Enumeration and Limit Laws of Topological Graphs

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Introduction

This work is devoted to the study of enumerative combinatorics, which means, informally speaking, that we are interested in counting objects. In this thesis we study enumerative properties of *graphs defined by minor conditions* and *graphs which are embedded in surfaces* (usually called *maps*). Both topics are active areas in discrete mathematics, and they have also many important applications in physics, algorithmics, probability and algebraic geometry, among other disciplines. By counting we mean that we are interested in obtaining either exact formulas, asymptotic estimates or probability limit distributions for different families of graphs and maps. The language used in all this thesis is the one introduced by P. Flajolet and R. Sedgewick in their reference book *Analytic Combinatorics* [19]. The philosophy of this framework consists of translating combinatorial decompositions into equations satisfied by the corresponding generating functions. Then we apply analytic methods in order to obtain asymptotic estimates and probability limit laws.

In Chapter 1 we introduce the objects we want to study, the main notions about generating functions and the language of the so called *Symbolic Method*. We introduce also the main analytic results (*Singularity Analysis*), in order to obtain asymptotic and probabilistic results. We discuss the *Method of Moments* and the decomposition of 2-connected graphs into 3-connected components.

In Chapter 2 we present a framework which explains the asymptotic enumeration and limit laws for families of graphs defined in terms of 3-connected components. This theory covers the asymptotic enumeration of planar graphs [26], series-parallel graphs [9], $K_{3,3}$-free graphs [24] and many others. This work complements, to a certain extent, the results in [11] and [20], where a general combinatorial framework for this problem is presented, but without entering into asymptotic analysis. The key point of this chapter consists of exploiting the seminal ideas of Tutte [49] about the decomposition of maps into 3-connected maps, as well as the analytic techniques developed in [26] in order to enumerate planar graphs. Within this background, we study the behaviour of the singularities of the generating functions derived from combinatorial decompositions. Our first result shows that if $T(x, z)$ is the generating function of the 3-connected components (x marks vertices and z marks edges), then the asymptotic enumeration depends crucially on the singular behaviour of $T(x, z)$. The proofs in this part are based on a careful analysis of singularities.

Applying singularity analysis we show that several basic parameters converge in distribution either to a normal law or to a Poisson law. In particular, the number of edges, number of blocks and number of cut vertices are asymptotically normal with linear mean and variance. This is also the case for the number of special copies of a fixed graph or a fixed block in the class. On the other hand, the number of connected components converges to a discrete Poisson law.

We also study extremal parameters. We start with the size of the largest block, or the largest 2-connected component. In this case we find a striking difference depending on the class of graphs. For planar graphs there is asymptotically almost surely a block of linear size, and the remaining blocks are of order $O(n^{2/3})$. For series-parallel graphs there is no block of linear size. A similar dichotomy occurs when considering the size of the largest 3-connected component. These results are proved adapting the main results of [3] about the size of the *core* of families of maps defined by a composition scheme. For planar graphs we prove the following precise result: if $\mathbf{X}_n$ is the number of vertices of the largest block in random planar graphs with n vertices, then

$$\mathbf{p}\left(\{\mathbf{X}_n = \alpha n + xn^{2/3}\}\right) \sim n^{-2/3} c g(cx),$$

where $\alpha \approx 0.95982$ and $c \approx 128.35169$ are well-defined analytic constants, and $g(x)$ is the so called *Airy distribution of the map type*, which is closely related to a stable law of index $3/2$. Moreover, the size of the second largest block is $O(n^{2/3})$. The giant block is uniformly distributed among the planar 2-connected graphs with the same number of vertices, hence according to the results in [5] it has about $2.2629 \cdot 0.95982\, n = 2.172\, n$ edges, with deviations of order $O(n^{1/2})$. With respect to the largest 3-connected component in a random planar graph, we show that it has ηn vertices and ζn edges, where $\eta \approx 0.7346$ and $\zeta \approx 1.7921$ are again well-defined.

Our techniques allow us to study graphs with a given density, or average degree. Parameters like the number of components or the number of blocks can be analyzed too when the edge density varies. It turns out that the family of planar graphs with density $\mu \in (1,3)$ shares the main characteristics of planar graphs. This is also the case for series-parallel graphs, where $\mu \in (1,2)$ since maximal graphs in this class have only $2n-3$ edges. We present examples of critical phenomena by a suitable choice of the family $\mathcal{T}$ of 3-connected components. In the associated closed class $\mathcal{G}$, graphs below a critical density μ_0 behave like series-parallel graphs, and above μ_0 they behave like planar graphs, or conversely. We even have examples with more than one critical value.

In Chapter 3 we study the enumeration of a certain family of maps over the projective plane. In particular, let $\mathbb{P}_1$ be the real projective plane, obtained by adding a cross-cap to the sphere. We fix a polygon Q in $\mathbb{P}_1$, that is, a simple contractible closed curve, in which n points are labelled $1, 2, \ldots, n$ circularly. By a *triangulation of a polygon in the projective plane* we mean a 2-cell decomposition of the outside of Q into triangles using as vertices only the n labelled points, such that two intersecting triangles meet only in a common vertex or in a common edge (i.e., a simplicial decomposition). The number of triangulations of the Möbius band was first determined by Edelman and Reiner in [16]. We reprove the same result with a different approach, using the Symbolic Method for handling generating functions [19]. More generally, we deal with decompositions of a polygon into quadrangles, pentagons, and so on, and also in unrestricted dissections. In each case we require that two cells of a decomposition intersect either at a vertex or at an edge. We believe our proof is more transparent and moreover this approach allows us to solve other related problems which appear difficult to obtain using recurrence equations as in [16].

