x)$ is the Airy function, a particular solution of the differential equation $y'' - xy = 0$. An explicit series expansion is (see Equation (2) in [1])

$$g(x) = \frac{1}{\pi x} \sum_{n\geq 1} (-3^{2/3}x)^n \frac{\Gamma(1 + 2n/3)}{n!} \sin(-2n\pi/3).$$

A plot of $g(x)$ is shown in Figure 1. We remark that the left tail (as $x \to -\infty$) decays polynomially while the right tail (as $x \to +\infty$) decays exponentially. We are in case (2) of Proposition 3.10. In this situation we have $\rho = \psi(R)$ and $H(\rho) = R$, which is a *critical* composition scheme. We need Theorem 5 of [1] and the discussion preceding it, which we rephrase in the following proposition.

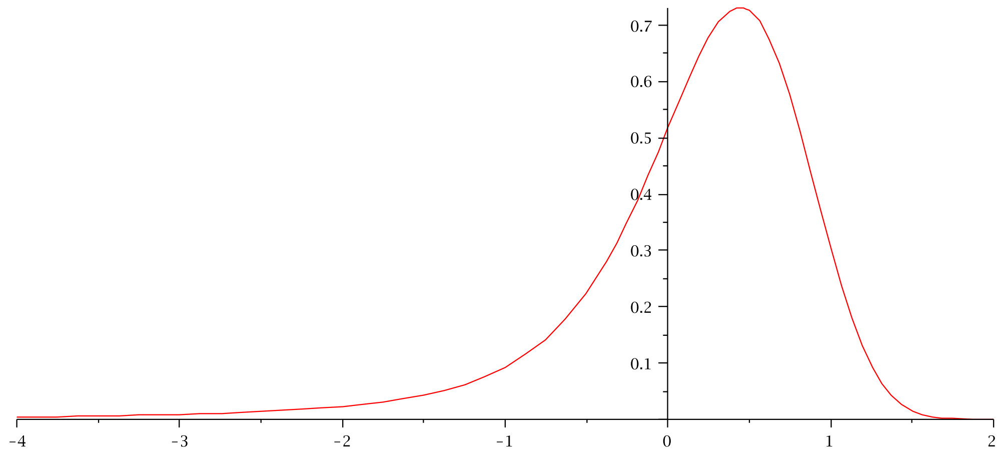

FIGURE 1. The Airy distribution.

Proposition 5.3. *Consider the composition scheme $Q(uH(x))$. Let R, ρ be the radius of convergence of Q and H respectively. Assume that Q and H satisfy the critical condition $H(\rho) = R$, and that $H(x)$ and $Q(z)$ have a unique singularity at ρ and R in their respective discs of convergence. Moreover, the singularities of $H(x)$ and $Q(z)$ are of type $3/2$, that is,*

$$\begin{aligned} H(x) &= H_0 + H_2X^2 + H_3X^3 + O(X^4), \\ Q(z) &= Q_0 + Q_2Z^2 + Q_3Z^3 + O(Z^4), \end{aligned}$$

where $X = \sqrt{1 - x/\rho}$, $Z = \sqrt{1 - z/R}$. Let α_0 and M_3 be

$$\alpha_0 = -\frac{H_0}{H_2}, \qquad M_3 = -\frac{Q_2H_3}{R} + Q_3\alpha_0^{-3/2}.$$

Then the asymptotic distribution of the size of the Q-core in $Q(uH(z))$ has two different modes. With probability $p_s = -Q_2H_3/(RM_3)$ the core has size $O(1)$, and with probability $1 - p_s$ the core follows a continuous limit Airy distribution concentrated at $\alpha_0 n$. More precisely, let Y_n be the size of the Q-core of a random element of size n of $Q(uH(z))$.

(a) *For fixed k,*

$$\mathbf{P}\left(Y_n = k\right) \sim \frac{H_3}{M_3} kR^{k-1}[z^k]Q(z).$$

(b) *For $k = \alpha_0 n + xn^{2/3}$ with $x = O(1)$,*

$$n^{2/3}\mathbf{P}\left(Y_n = k\right) \sim \frac{Q_3\alpha_0^{-3/2}}{M_3} cg(cx), \qquad c = \frac{1}{\alpha_0}\left(\frac{-H_2}{3H_3}\right)^{2/3},$$

where $cg(cx)$ is the Airy distribution of parameter c.

In particular, we have $\mathbb{E}\left[X_n\right] \sim \alpha_0 n$. The parameter c quantifies in some sense the dispersion of the distribution (not the variance, since the second moment does not exist). Note that the asymptotic probability that the core has size $O(1)$ is

$$p_s = \sum_{k=0}^{\infty} \mathbf{P}\left(X_n = k\right) \sim \frac{H_3}{M_3} \sum_{k=0}^{\infty} kR^{k-1}[z^k]Q(z) = \frac{H_3}{M_3} Q'(R) = \frac{H_3}{M_3}\left(\frac{-Q_2}{R}\right),$$

and that the asymptotic probability that the core has size $\Theta(n)$ is

$$\frac{Q_3\alpha_0^{-3/2}}{M_3} = 1 - p_s.$$

Now we state the main result in this section. Recall that for a planar-like class of graphs we have

$$B(X) = B_0 + B_2X^2 + B_4X^4 + B_5X^5 + O(X^6),$$

where R is the dominant singularity of $B(x)$ and $X = \sqrt{1 - x/R}$.

Theorem 5.4. *Let $\mathcal{G}$ be a planar-like class, and let X_n be the size of the largest block in a random connected graph $\mathcal{G}$ with n vertices. Then*

$$\mathbf{P}\left(X_n = \alpha n + xn^{2/3}\right) \sim n^{-2/3}cg(cx),$$

where

$$\alpha = \frac{R - 2B_4}{R}, \qquad c = \left(\frac{-2R}{15B_5}\right)^{2/3},$$

and $g(x)$ is as in (15). Moreover, the size of the second largest block is $O(n^{2/3})$. In particular, for the class of planar graphs we have $\alpha \approx 0.95982$ and $c \approx 128.35169$.

Proof. The composition scheme in our case is $B'(uxC'(x))$. In the notation of the previous proposition, we have $Q(x) = B'(x)$ and $H(x) = xC(x)$.

The size of the core is obtained as a direct application of Proposition 5.3. The exact values for planar graphs have been computed using the known singular expansions for $B(x)$ and $C(x)$ given in the appendix of [15].

For the size of the largest block, one can adapt an argument from [1], implying that the probability that the core has linear size while not being the largest block tends to 0 exponentially fast. It follows that the distribution of the size of the largest block is exactly the same as the distribution of the core in the linear range. □

The main conclusion is that for planar-like classes of graphs (and in particular for planar graphs) there exists a unique largest block of linear size, whose expected value is asymptotically αn for some computable constant α. The remaining block are of size $O(n^{2/3})$. This is in complete contrast with series-parallel graphs, where we have seen that there are only blocks of sublinear size.

Remark. An observation that we need later, is that if the largest block L has N vertices, then it is uniformly distributed among all the 2-connected graphs in the class. This is because the number of graphs of given size whose largest block is L depends only on the number of vertices of L, and not on its isomorphism type.

We can also analyze the size of the largest block for graphs with a given edge density, or average degree. We state a precise result for planar graphs, which is an important example.

