Outerplanar Obstructions for the Feedback Vertex Set

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Abstract

For $k \geq 1$, let $\mathcal{F}_k$ be the class containing every graph that contains k vertices meeting all its cycles. The *minor-obstruction set* for $\mathcal{F}_k$ is the set $\mathbf{obs}(\mathcal{F}_k)$ containing all minor-minimal graph that does not belong to $\mathcal{F}_k$. We denote by $\mathcal{Y}_k$ the set of all outerplanar graphs in $\mathbf{obs}(\mathcal{F}_k)$. In this paper, we provide a precise characterization of the class $\mathcal{Y}_k$. Then, using the symbolic method, we prove that $|\mathcal{Y}_k| \sim \alpha \cdot k^{-5/2} \cdot \rho^{-k}$ where $\alpha \doteq 0.02602193$ and $\rho^{-1} \doteq 14.49381704$.

Keywords: Graph minors, outerplanar graphs, obstructions, feedback vertex set, graph enumeration, singularity analysis.

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1 Introduction

All graphs in this paper are simple. Given an edge $e = \{x, y\}$ of a graph G, the graph G/e is obtained from G by contracting the edge e; that is, to get G/e we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a subgraph of G after a sequence of edge-contractions is said to be a *minor* of G. Given a graph class $\mathcal{G}$, we define its minor-obstruction set as the set of all minor-minimal graphs that do not belong in $\mathcal{G}$ and we denote it as $\mathbf{obs}(\mathcal{G})$. By the Robertson and Seymour theorem [7], it follows that $\mathbf{obs}(\mathcal{G})$ is a finite set for every graph class $\mathcal{G}$. An active field of research in Graph Minors Theory is to characterize or (upper or lower) bound the size of the obstruction set of certain graphs classes (the first result of this kind was the Kuratowski-Wagner theorem concerning planar graphs).

Given a graph G, and a vertex set $S \subseteq V(G)$, we say that S is a *feedback vertex set* of G if $G \setminus S$ is a acyclic. We denote by $\mathbf{fvs}(G)$ the minimum k for which G contains a feedback vertex set of size k. For any non-negative integer k, we denote as $\mathcal{F}_k = \{G \mid \mathbf{fvs}(G) \leq k\}$ (i.e. the class of graphs that contain a feedback vertex set of size at most k). We define $\mathbf{obs}(\mathcal{F}_k)$ as the set of all minor-minimal graphs not contained in $\mathcal{F}_k$. Again by the Robertson and Seymour theorem, it is known that $\mathbf{obs}(\mathcal{F}_k)$ is finite for any k. Complete characterizations of $\mathbf{obs}(\mathcal{F}_k)$ have been provided for $k \leq 2$ in [3]. However, as remarked in [3], the number of obstructions for bigger values of k seems to grow quite rapidly. In this paper we provide a precise characterization of all outerplanar obstructions for every $k \geq 1$ and we use the symbolic method developed by Flajolet and Sedgewick [4] to asymptotically count them. We remark that, so far, such type of characterizations are known only for the acyclic obstructions of classes of bounded pathwidth [8] and its variations (search number [6], proper-pathwidth [8], linear-width [9]). Moreover, this is the first time where an asymptotic enumeration of such a class is derived.

2 Definition of the classes $\mathcal{C}_k$ and $\mathcal{Y}_k$

For any integer $r \geq 3$, we denote by C_r the cycle on r vertices. We also define A_r to be the graph obtained by C_r if we add r vertices in C_r and then connect each of them with a pair of adjacent vertices in C_r. Let $G_1, \ldots, G_q$ be a sequence of graphs. We define the class $\Delta(G_1, \ldots, G_q)$ as the set containing any graph G that can be constructed as follows: Take a cycle C_{q+1} of length $q + 1$ with vertex set $\{v_0, \ldots, v_q\}$ and for each $i \in \{1, \ldots, q\}$, remove from G_i a vertex u_i with exactly two adjacent neighbors, identify $N_{G_i}(u_i)$ with the vertices $\{v_{i-1}, v_i\}$ of C_{q+1} and remove multiple edges when appear. We define the graph classes $\mathcal{C}_k, \mathcal{Y}_k$, $k \geq 1$ as follows:

$$\begin{aligned}
\mathcal{C}_k = & \{A_{2k+1}\} \cup \{G \mid G \in \Delta(G_1, \ldots, G_q) \text{ for } G_i \in \mathcal{C}_{k_i}, i \in \{1, \ldots, q\} \\
& \quad \text{where} \sum_{i=1,\ldots,q} k_i = k \text{ and } \prod_{i=1,\ldots,q} k_i > 0\} \text{ and} \\
\mathcal{Y}_k = & \{G \mid G \text{ is the disjoint union of } G_1, \ldots, G_q \text{ for } G_i \in \mathcal{C}_{k_i}, i \in \{1, \ldots, q\} \\
& \quad \text{where} \sum_{i=1,\ldots,q} k_i = k \text{ and } \prod_{i=1,\ldots,q} k_i > 0\}.
\end{aligned}$$

Our first result is the following precise characterization of the (connected) outerplanar graphs in $\mathbf{obs}(\mathcal{F}_k)$, for every $k \geq 1$.