In this chapter we obtain the generating functions and precise asymptotic estimates for the numbers of polygon dissections of various kinds. Finally, we derive limit laws for two parameters of interest: the number of cyclic triangles in triangulations, and the number of cells in arbitrary polygon dissections. In the second case we obtain a classical normal law, whereas in the first case the limit law is the *absolute value* of a normal law.

A parallel study is made in Chapter 4, but now studying maps on the cylinder. We study simplicial decompositions of the cylinder, with the restriction that all vertices lie on the boundary, and vertices on each polygon are labelled circularly. One may try to use the same strategy used to enumerate simplicial decompositions over the projective plane (combinatorial surgery and inclusion-exclusion arguments over generating functions). Unfortunately, the cases to be studied growth notably, and computations for general families becomes extremely involved. The idea used in order to study triangulations on the cylinder is that the dual of triangulations of the projective plane are extremely simple, and this can be adapted to the cylinder. The aim of this chapter consists of exploiting this second point of view in order to obtain the enumeration for simplicial decompositions, dissections into polygons with a fixed degree, and unrestricted dissections.

We are also able to obtain limit distributions for several parameters on the cylinder, giving rise in all cases to non-Gaussian limit laws. In all cases, we obtain closed formulas for the density probability functions. These are related to classical functions, such as the *complementary error function* and *Bessel functions of the first kind*. In this study we need to use the classical *Laplace transform*, which is, in many cases, the right tool to deal with analytic equations.

We conclude this thesis in Chapter 5. We consider compact and connected surfaces with boundary in full generality. Observe that families studied in the previous chapters are particular cases of the problem treated here. In this case, we are not able to obtain explicit expressions for the generating functions. However, we can obtain precise asymptotic estimates for these families.

A map is *triangular* if every face has degree 3. Given a set $\Delta \subseteq \{1, 2, 3, \ldots\}$, a map is Δ-angular (or a Δ-map) if the degree of any face belongs to Δ. A map is a *dissection* if a face of degree k is incident with k distinct vertices, and the intersection of each pair of faces is either empty, a vertex or an edge. It is easy to see that triangular maps are dissections if and only if they have neither loops nor multiple edges. These maps are called also *simplicial decompositions* of $\mathbb{S}$. In this chapter we enumerate asymptotically the simplicial decompositions of an arbitrary surface $\mathbb{S}$ with boundaries. We shall consider the set $\mathcal{D}_{\mathbb{S}}(n)$ of rooted simplicial decomposition of $\mathbb{S}$ having n vertices, all of them lying on the boundary and prove the asymptotic estimate

$$|\mathcal{D}_{\mathbb{S}}(n)| \ \sim \ c(\mathbb{S}) \ n^{-3\chi(\mathbb{S})/2} \ 4^n,$$

where $c(\mathbb{S})$ is a constant which depends only on $\mathbb{S}$, and $\chi(\mathbb{S})$ is the Euler characteristic of $\mathbb{S}$.

We also study limit laws. We say that an edge is *non-structuring* if it belongs to the boundary of $\mathbb{S}$ or separates the surface into two parts, one of which is isomorphic to a disk (the other being isomorphic to $\mathbb{S}$); the other edges are called *structuring*. We determine the limit law for the number of structuring edges in simplicial decompositions. In particular, we show that the (random) number $\mathbf{U}_n(\mathcal{D}_{\mathbb{S}})$ of structuring edges in a uniformly random simplicial decomposition of a surface $\mathbb{S}$ with n vertices, rescaled by a factor $n^{-1/2}$, converges in distribution toward a continuous random variable (related to the Gamma distributions) which depends only on the Euler characteristic of $\mathbb{S}$.

We generalize the enumeration and limit law results to Δ- angular dissections for any set of degrees $\Delta \subseteq \{3, 4, 5, \ldots\}$, where all vertices lie on the boundary of the surface. Our results are obtained by exploiting a decomposition of the Δ-angular maps, which is reminiscent of Wright's work on graphs with fixed excess [52, 53], or of work by Chapuy, Marcus and Schaeffer on the enumeration of unicellular maps [12]. This decomposition easily translates into an equation satisfied by the corresponding generating function. We then apply classical enumeration techniques based on singularity analysis [19]. This generalisation also applies to the number of structuring edges in uniformly random Δ-angular dissections of a surface $\mathbb{S}$ with n vertices on its boundary.

As in Chapters 3 and 4, we deal with maps having all their vertices on the boundary of the surface. This is a sharp restriction which contrasts with most papers in map enumeration. In contrast, most of the literature on maps enumeration deals with maps having vertices outside the boundary of the underlying surface. We do not deal with this more general problem here. However, a remarkable feature of the asymptotic result obtained (and the generalisation we obtain for arbitrary set of degrees $\Delta \subseteq \{3, 4, 5, \ldots\}$) is the linear dependency of the polynomial growth exponent in the Euler characteristic of the underlying surface. Similar results were obtained by a recursive method for general maps by Bender and Canfield in [4] and for maps with certain degree constraints by Gao in [21]. This feature as also been re-derived for general maps using a bijective approach in [12].

Scientific publications This thesis is based on papers written by the author and, in some cases, co-authored with Olivier Bernardi, Omer Giménez or Marc Noy. All these papers have been published, submitted or in the way to be submitted:

[28] *Graph classes with given 3-connected components: asymptotic counting and critical phenomena.* **Electronical Notes in Discrete Mathematics 29 (2007) 521-529**. With Omer Giménez and Marc Noy.