Theorem 5.5. *For $\mu \in (1, 3)$, the largest block in random planar graphs with n vertices and $\lfloor \mu n \rfloor$ edges follows asymptotically an Airy law with computable parameters $\alpha(\mu)$ and $c(\mu)$.*

Proof. As discussed in [15], we choose a value $y_0 > 0$ depending on μ such that, if we give weight y_0^k to a graph with k edges, then only graphs with n vertices and μn edges have non negligible weight. If $\rho(y)$ is the radius of convergence of $C(x, y)$ as usual, the right choice is the unique positive solution y_0 of

$$-y\rho'(y)/\rho(y) = \mu, \tag{16}$$

Then we work with the generating function $xC(x, y_0)$ instead of $xC'(x)$. Again we have a critical composition scheme, and as in the proof of Theorem 5.4, the size of the largest block follows asymptotically an Airy law. □

Figure 2 shows a plot of the main parameter $\alpha(\mu)$ for planar graphs and $\mu \in (1, 3)$. When $\mu \to 3^-$ we see that $\alpha(\mu)$ approaches 1; the explanation is that a planar triangulation is 3-connected and hence has a unique block. When $\mu \to 1^+$, $\alpha(\mu)$ tends to 0, in this case because the largest block in a tree is just an edge.

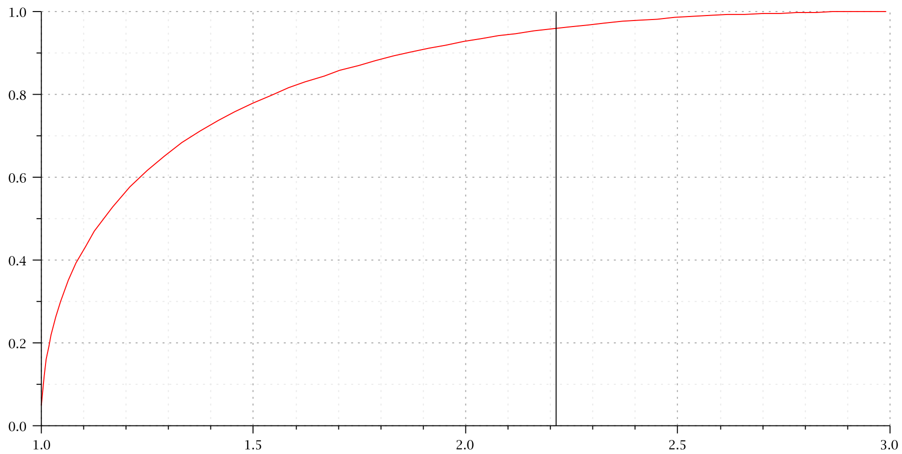

FIGURE 2. Size of largest block for planar graphs with μn edges, $\mu \in (1,3)$. The ordinate gives the value $\alpha(\mu)$ such that the largest block has size $\sim \alpha(\mu)n$. The value at κ is 0.9598 as in Theorem 5.4.

6. LARGEST 3-CONNECTED COMPONENT

Let us recall that, given a 2-connected graph G, the 3-connected components of G are those 3-connected graphs that are the support of h-networks in the network decomposition of G.

We have seen in Theorem 5.4 that the largest block in a random graph from a planar-like class is almost surely of linear size, and it is unique. In this section we prove a similar result for the largest 3-connected component in random connected graphs with n vertices. Again we obtain a limit Airy law, but the proof is more involved. We remark that for series-parallel-like classes there is no linear 3-connected component, just as for 2-connected components.

There are three main technical issues we need to address:

(1) We start with a connected graph G. We know from Theorem 5.4 that the largest block L of G is distributed according to an Airy law. We show that the largest 3-connected component T of L is again Airy distributed. Thus we have to *concatenate* two Airy laws, and we show that we obtain another Airy law with computable parameters. Our proof is based on the fact that the sum of two independent stable laws of the same index α (recall that the Airy law corresponds to a particular stable law of index $3/2$) is again an Airy law with computable parameters. In order to illustrate this step, we prove a result of independent interest: given a random planar map with m edges, the size of the largest 3-component is Airy distributed with expected value $n/9$.

(2) We also need to analyze the number of edges in the largest block L of a connected graph. The number of vertices of L is Airy distributed with known parameters; see Theorem 5.4. On the other hand, the number of edges in 2-connected graphs with N vertices is asymptotically normally distributed with expected value $\kappa_2 N$ (see Theorem 4.1). Thus we have to study a parameter normally distributed (number of edges) within the largest block, whose size (number of vertices) follows an Airy law. We show that the composition of these two limit laws gives rise to an Airy law for the number of edges in the largest block, again with computable parameters.

(3) The analysis of the largest block in random connected graphs is in terms of the number of vertices, but the analysis for the largest 3-connected component of a 2-connected graph is necessarily in terms of the number of edges. Thus we need a way to relate both models. This is done through a technical lemma that shows that two probability distributions on 2-connected graphs with m edges are asymptotically equivalent. This is the content of Lemma 6.6.

Our main result is the following. We state it for planar graphs, since this is the most interesting case and we can give explicitly the parameters, but it holds more generally for planar-like classes of graphs.

Theorem 6.1. *Let X_n be the number of vertices in the largest 3-connected component of a random connected planar graph with n vertices. Then*

$$\mathbf{P}\left(X_n = \alpha_2 n + x n^{2/3}\right) \sim n^{-2/3} c_2 g(c_2 x),$$

where $\alpha_2 \approx 0.7346$ and $c_2 \approx 3.1459$ are computable constants. Additionally, the number of edges in the largest 3-connected component of a random connected planar graph with n vertices also follows asymptotically an Airy law with parameters $\alpha_3 \approx 1.7921$ and $c_3 \approx 1.28956$.

The rest of the section is devoted to the proof of the theorem. The next three subsections address the technical points discussed above.

6.1. **Largest 3-connected component in random planar maps.** Recall that a planar map (we say just a map) is a connected planar graph together with a specific embedding in the plane. The size of largest k-components in several families of maps was thoroughly studied in [1]. Denote by $M(z)$, $B(z)$ and $C(z)$ the ordinary GFs associated to maps, 2-connected maps and 3-connected maps, respectively; in all cases, z marks edges. Let L_n be the random variable, defined over the set of maps with n edges, equal to the size of the largest 2-connected component. Let T_m be the random variable, defined over the set of 2-connected maps with m edges, equal to the size of the largest 3-connected component.

In [1] it is shown the following result:

Theorem 6.2. *The distribution of both L_n and T_m follows asymptotically an Airy law, namely*

$$\begin{aligned}
\mathbf{P}\left(L_n = a_1 n + x n^{2/3}\right) &\sim n^{-2/3} c_1 g(c_1 x), \\
\mathbf{P}\left(T_m = a_2 m + y m^{2/3}\right) &\sim m^{-2/3} c_2 g(c_2 y),
\end{aligned}$$

where $g(x)$ is the map Airy distribution, $a_1 = 1/3$, $c_1 = 3/4^{2/3}$, $a_2 = 1/3$, and $c_2 = 3^{4/3}/4$.

Proof. Here is a sketch of the proof. In both cases, the distribution arises from a critical composition scheme of the form $\frac{3}{2} \circ \frac{3}{2}$. The distribution of L_n is given by the scheme $B\left(z(1+M(z))^2\right)$, which reflects the fact that a map is obtained by gluing a map at each corner of a 2-connected map. In the second case, the result is obtained from the composition scheme $C\left(B(z)/z - 2\right)$, which reflects the fact that a 2-connected map is obtained by replacing each edge of a 3-connected map by a non-trivial 2-connected map (to complete the picture one must take also into account series and parallel compositions, but these play no role in the analysis of the largest 3-connected component, see [26] for details). □

Let X_n be the random variable equal to the size of the largest 3-connected component in maps of n vertices. In order to get a a limit law for X_n, we need a more detailed study of stable laws. In particular, Airy laws are particular examples of stable laws of index $3/2$. Our main reference is the forthcoming book [20]. The result we need is Proposition 1.17, which appears in [20, Section 1.6]. We rephrase it here in a form convenient for us.