Theorem 2.1 *Let $\mathcal{B}$ (resp $\mathcal{B}'$) be the class of all outerplanar (resp. connected outerplanar) graphs. Then, for every positive integer k, $\mathbf{obs}(\mathcal{F}_k) \cap \mathcal{B}' = \mathcal{C}_k$ and $\mathbf{obs}(\mathcal{F}_k) \cap \mathcal{B} = \mathcal{Y}_k$.*

3 A precise characterization

From [5, Lemma 31] and [2, Theorem 2] it is enough to prove Theorem 2.1 for biconnected graphs. We call a pair of vertices x, y in a graph G *simplicial* if they are the neighbours of some vertex of degree 2 in G. We say that a graph is *nice* if any simplicial pair of it is contained in some feedback vertex set of G of size $\mathbf{fvs}(G)$. An edge of a biconnected outerplanar graph is a *side edge* if $G \setminus e$ is not biconnected and both its endpoints have degree bigger than 2.

Lemma 3.1 *Let $G_i, i = 1, \ldots, q$ be nice graphs where $G_i \in \mathbf{obs}(\mathcal{F}_{k_i}), i = 1, \ldots, q$. Let also $G \in \Delta(G_1, \ldots, G_q)$. Then G is nice and belongs in $\mathbf{obs}(\mathcal{F}_k)$ where $k = k_1 + \cdots + k_q$.*

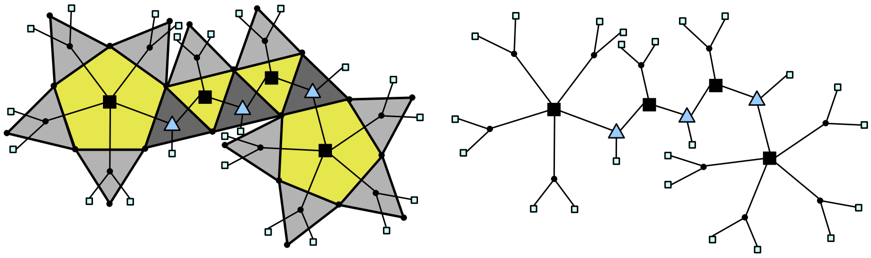
Lemma 3.2 *Let G be a graph in $\mathbf{obs}(\mathcal{F}_k) \cap \mathcal{B}'$. Then none of the faces of G is incident to more than one side-edge. Moreover, the only graph in $\mathbf{obs}(\mathcal{F}_k) \cap \mathcal{B}'$ without side-edges is A_{2k+1}.*

Lemma 3.3 *Let k be a positive integer, let $G \in \mathbf{obs}(\mathcal{F}_k) \cap \mathcal{B}'$ containing a side edge $e = \{v_0, v_q\}$ and let $G_i, i = 1, \ldots, q$ be graphs such that $G = \Delta(G_1, \ldots, G_q)$ in a way that e belongs in the central cycle C of G. Then $G_i \in \mathbf{obs}(\mathcal{F}_{k_i}), i = 1, \ldots, q$ where $\sum_{i=1,\ldots,q} k_i = k$.*

Proof of Theorem 2.1. Observe that A_{2k+1} is a nice graph and a member of $\mathbf{obs}(\mathcal{F}_k), k \geq 1$. The definition of $\mathcal{C}_k$, the fact that all graphs in $\mathcal{C}_k$ are outerplanar, and Lemma 3.1 implies that $\mathbf{obs}(\mathcal{F}_k) \cap \mathcal{B}' \supseteq \mathcal{C}_k$. From Lemma 3.2 and 3.3 we have that $\mathbf{obs}(\mathcal{F}_k) \cap \mathcal{B}' \subseteq \mathcal{C}_k$ and Theorem 2.1 follows.

4 Enumeration

The class of graphs that we have defined can be enumerated using the language of generating functions, and more concretely using symbolic methods introduced in [4]. The strategy we use is to consider embeddings of these graphs in the plane, taking care of possible symmetries of the graphs. Consequently, the problem of counting graphs has become a problem on counting a family of unrooted maps, up to rotations and reflections. To deal with this problem, we associate to each unrooted map a plane tree with 4 types of vertices ($\blacksquare$, $\triangle$, $\bullet$ and $\square$), which is obtained exploiting the vertex associated to the unbounded face of the initial map. These new vertices are represented using $\square$-vertices.



Additionally, we use the dissymmetry Theorem for trees [1] which provides a way to count unrooted trees in terms of rooted trees (up to symmetries).