[27] *Graph classes with given 3-connected components: asymptotic enumeration and random graphs.* In preparation. With Omer Giménez and Marc Noy.

[41] *Counting polygon dissections in the projective plane.* **Advances in Applied Mathematics 41 (2008) 599-619**. With Marc Noy.

[45] *Enumeration and limit laws of dissections in a cylinder.* In preparation.

[7] *Counting simplicial decompositions of surfaces with boundaries.* Submitted. With Olivier Bernardi.

Contents

1

Background and definitions

In this chapter we introduce the main definitions about the objects we want to study. We introduce also the language of the generating functions, and the analytic tools to deal with them. We make a brief review of probability theory, and how to apply generating function techniques in this context. At the end, we introduce the basic notions for decomposing graphs into 3-connected components, which is a key point in the development of Chapter 2.

1.1 Mathematical structures

In this section we recall definitions in the context of graph theory, and we set our notation about this subject. We introduce also the basic concepts needed about surfaces and embedded graphs (i.e., maps).

1.1.1 Graphs

Notation in Graph Theory is not uniform in the literature. In this part we recall and fix the basic definitions we use in the rest of this thesis. Our main reference for this part is Chapter 1 of Diestel's book [14].

A *labelled graph* G is defined by a pair $(V(G), E(G))$, where $V = V(G)$ is the *vertex set* of G and $E = E(G)$ is the *edge set* of G. We always consider finite graphs, and we use the set $[n] = \{1, 2, \ldots, n\}$ to denote the set of vertices of G. If v and w are the vertices of G which define the edge e of E, then we write $e = \overline{vw}$. the degree of a vertex v is denoted by $\deg(v)$. An edge with the same endpoints is called a *loop*. A *simple* graph is a graph without loops. An *unlabelled graph* is a class of labelled graphs up to permutations of the labels of the vertex set. A graph G is *connected* if there is a path between each pair of vertices. Every connected maximal subgraph of G is a *connected component* of G. A graph G is *k-connected* if $|V| > k$ and if for every set $X \subset V$ with $|X| < k$, $G - X$ is connected. For a k-connected graph G which is not $(k+1)$-connected, a set of vertices $v_1, v_2, \ldots, v_k$ which disconnects G is called a k-cut of G. If $k = 1$, this set is also called a *cut vertex* of G. A connected graph without cycles is called a *tree*. Vertices in a tree with degree 1 are called *leaves*. In particular, if v is a vertex of a tree which is not a leave, then $G - \{v\}$ is disconnected. A tree with a distinguished vertex is called a *rooted* tree, and the distinguished vertex is called the *root* of the tree.

Let us introduce some notation for specific families of graphs. The cycle with n vertices is denoted by C_n. The *wheel graph* on n vertices W_n is obtained from the cycle graph on $n-1$ vertices joining the remaining vertex with all the points that belong to the original cycle. The *complete* graph on n vertices is denoted by K_n. Finally, *r-partite* graphs are denoted by $K_{s_1, s_2, \ldots, s_r}$. In the particular case of $r = 2$, graphs are called *bipartite*. Some examples are shown in Figure 1.1.

To finish, let us recall the concept of *graph minor*. Let e be an edge of G. The *contraction* of G by e is the graph obtained from G by identifying the ends of e, and removing possible multiple edges.

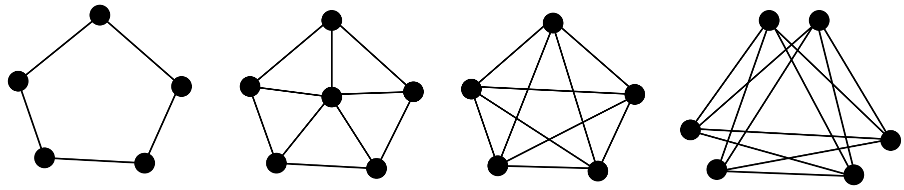


Figure 1.1 Examples, from left to right, of C_5, W_6, K_5 and $K_{2,2,2}$.

We say that G' is a graph minor of G if G' is obtained by a sequence of edge contractions from a subgraph of G. In Figure 1.2, a graph with K_4 as a minor is shown.

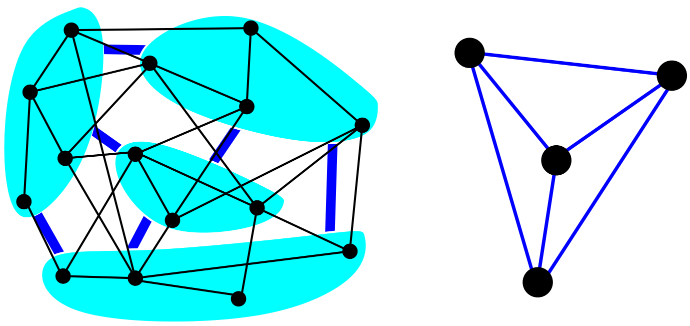


Figure 1.2 A graph which has K_4 as a minor.

We say that a family of graphs is *G-minor free* (or excludes G as a minor) if no graph of the family has G as a minor. The family of *planar* graphs consists in graphs which can be drawn on the sphere without crossings. An equivalent characterization for this family is given by Kuratowski's Theorem [14], which asserts that the family of planar graphs is the family of $\{K_{3,3}, K_5\}$-minor free graphs. A graph is *series-parallel* if it is K_4-minor free. Equivalently, a connected series-parallel graph can be obtained from a tree by series operations and parallel operations.