Proposition 6.3. *Let Y_1 and Y_2 be independent Airy distributions, with density probability functions $c_1 g(c_1 x)$ and $c_2 g(c_2 x)$. Then $Y_1 + Y_2$ follows an Airy distribution with density probability function $cg(cx)$, where $c = \left(c_1^{-3/2} + c_2^{-3/2}\right)^{-2/3}$.*

Proof. We use the same notation as in [20]. A stable law is characterized by its *stability factor* $\alpha \in [0, 2)$, its *skewness* $\beta \in [-1, 1]$, its *factor scale* $\gamma > 0$, and its *location parameter* $\delta \in \mathbb{R}$. A stable random variable with this parameters is written in the form $S(\alpha, \beta, \gamma, \delta; 1)$ (the constant 1 refers to the type of the parametrization; we only deal with this type). Proposition 1.17 in [20] states that if $S_1 = S(\alpha, \beta_1, \gamma_1, \delta_1; 1)$ and $S_2 = S(\alpha, \beta_2, \gamma_2, \delta_2; 1)$ are independent random variables, then $S_1 + S_2 = S(\alpha, \beta, \gamma, \delta, 1)$, with

$$\beta = \frac{\beta_1\gamma_1^\alpha + \beta_2\gamma_2^\alpha}{\gamma_1^\alpha + \gamma_2^\alpha},\ \gamma^\alpha = \gamma_1^\alpha + \gamma_2^\alpha,\ \delta = \delta_1 + \delta_2. \tag{17}$$

Let us identify the Airy distribution with density probability function $cg(cx)$ within the family of stable laws as defined. By definition, the stability factor is equal to $3/2$. Additionally, $\beta = -1$: this is the unique value that makes that a stable law decreases exponentially fast (see Section 1.5 of [20]). The value of the location parameter δ coincides with the expectation of the random variable, hence $\delta = 0$ (see Proposition 1.13). Finally, the factor scale can be written in the form γ_0/c, for a suitable value of γ_0 (the one which corresponds with the normalized Airy distribution with density $g(x)$). Since $Y_1 = S(3/2, -1, \gamma_0/c_1, 0; 1)$ and $Y_2 = S(3/2, -1, \gamma_0/c_2, 0; 1)$, the result follows from (17). □

Theorem 6.4. *The size X_n of the largest 3-connected component in a random map with n edges follows asymptotically an Airy law of the form*

$$\mathbf{P}\left(X_n = an + zn^{2/3}\right) \sim n^{-2/3}cg(cz),$$

where $g(z)$ is the Airy distribution and

$$a = a_1a_2 = 1/9, \qquad c = \left(\left(\frac{c_1}{a_2}\right)^{-3/2} + c_2^{-3/2}a_1\right)^{-2/3} \approx 1.71707.$$

Proof. Let us estimate $n^{2/3}\mathbf{P}\left(X_n = an + zn^{2/3}\right)$ for large n. Considering the possible values size of the largest 2-connected component, we obtain

$$n^{2/3}\mathbf{P}\left(X_n = an + zn^{2/3}\right) = n^{2/3}\sum_{m=1}^{\infty}\mathbf{P}\left(L_n = m\right)\mathbf{P}\left(T_m = an + zn^{2/3}\right).$$

In the previous equation we have used the fact that the largest 2-connected component is distributed uniformly among all 2-connected maps with the same number of edges; this is because the number of ways a 2-connected map M can be completed to a map of given size depends only on the size of M.

Notice that X_n and T_m are *integer* random variables, hence the previous equation should be written in fact as

$$n^{2/3}\mathbf{P}\left(X_n = \lfloor an + zn^{2/3}\rfloor\right) = n^{2/3}\sum_{m=1}^{\infty}\mathbf{P}\left(L_n = m\right)\mathbf{P}\left(T_m = \lfloor an + zn^{2/3}\rfloor\right).$$

Let us write $m = a_1n + xn^{2/3}$. Then $an + zn^{2/3} = a_2m + ym^{2/3} + o\left(m^{2/3}\right)$, where $y = a_1^{-2/3}(z - a_2x)$. Observe that when we vary m in one unit, we vary x in $n^{-2/3}$ units. Let $x_0 = (1 - a_1n)n^{-2/3}$, so that $a_1n + x_0n^{2/3} = 1$ is the initial term in the sum. The previous sum can be written in the form

$$n^{2/3}\sum_{x=x_0+\ell n^{-2/3}}\mathbf{P}\left(L_n = a_1n + xn^{2/3}\right)\mathbf{P}\left(T_m = a_2m + a_1^{-2/3}(z - a_2x)m^{2/3}\right).$$

where the sum is for all values $\ell \geq 0$. From Theorem 6.2 it follows that

$$\begin{aligned}
& n^{2/3}\sum_{x=x_0+\ell n^{-2/3}}\mathbf{P}\left(L_n = a_1n + xn^{2/3}\right)\mathbf{P}\left(T_m = a_2m + a_1^{-2/3}(z - a_2x)m^{2/3}\right) \\
\sim\ & n^{2/3}\sum_{x=x_0+\ell n^{-2/3}} n^{-2/3}c_1g(c_1x)\ m^{-2/3}c_2g\left(c_2a_1^{-2/3}(z - a_2x)\right) \\
\sim\ & \frac{1}{n^{2/3}}\sum_{x=x_0+\ell n^{-2/3}} c_1g(c_1x)\ c_2a_1^{-2/3}g\left(c_2a_1^{-2/3}(z - a_2x)\right).
\end{aligned}$$

In the last equality we have used that $m^{-2/3} = (a_1 n)^{-2/3}(1+o(1))$. Now we approximate by an integral:

$$n^{-2/3} \sum_{x=x_0+\ell n^{-2/3}} c_1 g(c_1 x)\ c_2 a_1^{-2/3} g\left(c_2 a_1^{-2/3}(z-a_2 x)\right)$$
$$\sim \int_{-\infty}^{\infty} c_1 g(c_1 x)\ c_2 a_1^{-2/3} g\left(c_2 a_1^{-2/3}(z-a_2 x)\right) dx$$

The previous estimate holds uniformly for x in a bounded interval. Now we set $a_2 x = u$, and with this change of variables we get

$$\int_{-\infty}^{\infty} \frac{c_1}{a_2} g\left(\frac{c_1}{a_2} u\right)\ c_2 a_1^{-2/3} g\left(c_2 a_1^{-2/3}(z-u)\right) du.$$

This convolution can be interpreted as a sum of stable laws with parameter $3/2$ in the following way. Let Y_1 and Y_2 be independent random variables with densities $\frac{c_1}{a_2} g\left(\frac{c_1}{a_2} u\right)$ and $c_2 a_1^{-2/3} g\left(c_2 a_1^{-2/3}(z-u)\right)$, respectively. Then, the previous integral is precisely $\mathbf{P}\left(Y_1 + Y_2 = z\right)$, and the result follows from Proposition 6.3. □

Remark. The previous theorem can be obtained, alternatively, using the machinery developed in [1]. The two composition schemes $B(zM(z)^2)$ and $C(B(z)/z-2)$ can be concatenated algebraically into a single composition scheme $C(B(zM(z)^2)/z-2$. This is again a critical scheme with exponents 3/2 and an Airy law follows from the general scheme in [1]. The parameters can be computed using the singular expansions of $M(z)$, $B(z)$, $C(z)$ at their dominant singularities which are, respectively, equal to 1/12, 4/27 and 1/4. We have performed the corresponding computations in complete agreement with values obtained in Theorem 6.4. We have chosen the present proof since the same ideas are used later in the case of graphs, where no explicit composition seems available.