The main result in this part is the following Theorem, which encapsulates all the enumeration of the family of graphs in the corresponding generating function:

Theorem 4.1 *Let $\boldsymbol{W}(z,u)$ an analytic function at the origin that safisfies the functional equation $\boldsymbol{W}(z,u) = \frac{\left(z^2-z+z/(1-\boldsymbol{W}(z,u))^2\right)^2 u}{1-(z^2-z+z/(1-\boldsymbol{W}(z,u))^2)^2 u}$. Define the functions $\boldsymbol{B}(z,u) = z^2 + z/(1-\boldsymbol{W}(z,u))^2 - z$ and $\boldsymbol{P}(z,u) = \sum_{k=0}^{\infty} \mu(2k+1)u^k \left(z^{4k+2} + \frac{z^{2k+1}\boldsymbol{W}(z^{4k+2},u^{4k+2})}{1-\boldsymbol{W}(z^{4k+2},u^{4k+2})}\right) \frac{\boldsymbol{B}(z^{4k+2},u^{4k+2})u^{2k+1}}{1-\boldsymbol{B}(z^{4k+2},u^{4k+2})u^{2k+1}}$ where μ is the Möbius function. Then, the number of graphs in $\mathcal{C}_k$ with n vertices is counted by the following generating function: $\boldsymbol{T}(z,u) = \frac{1}{2}z\frac{\boldsymbol{W}(z,u)^2}{1-\boldsymbol{W}(z,u)} + \frac{1}{2}z\left(1+\frac{\boldsymbol{B}(z^2,u^2)u}{1-\boldsymbol{B}(z^2,u^2)u}\right)\frac{\boldsymbol{W}(z^2,u^2)}{1-\boldsymbol{W}(z^2,u^2)} + \frac{1}{4\sqrt{u}}\sum_{d=0}^{\infty}\frac{\varphi(2d+1)}{2d+1}\log\left(\frac{1+u\sqrt{u}\boldsymbol{B}(z^{2d+1},u^{2d+1})}{1-u\sqrt{u}\boldsymbol{B}(z^{2d+1},u^{2d+1})}\right) - \frac{1}{2}\boldsymbol{B}(z,u) + \frac{1}{2}\sum_{d=0}^{\infty}u^d\boldsymbol{P}(z^{2d+1},u^{2d+1}) - \frac{1}{2}\boldsymbol{W}(z,u)(\boldsymbol{B}(z,u)-z^2) - \frac{1}{2}\frac{\boldsymbol{B}(z^2,u^2)u}{1-\boldsymbol{B}(z^2,u^2)u}\frac{z\boldsymbol{W}(z^2,u^2)}{1-\boldsymbol{W}(z^2,u^2)}$ where φ is the Euler function. Additionally, the number of graphs in $\mathcal{Y}_k$ with n vertices is counted by the following generating function $\boldsymbol{g}(z,u) = \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\boldsymbol{T}(z^m,u^m)\right)$.*

The first terms in the series expansion of $\mathbf{T}(z,u)$ are $z^6u + (z^9+z^{10})u^2 + (3z^{12}+2z^{13}+z^{14})u^3 + (12z^{15}+16z^{16}+5z^{17}+z^{18})u^4 + (52z^{18}+117z^{19}+68z^{20}+9z^{21}+z^{22})u^5$, which can be obtained from the expression of Theorem 4.1

truncating the infinite series (for instance, up to $d = 50$). Similarly, the first terms in the series expansion of $\mathbf{g}(z,u)$ are $z^6u + (z^9 + z^{10} + z^{12})\,u^2 + (3z^{12} + 2z^{13} + z^{14} + z^{15} + z^{16} + z^{18})\,u^3 + (12z^{15} + 16z^{16} + 5z^{17} + 5z^{18} + 3z^{19} + 2z^{20} + z^{21} + z^{22} + z^{24})u^4 + (52z^{18} + 117z^{19} + 68z^{20} + 24z^{21} + 22z^{22} + 8z^{23} + 6z^{24} + 3z^{25} + 2z^{26} + z^{27} + z^{28} + z^{30})u^5$.

If we write in the previous Theorem $z = 1$, we obtain the generating functions for $|\mathcal{C}_k|$ and $|\mathcal{Y}_k|$. Using analytic tools and the Transfer Theorems introduced in [4], we obtain precise assympotics estimations for coefficients of these generating functions. In our work, we prove the following Theorem:

Theorem 4.2 *Asymptotically,* $[u^k]\,\boldsymbol{T}(1,u) = t_k$ *and* $[u^k]\boldsymbol{g}(1,u) = g_k$ *are equal to* $t_k \sim Ck^{-5/2}\rho^{-k}\left(1 + O\left(k^{-1}\right)\right)$ *and* $g_k \sim C'k^{-5/2}\rho^{-k}\left(1 + O\left(k^{-1}\right)\right)$ *where* $\rho = 1/512(29701 - 4633\sqrt{41}) \doteq 0.06899494$ *(and* $\rho^{-1} \doteq 14.49381704$*),* $C \doteq 0.02389878$ *and* $C' \doteq 0.02602193$.

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