1.1.2 Surfaces

Our main reference in this part is the monograph of Mohar and Thomassen [38]. In all cases, *surfaces* are compact (bounded and closed) connected 2-manifolds (locally homeomorphic to disks) without boundary. We also consider surfaces *with boundaries*, which are obtained from surfaces without boundary by removing the interior of a finite number of disjoint disks. We denote the boundary of $\mathbb{S}$ by $\partial\mathbb{S}$, and the number of connected components of $\partial\mathbb{S}$ by $\beta(\mathbb{S})$. By the *Classification Theorem for Surfaces* (see [34] for a general topological treatment) a surface $\mathbb{S}$ without boundary is determined, up to homeomorphism, by its *Euler characteristic* $\chi(\mathbb{S})$ and by whether it is oriented or not. More precisely, oriented surfaces are obtained by adding $g \geq 0$ *handles* to the sphere (which is denoted by $\mathbb{S}^2$), obtaining the *torus of genus g* (or shortly *g-torus*) denoted by $\mathbb{T}_g$, with Euler genus equal to $\chi(\mathbb{T}_g) = 2 - 2g$. On the other hand, non-oriented surfaces are obtained by adding $h > 0$ *cross-caps* to the sphere, hence obtaining a non-oriented surface $\mathbb{P}_h$. In this second case, the Euler characteristic satisfies $\chi(\mathbb{P}_h) = 2 - h$. For a surface $\mathbb{S}$ with boundaries, we denote by $\overline{\mathbb{S}}$ the surface (without boundary) obtained from $\mathbb{S}$ by gluing a disk on each of the $\beta(\mathbb{S})$ boundaries. An easy calculus shows that $\chi(\overline{\mathbb{S}})$ is equal to $\chi(\mathbb{S}) + \beta(\mathbb{S})$.

1.1.3 Maps

The main definitions in this part appear in the book of Lando and Zvonkin [32]. Let $\mathbb{S}$ be a surface without boundary. A *map* on $\mathbb{S}$ is a subdivision of $\mathbb{S}$ into 0-dimensional sets (*vertices* of the map), 1-dimensional contractible sets (*edges* of the map) and 2-dimensional contractible open sets (*faces* of the map). For a map M on $\mathbb{S}$, We denote by $v(M)$, $e(M)$ and $f(M)$ the set of vertices, edges and faces of a map M. The value $|f(M)|+|v(M)|-|e(M)|$ coincides with the Euler characteristic of $\mathbb{S}$. Maps are considered up to orientation preserving homeomorphisms of the underlying surface, preserving the combinatorial structure of the map (incidences between vertices, edges and faces). In Figure 1.4 a subdivision of the torus which is not a map, and a map, are shown.

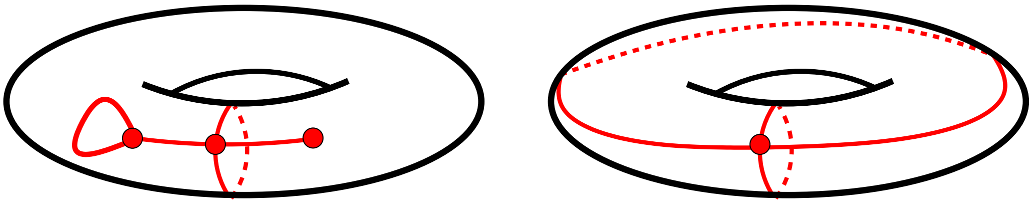


Figure 1.3 A subdivision of the torus which is not a map, and a map.

Let us introduce some particular terminology for maps. An edge of a map has two *ends* (incidence with a vertex) and either one or two *sides* (incidence with a face). A map is *rooted* if an end and a side of an edge are distinguished as the *root-end* and *root-side* respectively. Rooting of maps on oriented surfaces usually omits the choice of a root-side because the underlying surface is oriented and maps are considered up to orientation preserving homeomorphism. Our choice of a root-side is equivalent in the oriented case to the choice of an orientation of the surface. The vertex, edge and face defining these incidences are the *root-vertex*, *root-edge* and *root-face*, respectively. Rooted maps are considered up to homeomorphism preserving the root-end and root-side. In figures (see, for instance, Figure 1.4), the root-edge is indicated as an oriented edge pointing away from the root-end and crossed by an arrow pointing toward the root-side.

The *dual map* M^* of a map M on a surface without boundary is a map obtained by drawing the vertices of M^* in each face of M and edges of M^* across each edge of M. If the map M is rooted, the root-edge of M^* corresponds to the root-edge e of M; the root-end and root-side of M^* correspond respectively to the side and end of e which are not the root-side and root-end of M. In the second picture of Figure 1.4 this construction is shown for a concrete map on the plane (i.e., over the sphere). It is immediate to show that $M^{**}=M$.

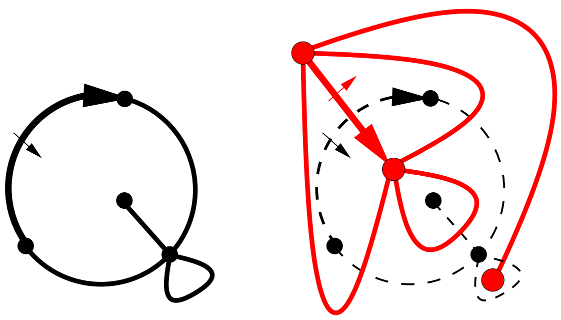


Figure 1.4 A rooted map on the plane (in black), and its associated dual map (in red).

1.2 Enumeration and Generating Functions

A technique to deal with enumerative problems is the use of generating functions. We use the language and the methodology introduced by Flajolet and Sedgewick in the context of *analytic combinatorics*. The main reference is the book [19].