6.2. **Number of edges in the largest block of a connected graph.** As discussed above, we have a limit Airy law X_n for the number of vertices in the largest block L in a random connected planar graph. In order to analyze the largest 3-connected component of L, we need to express X_n in terms of the number of edges. This amounts to combine the limit Airy law with a normal limit law, leading to slightly modified Airy law. The precise result is the following.

Theorem 6.5. *Let Z_n be the number of* edges *in the largest block of a random connected planar graph with n vertices. Then*

$$\mathbf{P}\left(Z_n = \kappa_2 \alpha n + z n^{2/3}\right) \sim n^{-2/3} \frac{c}{\kappa_2} g\left(\frac{c}{\kappa_2} z\right),$$

here α and c are as in Theorem 5.4, and $\kappa_2 \approx 2.26288$ is the constant for the expected number of edges in random 2-connected planar graphs, as in Theorem 4.1.

Proof. Let X_n be, as in Theorem 5.4, the number of vertices in the largest block. In addition, let Y_N be the number of edges in a random 2-connected planar graph with N vertices. Then

$$\begin{aligned}\mathbf{P}\left(Z_n = \kappa_2 \alpha n + z n^{2/3}\right) = \\ \textstyle\sum_{x=x_0+\ell n^{-2/3}} \mathbf{P}\left(X_n = \alpha n + x n^{2/3}\right) \mathbf{P}\left(Y_{\alpha n + x n^{2/3}} = \kappa_2 \alpha n + z n^{2/3}\right),\end{aligned} \tag{18}$$

with the same convention for the index of summation as in the previous section.

Since Y_N is asymptotically normal (Theorem 4.1)

$$\mathbf{P}\left(Y_N = \kappa_2 N + y N^{1/2}\right) \sim N^{-1/2} h(y),$$

where $h(y)$ is the density of a normal law suitably scaled. If we take $N = \alpha n + x n^{2/3}$, then

$$\kappa_2 N = \kappa_2 \alpha N + \kappa_2 x n^{2/3}.$$

As a consequence, the significant terms in the sum in (18) are concentrated around $\alpha n+(z/\kappa_2)n^{2/3}$, within a window of size $N^{1/2}=\Theta(n^{1/2})$. Thus we can conclude that

$$\mathbf{P}\left(Z_n=\kappa_2\alpha n+zn^{2/3}\right)\sim\frac{1}{\kappa_2}\mathbf{P}\left(X_n=\lfloor\alpha n+(z/\kappa_2)n^{2/3}\rfloor\right)\sim n^{-2/3}\frac{c}{\kappa_2}g\left(\frac{c}{\kappa_2}z\right),$$

where c is the constant in Theorem 5.4. The factor $\frac{1}{\kappa_2}$ in the middle arises since in $\lfloor\kappa_2\alpha n+zn^{2/3}\rfloor$ we have steps of length $n^{-2/3}$, whereas in $\lfloor\alpha n+(z/\kappa_2)n^{2/3}\rfloor$ they are of length $n^{-2/3}/\kappa_2$. □

6.3. **Probability distributions for 2-connected graphs.** In this section we study several probability distributions defined on the set of 2-connected graphs of m edges. The first distribution X_n^m (in fact, a family of probability distributions, one for each n) models the appearance of largest blocks with m edges in random connected graphs with n vertices. The second one Y_m is a weighted distribution where each 2-connected graph with m edges receives a weight according to the number of vertices, and for which it is easy to obtain an Airy law for the size of the largest 3-connected component. We show that these two distributions are asymptotically equivalent in a suitable range. In particular, it follows that the Airy law of the latter distribution also occurs in the former one. We start by defining precisely both distributions. We use capital letters like X_n^m and Y_m to denote random variables whose output is a graph in the corresponding universe of graphs, so that our distributions are associated to these random variables. We find this convention more transparent than defining the associated probability measures.

Let n,m be fixed numbers, and let $\mathcal{C}_n^m$ denote the set of connected graphs on n vertices such that their largest block L has m edges. The first probability distribution X_n^m is the distribution of L in a graph of $\mathcal{C}_n^m$ chosen uniformly at random. That is, if B is a 2-connected graph with m edges, and $\mathcal{C}_n^B\subseteq\mathcal{C}_n^m$ denotes the set of connected graphs that have $L=B$ as the largest block, then

$$\mathbf{P}\left(X_n^m=B\right)=\frac{|\mathcal{C}_n^B|}{|\mathcal{C}_n^m|}.$$

Let m be a fixed number. The second probability distribution Y_m assigns to a 2-connected graph B of m edges and k vertices the probability

$$\mathbf{P}\left(Y_m=B\right)=\frac{R^k}{k!}\frac{1}{[y^m]B(R,y)},\tag{19}$$

where R is the radius of convergence of the exponential generating function $B(x)=B(x,1)$ enumerating 2-connected graphs. It is clear that $[y^m]B(R,y)$ is the right normalization factor.

Now we state precisely what we mean when we say that these two distributions are asymptotically equivalent in a suitable range. In what follows, α and κ_2 are the multiplicative constants of the expected size of the largest block and the expected number of edges in a random connected graph (see Theorems 5.4 and 4.1). We denote by $V(G)$ and $E(G)$ the set of vertices and edges of a graph G.

Lemma 6.6. *Fix positive values $\bar{y},\bar{z}\in\mathbb{R}^+$. For fixed m, let I_n and I_k denote the intervals*

$$\begin{aligned}I_k&=\left[\frac{1}{\kappa_2}m-\bar{z}m^{1/2},\frac{1}{\kappa_2}m+\bar{z}m^{1/2}\right],\\ I_n&=\left[\frac{1}{\alpha\kappa_2}m-\bar{y}m^{2/3},\frac{1}{\alpha\kappa_2}m+\bar{y}m^{2/3}\right].\end{aligned}$$

Then, for $n\in I_n$, the probability distributions Y_m and X_n^m are asymptotically equal on graphs with $k\in I_k$ vertices, with uniform convergence for both k and n. That is, there exists a function $\epsilon(m)$ with $\lim_{m\to\infty}\epsilon(m)=0$ such that, for every 2-connected graph B with m edges and $k\in I_k$ vertices, and for every $n\in I_n$, it holds that

$$\left|\frac{\mathbf{P}\left(X_n^m=B\right)}{\mathbf{P}\left(Y_m=B\right)}-1\right|<\epsilon(m).$$

Proof. Fix $y \in [-\bar{y}, \bar{y}]$, and let $n = \lfloor \frac{1}{\alpha\kappa_2} m + ym^{2/3} \rfloor \in I_n$. First, we prove that X_n^m and Y_m are concentrated on graphs with $k = \frac{1}{\kappa_2} m + O(m^{1/2})$ vertices, that is,

$$\mathbf{P}\left(\frac{1}{\kappa_2} m - zm^{1/2} \leq |V(X_n^m)| \leq \frac{1}{\kappa_2} m + zm^{1/2}\right)$$

goes to 1 when $z, m \to \infty$, and the same is true for Y_m. Then, we show that X_n^m and Y_m are asymptotically *proportional* for graphs on $k \in I_k$ vertices. A direct consequence of both facts is that X_n^m and Y_m are asymptotically *equal* in I_k, since the previous results are valid for arbitrarily large $\bar{z}$.