Let $\mathcal{A}$ be a set of objects, and let $|\cdot|$ be an application from $\mathcal{A}$ to $\mathbb{N}$. A pair $(\mathcal{A}, |\cdot|)$ is called a *combinatorial class*, and if $a \in \mathcal{A}$, $|a|$ is the *size* of a. We restrict ourselves to the study of *admissible* combinatorial classes, i.e. for every n the number of elements in $\mathcal{A}$ with size n is finite. Let $\mathcal{A}(n)$ be the set of elements of size n. We define the formal power series $A(z) = \sum_{n=0}^{\infty} |\mathcal{A}(n)| z^n = \sum_{a\in\mathcal{A}} z^{|a|} = \sum_{n=0}^{\infty} a_n z^n$. Conversely we write $[z^n]A(z) = |\mathcal{A}(n)| = a_n$. We say that $A(z)$ is the *ordinary generating function* (OGF) associated to the combinatorial class $(\mathcal{A}, |\cdot|)$. If $(\mathcal{B}, \|\cdot\|)$ is another combinatorial class with OGF $B(z) = \sum_{n\geq 0} b_n z^n$, we write $B(z) \leq A(z)$ if and only if the inequality $b_n \leq a_n$ is true for every value of n. The consideration of additional parameters over the combinatorial classes gives rise to *multivariate* GFs.

The *Symbolic Method* is a tool that provides systematic rules to translate set conditions between combinatorial classes into algebraic conditions between generating functions. We introduce the basic classes and combinatorial constructions, as well as their translation into the GF language. The *neutral* class $\mathcal{E}$ is made of a single object of size 0, and its generating function is $e(z) = 1$. The *atomic* class $\mathcal{Z}$ is made of a single object of size 1, and its associated GF is $Z(z) = z$. The *union* $\mathcal{A} \cup \mathcal{B}$ of two classes $\mathcal{A}$ and $\mathcal{B}$ refers to the disjoint union of classes (and the corresponding induced size). The *cartesian product* $\mathcal{A} \times \mathcal{B}$ of two classes $\mathcal{A}$ and $\mathcal{B}$ is the set of pairs (a, b) where $a \in \mathcal{A}$, and $b \in \mathcal{B}$. The size of (a, b) is the sum of size of a and b. The *sequence* $\mathrm{Seq}(\mathcal{A})$ of a set $\mathcal{A}$ corresponds with the set $\mathcal{E} \cup \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup (\mathcal{A} \times \mathcal{A} \times \mathcal{A}) \cup \dots$. The *multiset* construction $\mathrm{Mul}(\mathcal{A})$ corresponds to $\mathrm{Seq}(\mathcal{A}) / \smile$, where $(a_1, a_2, \dots, a_r) \smile (\widehat{a}_1, \widehat{a}_2, \dots, \widehat{a}_r)$ if and only if there exists a permutation τ of $\{1, \dots, r\}$ such that, for all i, $a_i = \widehat{a}_{\tau(i)}$. The size of an element $(a_1, \dots, a_s)$ in either $\mathrm{Seq}(\mathcal{A})$ or $\mathrm{Mul}(\mathcal{A})$ is the sum of sizes of the elements a_i. The *pointing* operator over a class $\mathcal{A}$ works in the following way: for each element $a \in \mathcal{A}$, such that $|a| = n$, the pointing operator consists in distinguishing one of the n atoms that compounds a. Finally, the *substitution* of $\mathcal{B}$ in the class $\mathcal{A}$ consists in substituting each atom of every element of $\mathcal{A}$ by an element of $\mathcal{B}$.

A technical refinement is needed when we deal with labelled structures (i.e. combinatorial classes where each element a in the class has attached $|a|$ different labels in the set $\{1, \dots, |a|\}$). Many of the previous combinatorial operations must be redefined to deal with labels. For a labelled combinatorial class $(\mathcal{A}, |\cdot|)$ (which is assumed to be admissible) we define the *exponential generating function* associated to $\mathcal{A}$ (EGF) as the formal power series $A(z) = \sum_{a\in\mathcal{A}} z^{|a|}/|a|! = \sum_{n=0}^{\infty} a_n z^n/n!$. The introduction of the term $n!$ provides a way to deal with labels. All the previous operations can be translated easily to labelled structures. Definition of classes are quite different compared with ordinary classes. For instance, instead of considering the product of classes we consider the *labelled product*: let $(\mathcal{A}, |\cdot|)$ and $(\mathcal{B}, \|\cdot\|)$ be labelled combinatorial classes, and we define $\mathcal{A} * \mathcal{B}$ as the set of all possible labellings of the pairs of the form (a, b), $a \in \mathcal{A}$ and $b \in \mathcal{B}$. This specification is translated in the language of EGF in the following way:

$$\sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} \binom{|a| + \|b\|}{|a|} \frac{z^{|a|+\|b\|}}{(|a| + \|b\|)!} = \sum_{a\in\mathcal{A}} \frac{z^{|a|}}{|a|!} \sum_{b\in\mathcal{B}} \frac{z^{\|b\|}}{\|b\|!} = A(z)B(z).$$

Notice that the multiset operator do not have sense in this context, because there do not appear repeated elements (due to labels). All the other operations specified for OGFs can be rephrased in the context of EGF using the labelled product. In Table 1.1 all the constructions for both type of generating functions are shown. The particular construction $\mathrm{Set}(\mathcal{A})$ refers to the class made of sets of elements of $\mathcal{A}$.

1.2.1 An example: dissections of the disk

As an example, we apply this machinery in the enumeration of the number of triangulations of a polygon. This is a well studied problem and the solution is known since Euler's time. The reader

Construction		OGF	EGF
Union	$\mathcal{A} \cup \mathcal{B}$	$A(z) + B(z)$	$A(z) + B(z)$
Product	$\mathcal{A} \times \mathcal{B}$	$A(z) \cdot B(z)$	$-$
Labelled Product	$\mathcal{A} * \mathcal{B}$	$-$	$A(z) \cdot B(z)$
Sequence	$\mathrm{Seq}\,(\mathcal{A})$	$\frac{1}{1-A(z)}$	$\frac{1}{1-A(z)}$
Multiset	$\mathrm{Mul}\,(\mathcal{A})$	$\exp\left(\sum_{r=1}^{\infty} \frac{1}{r} A(z^r)\right)$	$-$
Set	$\mathrm{Set}\,(\mathcal{A})$	$-$	$\exp\left(A(z)\right)$
Pointing	$\mathcal{A}^{\bullet}$	$z\frac{\partial}{\partial z}A(z)$	$z\frac{\partial}{\partial z}A(z)$
Substitution	$\mathcal{A} \circ \mathcal{B}$	$A\left(B(z)\right)$	$A\left(B(z)\right)$

Table 1.1 Translation of combinatorial specifications into algebraic conditions using the Symbolic Method.

can consult [17] for more constructions over a disk. Consider a disk with n vertices on its boundary, which are labelled in counter clockwise order (we can also consider an oriented edge of the polygon, which corresponds with edge $\overline{12}$). We consider edges that join vertices on the boundary of a disk , with the restriction that each pair of them does not cross. This second point of view is more natural in the context of map enumeration (we can consider the oriented arrow as the root of the resulting map). We say that the polygon is rooted or labelled.

We say that a decomposition of a labelled polygon is a *triangulation* if and only if each face is a triangle. More generally, given a set $\Delta \subseteq \mathbb{N} - \{1,2\}$, a decomposition of a labelled polygon is a Δ-*dissection* if the degree of each face belongs to Δ. In particular, a triangulation is a $\{3\}-$dissection of a disk . For the special case when $\Delta = \mathbb{N} - \{1,2\}$, Δ-dissections are called *unrestricted dissections*.

The number of triangulations of a rooted polygon can be obtained using the Symbolic Method. Let $(\mathcal{C}, |\cdot|)$ be the class of triangulations of a rooted polygon, where the size of a triangulation is the number of triangles in which it decomposes. Observe that a triangulation is either an edge or a proper triangulation. In the second case, a proper triangulation can be written as a pair of triangulations and the root triangle. See Figure 1.5 for an example of this combinatorial decomposition.

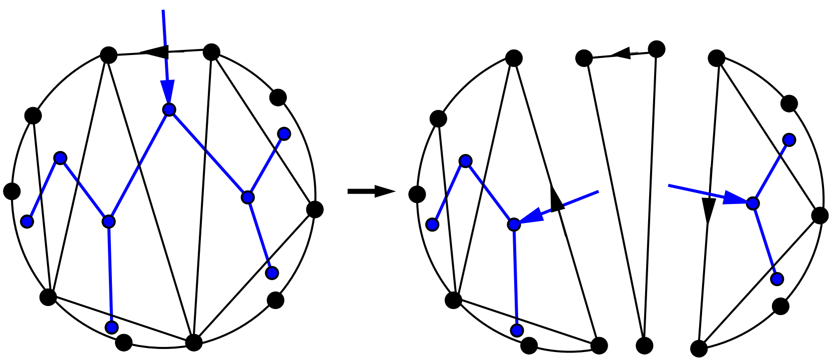


Figure 1.5 A decomposition for a triangulation, and the associated dual tree.

These conditions are translated into the formal equation $\mathcal{C} = (\bullet - \bullet) \cup (\mathcal{C} \times \triangle \times \mathcal{C})$. The Symbolic Method gives $C(z) = 1 + zC(z)^2$, whose solution is the *Catalan function* $C(z) = (1 - \sqrt{1-4z})/(2z)$. Developing the term $\sqrt{1-4z}$ as a series around $z = 0$ we obtain the explicit expression for the *Catalan numbers* $[z^n]C(z) = C(n) = \frac{1}{n+1}\binom{2n}{n}$.

The Symbolic Method provides also a tool to deal with the case of Δ-dissections in full generality. Let $(\mathcal{C}_\Delta, |\cdot|)$ be the combinatorial class of Δ-dissections of a polygon, where the size is the number of regions in which the polygon decomposes. The combinatorial decomposition of the class works similarly: suppose that the root polygon has $\delta \in \Delta$ sides. The root polygon separates the disk into $\delta - 1$ disjoint Δ-dissections. Summing over all the possibilities for δ, we obtain the following equation:

$$C_\Delta(z) = 1 + z \sum_{\delta \in \Delta} C_\Delta(z)^{\delta - 1}. \tag{1.1}$$

Notice that the case $\Delta = \{k+1\}$ corresponds with decompositions of a disk in $k+1$-agons. We denote by $C_{k+1}(z)$ the GF for this family. We obtain in this particular situation that $C_{k+1}(z)$ satisfy the equation $C_{k+1}(z) = 1 + zC_{k+1}^k(z)$. This equation can we written also in terms of vertices. By Euler relation, a dissection into n $(k+1)$-agons has $(k-1)n + 2$ vertices. Consequently, the equation $C_{k+1}(z) = 1 + zC_{k+1}^k(z)$ is translated into $C_{k+1}(x) = x^2 + x^{1-k}C_{k+1}^k(x)$ (x is used to mark vertices).