We start by considering the probability distribution Y_m. If we add (19) over all the $b_{k,m}$ 2-connected graphs with k vertices and m edges, we get

$$\mathbf{P}\left(|V(Y_m)| = k\right) = b_{k,m} \frac{R^k}{k!} \frac{1}{[y^m]B(R, y)}.$$

On the one hand, the value $[y^m]B(R, y)$ is a constant that does not depend on k. On the other hand, since the numbers $b_{k,m}$ satisfy a local limit theorem (the proof is the same as in [15]), it follows that the numbers $b_{k,m}R^k/k!$ follow a normal distribution concentrated at $k = 1/\kappa_2$ on a scale $m^{1/2}$, as desired.

We show the same result for X_n^m. Let B be a 2-connected graph with k vertices and m edges. We write the probability that a graph drawn according to X_n^m is B as a conditional probability on the largest block L_n of a random connected graph of n vertices. In what follows, $v(L_n)$ and $e(L_n)$ denote, respectively, the number of vertices and edges of L_n.

$$\begin{aligned} \mathbf{P}\left(X_n^m = B\right) &= \mathbf{P}\left(L_n = B \;\mid\; e(L_n) = m\right) \\ &= \frac{\mathbf{P}\left(L_n = B,\ e(L_n) = m\right)}{\mathbf{P}\left(e(L_n) = m\right)} = \frac{\mathbf{P}\left(L_n = B\right)}{\mathbf{P}\left(e(L_n) = m\right)}. \end{aligned}$$

Note that in the last equality we drop the condition $e(L_n) = m$ because it is subsumed by $L_n = B$.

The probability that the largest block L_n is B is the same for all 2-connected graphs on k vertices. Hence, if b_k denotes the number of 2-connected graphs on k vertices, we have

$$\mathbf{P}\left(X_n^m = B\right) = \frac{1}{b_k} \frac{\mathbf{P}\left(v(L_n) = k\right)}{\mathbf{P}\left(e(L_n) = m\right)},$$

If we sum over all the $b_{k,m}$ 2-connected graphs B with k vertices and m edges, we finally get the probability of X_n^m having k vertices,

$$\mathbf{P}\left(|V(X_n^m)| = k\right) = \frac{b_{k,m}}{b_k} \frac{\mathbf{P}\left(v(L_n) = k\right)}{\mathbf{P}\left(e(L_n) = m\right)}.$$

For fixed n, m, the numbers $\mathbf{P}\left(V(L_n) = k\right)$ follow an Airy distribution of scale $n^{2/3}$ concentrated at $k_1 = \alpha n$ (see Section 5.2), and the numbers $b_{k,m}/b_k$ are normally distributed around $k_2 = m/\kappa_2$ on a scale $m^{1/2}$. The choice of n makes k_1 and k_2 coincide but for a lower-order term $O(m^{2/3})$; hence, it follows that $\mathbf{P}\left(|V(X_n^m)| = k\right)$ is concentrated at $k_2 = m/\kappa_2$ on a scale $m^{1/2}$, as desired.

Now that we have established concentration for both probability distributions, we just need to show that they are asymptotically proportional in the range $k = m/\kappa_2 + O(m^{1/2})$. This is easy to establish by considering asymptotic estimates. Indeed, we have

$$\mathbf{P}\left(X_n^m = B\right) = \frac{1}{b_k} \frac{\mathbf{P}\left(v(L_n) = k\right)}{\mathbf{P}\left(e(L_n) = m\right)},$$

and since $\mathbf{P}\left(v(L_n) = k\right)$ is Airy distributed in the range $k = m/\kappa_2 + O(m^{2/3})$ and $b_k \sim b \cdot k^{-7/2} R^{-k} k!$, it follows that

$$\mathbf{P}\left(X_n^m = B\right) \sim b^{-1} \cdot k^{7/2} \frac{R^k}{k!} \frac{g(x)}{\mathbf{P}\left(e(L_n) = m\right)},$$

where x is defined as $(k-\alpha n)n^{-2/3}$ and $g(x)$ is the Airy distribution of the appropriate scale factor. Let us compare it with the exact expression for the probability distribution Y_m, that is,

$$\mathbf{P}\left(Y_m = B\right) = \frac{R^k}{k!}\frac{1}{[y^m]B(R,y)}.$$

Clearly, both expressions coincide in the high order terms R^k and $1/k!$. The remaining terms are either constants like b, $\mathbf{P}\left(e(L_n)=m\right)$ and $[y^m]B(R,y)$, or expressions that are asymptotically constant in the range of interest. This is the case for $k^{7/2}$, which is asymptotically equal to $((1/\kappa_2)m)^{7/2}$. And also for $g(x)$, which is asymptotically equal to $g(y(\alpha\kappa_2)^{2/3})$, since $x=(k-\alpha n)n^{-2/3}$ and $n=\frac{1}{\alpha k_2}m+ym^{2/3}$ implies that

$$\begin{aligned} x &= \left(\frac{1}{k_2}m + O(m^{1/2}) - \frac{\alpha}{\alpha k_2}m + ym^{2/3}\right)n^{-2/3} \\ &= \left(ym^{2/3}\right)\left(\frac{m}{\alpha\kappa_2}\right)^{-2/3} + o(1) \\ &= y(\alpha\kappa_2)^{2/3} + o(1) \end{aligned}$$

in the given range. Hence, both distributions are asymptotically proportional in the given range.

Thus, we have shown the result when n is linked to m by $n=\frac{1}{\alpha\kappa_2}m+ym^{2/3}$, for any y. Clearly, uniformity holds when y is restricted to a compact set of $\mathbb{R}$, like $[-\bar{y},\bar{y}]$. □

6.4. **Proof of the main result.** In order to prove Theorem 6.1, we have to concatenate two Airy laws. The first one is the number of edges in the largest block, given by Theorem 6.5. The second is the number of edges in the largest 3-connected component of a random 2-connected planar graph with a given number of edges. This is a gain an Airy law produced by the composition scheme $T_z(x, D(x,y))$, which encodes the combinatorial operation of substituting each edge of a 3-connected graph by a network (which is essentially a 2-connected graph rooted at an edge). However, this scheme is relative to the variable y marking edges. In order to have a legal composition scheme we need to take a fixed value of x. The right value is $x=R$, as shown by Lemma 6.6 Indeed, taking $x=R$ amounts to weight a 2-connected graph G with m edges with $R^k/k!$, where k is the number of vertices in G. Thus the relevant composition scheme is precisely $T_z(R, uD(R,y))$, where u marks the size of the 3-connected core. Formally, we can write it as the scheme

$$C(uH(y)), \qquad H(y) = D(R,y), \qquad C(y) = T_z(R,y).$$

The composition scheme $T_z(R, D(R,y))$ is critical with exponents $3/2$, and an Airy law appears. In order to compute the parameters we need the expansion of $D(R,y)$ at the dominant singularity $y=1$, which is of the form

$$D(R,y) = \widetilde{D}_0 + \widetilde{D}_2 Y^2 + \widetilde{D}_3 Y^3 + O(Y^4), \tag{20}$$

and $Y=\sqrt{1-y}$. The different $\widetilde{D}_i$ can be obtained in the same way as in Proposition 3.7.

Proposition 6.7. *Let W_m be the number of edges in the largest 3-connected component of a 2-connected planar graph with m edges, weighted with $R^k/k!$, where k is the number of vertices. Then*

$$\mathbf{P}\left(W_m = \beta n + zn^{2/3}\right) \sim n^{-2/3}c_2 g\left(c_2 z\right),$$

where $\beta = -\widetilde{D}_0/\widetilde{D}_2 \approx 0.82513$ and $c_2 = -\widetilde{D}_2/\widetilde{D}_0\left(-\widetilde{D}_2/3\widetilde{D}_3\right)^{2/3} \approx 2.16648$, and the $\widetilde{D}_i$ are as in Equation (20).