To conclude, if $\Delta = \mathbb{N} - \{1, 2\}$, the degree of each face is unrestricted. We denote by $D(x)$ its generating function, where x counts the number of vertices. Define also $D(u, x)$ as the bivariate generating function where the parameter u is used to mark the number of faces of each unrestricted dissection. An implicit equation for $D(u, x)$ is deduced in [17], giving rise to the equation

$$(1 + u)D(u, x)^2 - x(1 + x)D(u, x) + x^3 = 0. \tag{1.2}$$

1.3 Analytic combinatorics

Often we are not able to obtain an exact enumeration of combinatorial classes, and we need to introduce additional structures and techniques to obtain asymptotic estimates for the sequences of numbers a_n. The techniques come from analysis, and are briefly introduced here.

We say that two sequences of numbers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are of the same *exponential order* if $\limsup |a_n|^{1/n}$ and $\limsup |b_n|^{1/n}$ coincide. Under these assumptions, we write $a_n \bowtie b_n$. The limit $\limsup |a_n|^{1/n}$ (which we assume finite) is the *exponential growth* or the *exponential order* of the sequence $(a_n)_{n\geq 0}$. If R is the exponential growth of $(a_n)_{n\geq 0}$, then $a_n = \theta(n) \cdot R^n$, where $\limsup |\theta(n)|^{1/n} = 1$. The term $\theta(n)$ is called the *subexponential term* of $(a_n)_{n\geq 0}$. Sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are called *asymptotically equivalent* if $\lim a_n/b_n$ exists and is equal to 1. We denote this fact writing $a_n \sim b_n$. We also write $A(z) \sim B(z)$ if $[z^n]A(z) \sim [z^n]B(z)$.

Estimates of the exponential order and the subexponential growth of a sequence $(a_n)_{n\geq 0}$ can be obtained often from the associated generating function. The key point consists in considering this formal object as an analytic function on a neighborhood of the origin. Using this approach, we can use powerful analytic tools to study the coefficients of the GF. The location of the singularity with smallest modulus gives the exponential growth of the sequence, and the nature of this singularity gives the subexponential term. More concretely, by Pringsheim's Theorem [19], the smallest singularity (if it exists) of a generating function $A(z)$ with positive coefficients is a positive real number. Let us assume that it exist and call it ρ. Then, the following theorem from complex analysis gives the desired growth order (see Theorem $IV.7$ in [19]):

Theorem 1.1 (Location of Singularities) *If $A(z)$ is analytic at 0, and its smallest singularity ρ is a positive real number, then*

$$[z^n]A(z) \bowtie \rho^{-n}.$$

The next step consists in refining this theorem in order to obtain the subexponential term. For $R > \rho > 0$ and $0 < \phi < \pi/2$, let $\Delta_\rho(\phi, R)$ be the set $\{z \in \mathbb{C} : |z| < R,\, z \neq \rho,\, |\mathrm{Arg}(z - \rho)| > \phi\}$. We call a set of this type a *dented domain* or a *domain dented* at ρ. The typical shape of a dented domain is shown in Figure 1.6.

Let $A(z)$ and $B(z)$ be GFs whose smallest singularity is the real number ρ. We write

$$A(z) \sim_{z\to\rho} B(z)$$

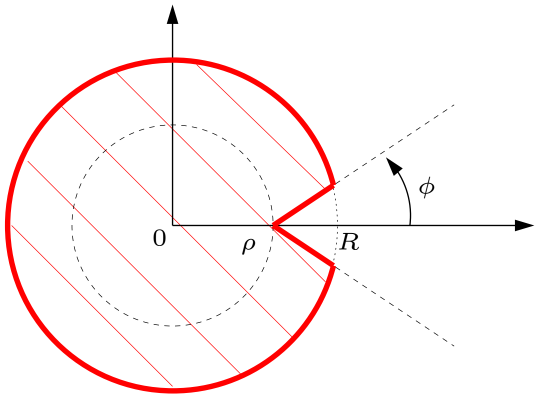


Figure 1.6 A typical dented domain.

if $\lim_{z\to\rho} A(z)/B(z) = 1$. We obtain the asymptotic expansion of $[z^n]A(z)$ by *transfering* the behaviour of $A(z)$ around its singularity from a simpler function $B(z)$, from which we know the asymptotic behaviour of their coefficients. This is the philosophy of the so called *Transfer Theorems* developed by Flajolet and Odlyzko [18].

In our work we use a mixture of Theorems $VI.1$ and $VI.3$ from [19]:

Theorem 1.2 (Transfer Theorem) *If $A(z)$ is analytic in a dented domain $\Delta = \Delta_\rho(\phi, R)$, where ρ is the smallest singularity of $A(z)$, and*

$$A(z) \underset{z\in\Delta, z\to\rho}{\sim} c\cdot\left(1-\frac{z}{\rho}\right)^{-\alpha} + o\left(\left(1-\frac{z}{\rho}\right)^{-\alpha}\right),$$

for $\alpha \notin \{0, -1, -2, \ldots\}$, then

$$a_n \; = c\cdot\frac{n^{\alpha-1}}{\Gamma(\alpha)}\cdot\rho^{-n}\left(1+o(n^{-1})\right),$$

where Γ is the Gamma function: $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$.

The previous theorem it is also true changing little-oh by big-Oh.

1.4 Limit laws

In this part we introduce the basic definitions from probability theory used in this thesis. A reference in this area is [29].