Proof. The proof is a direct application of the methods in Theorem 5.4. □

Proof of Theorem 6.1. Recall that X_n is the number of vertices in the largest 3-connected component of a random connected planar graph with n vertices. This variables aries as the composition of two random variables we have already studied. First we consider Z_n as in Theorem 6.5, which is the number of edges in the largest block, and then W_m as in Proposition 6.7.

The main parameter turns out to be $\alpha_2 = \mu\beta(\kappa_2\alpha)$, where

(1) α is for the expected number of vertices in the largest block;
(2) κ_2 is for the expected number of edges in 2-connected graphs;
(3) β is for the expected number of edges in the largest 3-connected component;
(4) μ is for the expected number of vertices in 3-connected graphs weighted according to $R^k/k!$ if k is the number of vertices.

The constant in 1. and 3. correspond to Airy laws, and the constants in 2. and 4. to normal laws.

Let Y_n be the number of edges in the largest 3-connected component of a random connected planar graph with n vertices (observe that our main random variable X_n is linked directly to Y_n after extracting a parameter normally distributed like the number of vertices). Then

$$\mathbf{P}\left(Y_n = \beta\kappa_2\alpha n + zn^{2/3}\right) = \sum_{m=1}^{\infty} \mathbf{P}\left(Z_n = m\right)\mathbf{P}\left(W_m = \beta\kappa_2\alpha n + zn^{2/3}\right).$$

This convolution can be analyzed in exactly the same way as in the proof of Theorem 6.4, giving rise to a limit Airy law with the parameters as claimed.

Finally, in order to go from Y_n to X_n we need only to multiply the main parameter by μ and adjust the scale factor. To compute μ we need the dominant singularity $\tau(x)$ of the generating function $T(x, z)$ of 3-connected planar graphs, for a given value of x (see Section 6 in [9]). Then

$$\mu = -R\tau'(R)/\tau(R).$$

Given that the inverse function $r(z)$ is explicit (see Equation (25) in [9]), the computation is straightforward.

7. Minor-closed classes

In this section we apply the machinery developed so far to analyse families of graphs closed under minors. A class of graphs $\mathcal{G}$ is minor-closed if whenever a graph is in $\mathcal{G}$ all its minors are also in $\mathcal{G}$. Given a minor-closed class $\mathcal{G}$, a graph H is an excluded minor for $\mathcal{G}$ if H is not in $\mathcal{G}$ but every proper minor is in $\mathcal{G}$. It is an easy fact that a graph is in $\mathcal{G}$ if and only if it does not contain as a minor any of the excluded minors from $\mathcal{G}$. According to the fundamental theorem of Robertson and Seymour, for every minor-closed class the number of excluded minors is finite [23]. We use the notation $\mathcal{G} = \mathrm{Ex}(H_1, \cdots, H_k)$ if $H_1, \ldots, H_k$ are the excluded minors of $\mathcal{G}$. If all the H_i are 3-connected, then $\mathrm{Ex}(H_1, \cdots, H_k)$ is a closed family. This is because if none of the 3-connected components of a graph G contains a forbidden, the same is true for G itself.

In order to apply our results we must know which connected graphs are in the set $\mathrm{Ex}(H_1, \cdots, H_k)$. There are several results in the literature of this kind. The easiest one is $\mathrm{Ex}(K_4)$, which is the class of series-parallel graphs. Since a graph in this class always contains a vertex of degree at most two, there are no 3-connected graphs. Table 1 contains several such results, due to Wagner, Halin and others (see Chapter X in [8]). The proofs make systematic use of Tutte's Wheels Theorem: a 3-connected graph can be reduced to a wheel by a sequence of deletions and contractions of edges, while keeping it 3-connected.

The 3-connected graphs when excluding W_5 and the triangular prism take longer to describe. For $K_3 \times K_2$ they are: $K_5, K_5^-, \{W_n\}_{n\geq 3}$, and the family G_Δ of graphs obtained from $K_{3,n}$ by adding any number of edges to the part of the bipartition having 3 vertices. For W_5 they are: K_4, K_5, the family G_Δ, the graphs of the octahedron and the cube Q, the graph obtained from Q by contracting one edge, the graph L obtained from $K_{3,3}$ by adding two edge in one of the parts of the bipartition, plus all the 3-connected subgraphs of the former list. Care is needed here for checking that all 3-connected graphs are included and for counting how many labellings each graph has.

Ex. minors	3-connected graphs	Generating function $T(x,z)$
K_4	$\emptyset$	0
W_4	K_4	$z^6x^4/4!$
K_5-e	$K_3, K_{3,3}, K_3\times K_2, \{W_n\}_{n\geq 3}$	$70z^9x^6/6! - \frac{1}{2}x(\log(1-z^2x)+2z^2x+z^4x^2)$
$K_5, K_{3,3}$	Planar 3-connected	$T_p(x,z)$
$K_{3,3}$	Planar 3-connected, K_5	$T_p(x,z)+z^{10}x^5/5!$
$K_{3,3}^+$	Planar 3-connected, $K_5, K_{3,3}$	$T_p(x,z)+z^{10}x^5/5!+10z^9x^6/6!$

TABLE 1. Classes of graphs defined from one excluded minor. $T_p(x,z)$ is the GF of planar 3-connected graphs.

Once we have the full collection of 3-connected graphs, we have $T(x,z)$ at our disposal. For the family of wheels we have a logarithmic term (see the previous table) and for the family G_Δ it is a simple expression involving $\exp(z^3x)$. We can then apply the machinery developed in this paper and compute the generating functions $B(x,y)$ and $C(x,y)$. For the last three entries in Table 1, the main problem is computing $B(x,y)$ and this was done in [15] and [14]; these correspond to the planar-like case. In the remaining cases $T(x,z)$ is either analytic or has a simple singularity coming from the term $\log(1-xz^2)$, and they correspond to the series-parallel-like case.

In Table 2 we present the fundamental constants for the classes under study. For a given class $\mathcal{G}$ they are: the growth constants ρ^{-1} of graphs in $\mathcal{G}$; the growth constant R^{-1} of 2-connected graphs in $\mathcal{G}$; the asymptotic probability p that a random graph in $\mathcal{G}$ is connected; the constant κ such that κn is the asymptotic expected number of edges for graphs in $\mathcal{G}$ with n vertices; the analogous constant κ_2 for 2-connected graphs in $\mathcal{G}$; the constant β such that βn is the asymptotic expected number of blocks for graphs in $\mathcal{G}$ with n vertices; and the constant δ such that δn is the asymptotic expected number of cut vertices for graphs in $\mathcal{G}$ with n vertices. The values in Table 2 have been computed with `Maple` using the results in sections 3 and 4.

Class	ρ^{-1}	R^{-1}	κ	κ_2	β	δ	p
Ex(K_4)	9.0733	7.8123	1.61673	1.71891	0.149374	0.138753	0.88904
Ex(W_4)	11.5437	10.3712	1.76427	1.85432	0.107065	0.101533	0.91305
Ex(W_5)	14.6667	13.5508	1.90239	1.97981	0.0791307	0.0760808	0.93167
Ex(K_5^-)	15.6471	14.5275	1.88351	1.95360	0.0742327	0.0715444	0.93597
Ex($K_3\times K_2$)	16.2404	15.1284	1.92832	1.9989	0.0709204	0.0684639	0.93832
Planar	27.2269	26.1841	2.21327	2.2629	0.0390518	0.0382991	0.96325
Ex($K_{3,3}$)	27.2293	26.1866	2.21338	2.26299	0.0390483	0.0382957	0.963262
Ex($K_{3,3}^+$)	27.2295	26.1867	2.21337	2.26298	0.0390481	0.0382956	0.963263

TABLE 2. Constants for a given class of graphs: ρ and R are the radius of convergence of $C(x)$ and $G(x)$, respectively; constants $\kappa, \kappa_2, \beta, \delta$ give, respectively, the first moment of the number of: edges, edges in 2-connected graphs, blocks and cut vertices; p is the probability of connectedness.

8. CRITICAL PHENOMENA

Let $\mathcal{G}$ be a closed family of graphs, and let $T(x,z)$ be the generating function associated to 3-connected graphs in $\mathcal{G}$, as in Theorem 3.2. Assume that $T_z(x,z)$ has singular exponent $\alpha=3/2$ at each singular point. Then, according to cases (2.1), (2.2) and (2.3) of Theorem 3.2, the singular exponents of $B(x,y_0)$ and $C(x,y_0)$, for fixed $y=y_0$, are either $5/2$ or $3/2$. However it could happen that, for a fixed class $\mathcal{G}$, we get different exponents depending on the value of y_0. When this happens for $B(x, y_0)$, we say that a *critical phenomenon* occurs for *2-connected* graphs in $\mathcal{G}$; if it happens for $C(x, y_0)$, we say that a critical phenomenon occurs for *connected* graphs in $\mathcal{G}$. Note that for such a phenomenon to occur, $T_z(x, z)$ must be of singular exponent $\alpha \geq 1$, since otherwise we would be in case (1) of Theorem 3.2, and the singular exponents of $B(x, y_0)$ and $C(x, y_0)$ would be 3/2 for all y_0. This already implies that there are no critical phenomena in the classes $\mathrm{Ex}(K_4)$, $\mathrm{Ex}(W_4)$, $\mathrm{Ex}(W_5)$, $\mathrm{Ex}(K_5^-)$ and $\mathrm{Ex}(K_3 \times K_2)$ discussed in the previous section.

When the generating function $T_z(x, z)$ has singular exponent $\alpha > 1$, the situation may be different. No critical phenomenon occurs for the class of planar graphs, since the singular exponents of $B(x, y_0)$ and $C(x, y_0)$ are 5/2 for any value of $y = y_0$, as proved in [15]. However, there exist classes where $T_z(x, z)$ has singular exponent $\alpha = 3/2$, and critical phenomena for 2-connected and connected graphs do occur; examples are presented later in this section. Incidentally, this will show that all three cases (2.1), (2.2) and (2.3) of Theorem 3.2 are in fact attainable.

Before giving the examples, let us discuss the significance of having a critical phenomenon. For simplicity, we only discuss it for connected graphs, but the situation is completely analogous for the 2-connected case. As mentioned in Theorem 5.5, fixing a constant value $y = y_0$ corresponds to weighting the edges so that almost all graphs, counted according to their weight, have $\mu(y_0)n + O(\sqrt{n})$ edges, where $\mu(y)$ is as in Equation (16). Setting $y_0 = 1$ corresponds to giving no weight, so that $\mu(1)$ gives the expected edge-density of all graphs in the class. If $\mathcal{G}$ is a class of graphs with a critical phenomena for connected graphs at $y = y_c$, then there is a critical edge density $\mu_c = \mu(y_c)$ such that *significant qualitative differences* are expected between connected graphs in $\mathcal{G}$ with more than μ_c edges per vertex, and those with less edges. As a rule, connected graphs in $\mathcal{G}$ will be planar-like for some densities (either $\mu > \mu_c$ or $\mu < \mu_c$, both situations may arise), and series-parallel in the complementary interval.

The most interesting difference between the two sides of a critical value $\mu_c = \mu(y_c)$ is the appearance of a block of linear size: graphs with density $\mu_1 = \mu(y_1)$ such that $C(x, y_1)$ has singular exponent 5/2 almost surely have a block of linear size, but if the singular exponent is 3/2, then the blocks have size $O(\log n)$. The critical density μ_c marks the transition between these two regimes. We remark that this transition is relatively smooth. When $\mu(y)$ approaches the critical value $\mu_c(y_c)$ on the planar-like case, the parameter $\alpha(y)$ that gives the size $\alpha(y)n$ of the largest block goes to 0; this is due to $\alpha(y) = -H_0(y)/H_2(y)$ by Theorem 5.3, and $H(x, y) = xC'(x, y)$ being of singular exponent 2/3 at $y = y_c$. In the series-parallel-like case, the expected size of the 2-connected components goes to infinite as y approaches y_c; this is due to Theorem 5.2 and the fact that $\tau(y) \to R(y)$ as $y \to y_c$.

Although our examples of classes with critical phenomena may look a bit unnatural from a combinatorial point of view, this situation is hardly surprising from an analytic point of view. We have two sources for the main singularity of $C(x, y)$ for a given value of y: either (a) it comes from the singularities of $B(x, y)$; or (b) it comes from a branch point of the equation defining $C(x, y)$. For planar graphs the singularity always comes from case (a), and for series-parallel graphs always from case (b). If there is a value y_0 for which the two sources coalesce, then we get a different singular exponent depending on whether $y < y_c$ or $y > y_c$.

Next we present some significant examples, and for this we need to recall the following result from Tutte [25]. Let a_n the number of (unlabelled) rooted planar triangulations with n vertices. Then

$$A(x) = \sum a_n x^n = x^2\theta(1 - 2\theta) - x^3,$$

where $\theta = \theta(x)$ is defined by

$$x = \theta(1 - \theta)^3,$$

and the term x^3 corresponds to a triangle, which in our context is not a 3-connected graph.

Triangulations. In this example, $\mathcal{T}$ is the family of planar triangulations (that is, maximal planar graphs). In order to apply Theorem 3.2 we need an explicit expression for the GF $T_z(x, z)$. Since a triangulation has $3n - 6$ edges, it follows that the number A_n of labelled triangulations rooted at a directed edge is given by

$$n!a_n = 4(3n - 6)A_n.$$

Indeed, there are $n!$ factorial ways to label a rooted triangulation, and there are $4(3n-6)$ ways to root a labelled triangulation. This gives

$$T_z(x,z) = \frac{1}{4z^7}A(xz^3).$$

The computations for this expression of $T_z(x,z)$ give a critical value $y_0 \approx 0.44686$, corresponding to a critical value $\mu_0 \approx 1.87551$ for the number of edges. In this case $B(x,y)$ has singular exponent $3/2$ when $\mu < \mu_0$, and has exponent $5/2$ when $\mu > \mu_0$. The intuitive explanation is the following: to obtain a dense 2-connected graph in this family it helps to use a large triangulation, and sparse graphs are obtained doing series and parallel connections on small triangulations. When $\mu = \mu_0$ we are precisely in the critical point, corresponding to case (3.1) in Theorem 3.2.

We also find a critical value for connected graphs, in this case at $y_0' \approx 0.45227$, corresponding to $\mu_0' \approx 1.73290$. Notice that μ_0' is the density for connected graphs, whereas μ_0 is for 2-connected graphs. This particular family, as well as the next ones, shows that all the situations described in cases (2) and (3) of Theorem 3.2 are in fact possible. More precisely:

- w hen $y < y_0$ we are in case (2.3);
- when $y = y_0$ we are in case (3.1);
- when $y_0 < y < y_0'$ we are in case (2.2);
- when $y = y_0'$ we are in case (3.2);
- when $y_0' < y$ we are in case (2.1).

In case (2.2), a random connected graph has no giant block, but random 2-connected graphs have a giant 3-connected component. This is not a contradiction, since we are analyzing different families of graphs. Notice that fixing a value of y gives different densities for connected and 2-connected graphs.

Cubic planar graphs. Now $\mathcal{T}$ is the family of 3-connected cubic planar graphs, which are precisely the planar duals of triangulations. If b_{2n} is the number of (unlabelled) rooted cubic planar maps with $2n$ vertices then, since a triangulation with n vertices has $2n-4$ faces, we have $a_n = b_{2n-4}$. It follows that

$$B(x) = \sum b_n x^n = \frac{1}{x^4}A(x^2).$$

As in the previous example we get

$$T_z(x,z) = \frac{1}{4z^7}B(xz^{3/2}).$$

For $B(x,y)$ we find a critical phenomenon at $y_0 \approx 0.074223$, but with the two regimes reversed: $B(x,y)$ has singular exponent $5/2$ when $y < y_0$ and $3/2$ when $y > y_0$. The corresponding critical value for the number of edges is $\mu_0 \approx 1.31726$. Since a cubic graph has only $3n/2$ edges, in this case denser graphs are obtained using small 3-connected components.

There is again a critical point for connected graphs too at $y_0' \approx 0.067075$ and $\mu_0' \approx 1.18441$. Notice that now $y_0' < y_0$, as opposed to the previous example. There are five regimes as before, but they appear in reverse order.

Two critical points. This example shows that more than one critical value may occur. This is done by adding a single dense graph to the family $\mathcal{T}$ in the first example. Let $\mathcal{T}$ be the family of triangulations plus the exceptional graph K_6. In terms of generating functions, it means to add the monomial $z^{15}x^6/6!$ to $T(x,z)$.

Then there are two critical values $y_0 \approx 0.44690$ and $y_1 \approx 108.88389$ for 2-connected graphs, and the corresponding edge densities are $\mu_0 \approx 1.87558$ and $\mu_1 \approx 3.49211$. This last value is close to $7/2$; this is the maximal edge density, which is approached by taking many copies of K_6 glued along a common edge. It turns out that $B(x,y)$ has exponent $3/2$ when $y < y_0$, $5/2$ when $y_0 < y < y_1$, and again $3/2$ for $y_1 < y$.

For connected graphs there are again two critical values $y_0' \approx 0.45232$ and $y_1' \approx 108.88387$, with corresponding edge densities $\mu_0' \approx 1.73298$ and $\mu_1' \approx 3.49209$. We have

$$y_0 < y_0' < y_1' < y_1$$

and the regimes are:

- when $y < y_0$ we are in case (2.3);
- when $y = y_0$ we are in case (3.1);
- when $y_0 < y < y_0'$ we are in case (2.2);
- when $y = y_0'$ we are in case (3.2);
- when $y_0' < y < y_1'$ we are in case (2.1);
- when $y = y_1'$ we are in case (3.2);
- when $y_1' < y < y_1$ we are in case (2.2);
- when $y = y_1$ we are in case (3.1);
- when $y_1 < y$ we are in case (2.3).

To conclude, notice that the regimes are symmetric with respect to the magnitude of y.

References

[1] C. Banderier, P. Flajolet, G. Schaeffer, M. Soria, Random Maps, Coalescing Saddles, Singularity Analysis, and Airy Phenomena, *Random Strucures Algorithms* **19** (2001), 194–246.

[2] E. A. Bender, Z. Gao, N. C. Wormald, The number of 2-connected labelled planar graphs, *Electron. J. Combin.* **9** 2002, R43.

[3] O. Bernardi, M. Noy, D. Welsh, Growth constants of minor-closed families of graphs, *J. Combin. Theory Ser. B* (to appear).

[4] M. Bodirsky, O. Giménez, M. Kang, and M. Noy, Enumeration and limit laws for series-parallel graphs, *European J. Combin.* **28** (2007), 2091–2105.

[5] M. Bodirsky, M. Kang, M. Löffler, C. McDiarmid, Random cubic planar graphs, *Random Structures Algorithms* **30** (2007), 78–94.

[6] G. Chapuy, E. Fusy, O. Giménez, B. Mohar, M. Noy, Asymptotic enumeration of graphs of given genus (in preparation).

[7] G. Chapuy, E. Fusy, M. Kang, B. Shoilekova, A Complete Grammar for Decomposing a Family of Graphs into 3-connected Components, *Electron. J. Combin.* **15** (2008), R148.

[8] R. Diestel, *Graph decompositions. A study in infinite graph theory*, Oxford U. Press, New York (1990).

[9] M. Drmota, O. Giménez, M. Noy, Degree distribution in random planar graphs (submitted).

[10] M. Drmota, O. Giménez, M. Noy, Vertices of given degree in series-parallel graphs, *Random Structures Algorithms* (to appear).

[11] P. Flajolet, R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge (2009).

[12] A. Gagarin, G. Labelle, P. Leroux, T. Walsh, Structure and enumeration of two-connected graphs with prescribed three-connected components, *Adv. in Appl. Math.* 43 (2009), 46–74.

[13] Z. Gao, N. C. Wormald, The size of the largest components in random planar maps, *SIAM J. Discrete Math.* **12**, (1999) 217-228.

[14] S. Gerke, O. Giménez, M. Noy, A. Weissl, On the number of $K_{3,3}$-minor-free and maximal $K_{3,3}$-minor-free graphs, *Electron. J. Combin.* **15** (2008), R114.

[15] O. Giménez, M. Noy, The number of planar graphs and properties of random planar graphs, *J. Amer. Math. Soc.* 22 (2009), 309–329.

[16] O. Giménez, M. Noy, Counting planar graphs and related families of graphs, in *Surveys in Combinatorics 2009*, Cambridge University Press, Cambridge, pp. 169–210.

[17] C. McDiarmid, Random graphs on surfaces, *J. Combin. Theory Ser. B* **98** (2008), 778–797.

[18] C. McDiarmid, A. Steger, D. J. A. Welsh, Random planar graphs, *J. Combin. Theory Ser. B* **93** (2005), 187–205.

[19] R. C, Mullin, P. J. Schellenberg, The enumeration of c-nets via quadrangulations, *J. Combin. Theory* **4** (1968), 259–276.

[20] J. P. Nolan, *Stable distributions.* Book in progress, first chapter available at `http://academic2.american.edu/ jpnolan/`.

[21] S. Norine, P. Seymour, R. Thomas, P. Wollan, Proper minor-closed families are small, *J. Combin. Theory Ser. B* **96** (2006), 754–757.

[22] K. Panagiotou, A. Steger, Maximal biconnected subgraphs of random planar graphs, Proc. ACM-SIAM Symposium on Discrete Algotihms (2009), pp. 432–440.

[23] N. Robertson and P. Seymour, Graph minors **I-XX**, J. Combin. Theory Ser. B (1983–2004).

[24] B. A. Trakhtenbrot, Towards a theory of non-repeating contact schemes (in Russian), *Trudi Mat. Inst. Akad. Nauk SSSR* **51** (1958), 226–269.

[25] W. T. Tutte, A census of planar triangulations, *Canad. J. Math.* **14** (1962), 21–38.

[26] W. T. Tutte, A census of planar maps, *Canad. J. Math.* **15** (1963), 249–271.

[27] W. T. Tutte, *Connectivity in graphs*, University of Toronto Press, Toronto (1966).

[28] T. R. S. Walsh, Counting labelled three-connected and homeomorphically irreducible two-connected graphs, *J. Combin. Theory Ser. B* **32** (1982), 1–11.

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