Let $(\Omega, \mathcal{P}(\Omega), \mathbf{p})$ be a probability space. Let $\mathbf{X}$ be a real *random variable* $\mathbf{X}$ defined over this probability space. The expression $\mathbf{p}\left(\{\mathbf{X}\le x\}\right)$ defines the *probability distribution function* of $\mathbf{X}$, which is denoted by $F_{\mathbf{X}}(x)$. If the derivative of this function exists, then the following equality holds,

$$\mathbf{p}\left(\{\mathbf{X}\le x\}\right) = F_{\mathbf{X}}(x) = \int_{-\infty}^{x} f_{\mathbf{X}}(s)ds\,,$$

where $f_{\mathbf{X}}(s)$ is the *density probability function* of $\mathbf{X}$. We denote by $\mathbb{E}\left[g(\mathbf{X})\right]$ the value

$$\mathbb{E}\left[g(\mathbf{X})\right] = \int_{-\infty}^{\infty} g(s)f_{\mathbf{X}}(s)ds.$$

In particular, $\mathbb{E}[\mathbf{X}]$ is the *expectation* of $\mathbf{X}$. The *variance* of $\mathbf{X}$ is $\sigma^2(\mathbf{X}) = \mathbb{E}[\mathbf{X}^2] - (\mathbb{E}[\mathbf{X}])^2$, and the *standard deviation* is $\sigma(\mathbf{X})$. The r-th *ordinary moment* (or shortly, the r-th moment) of $\mathbf{X}$ is $\mathbb{E}[\mathbf{X}^r]$, and the *factorial moment* is $\mathbb{E}[(\mathbf{X})_r] = \mathbb{E}[\mathbf{X}(\mathbf{X}-1)\dots(\mathbf{X}-r+1)]$.

There are many criteria for convergence of a sequence of random variables. Here we are concerned only with *convergence in distribution* (or *convergence in law*): for a sequence of random variables $(\mathbf{X}_n)_{n>0}$, such that each of them has a probability density function $f_{\mathbf{X}_n}(x)$, we say that the sequence tends in distribution (or in law) to a random variable $\mathbf{X}$, if the sequence of distribution probability functions $(F_{\mathbf{X}_n}(x))_{n>0}$ converges pointwise to the distribution function $F_{\mathbf{X}}(x)$ of $\mathbf{X}$. We denote this fact writing $\mathbf{X}_n \xrightarrow{d} \mathbf{X}$.

Probability is introduced in the framework of analytic combinatorics in the following way. Let $(\mathcal{A}, |\cdot|)$ be an admissible combinatorial class, and let $\chi : \mathcal{A} \to \mathbb{N}$ be a parameter. We define the bivariate generating function $A(u,z) = \sum_{a\in\mathcal{A}} u^{\chi(a)} z^{|a|} = \sum_{n,m=0}^{\infty} a_{m,n} u^m z^n$. In particular $A(1,z) = A(z)$, and $\sum_{m=0}^{\infty} a_{m,n} = a_n$. For each value of n, the parameter χ defines a random variable $\mathbf{X}_n$ over $\mathcal{A}(n)$ with discrete probability density function $\mathbf{p}(\{\mathbf{X}_n = m\}) = a_{m,n}/a_n$. This probabilities can be encapsulated into the *probability GF*:

$$p_n(u) = \frac{[z^n]A(u,z)}{[z^n]A(1,z)}.$$

The following result, which was obtained by Hwang [30] (the so called *Quasi-powers Theorem*), provides a direct way to deduce normal limit laws from singular expansions of generating functions. In other words, the Quasi-Powers Theorem gives sufficient conditions to assure normal limit laws in the context of analytic combinatorics. We rephrase here this result in a convenient way using the language of GFs :

Theorem 1.3 (Quasi-Powers Theorem) *Let $F(u,z)$ be a bivariate function that is analytic in both variables on a neighborhood of the point $(0,0)$, with nonnegative coefficients. Suppose that in the region $\mathcal{R} = \{|u-1| < \epsilon\} \times \{|z| \le r\}$ (for some $r, \epsilon > 0$) $F(u,z)$ admits a representation of the form*

$$F(u,z) = A(u,z) + B(u,z)C(u,z)^{-\alpha},$$

where A, B and C are analytic in $\mathcal{R}$, such that $C(1,z) = 0$ has a unique simple root $\rho < r$ in $|z| \le r$, and $B(1,\rho) \neq 0$. Additionally, neither $\partial_z C(1,\rho)$ nor $\partial_u C(1,\rho)$ are 0, so there exists a nonconstant function $\rho(u)$ analytic at $u = 1$ such that $C(u,\rho(u)) = 0$, $\rho = \rho(1)$. Finally, $\rho(u)$ is such that

$$\sigma^2 = -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2$$

is different from 0. Then, the random variable with probability GF

$$p_n(u) = \frac{[z^n]F(u,z)}{[z^n]F(1,z)}$$

converge in distribution to a normal random variable. The corresponding expectation μ_n and standard deviation σ_n converge asympotically to $-\rho'(1)/\rho(1)n$ and $\sigma\sqrt{n}$, respectively.

The Quasi-Powers Theorem does not apply if the limit law we are looking for is not a normal distribution. Fortunately, we have additional methods to obtain limit distributions. Using the previous notation, observe that

$$\frac{[z^n]\frac{\partial^r}{\partial u^r}A(1,z)}{[z^n]A(1,z)} = \frac{1}{a_n}\sum_{m=0}^{\infty} a_{m,n}(m)_r = \mathbb{E}[(\mathbf{X}_n)_r]. \tag{1.3}$$

In other words, factorial moments can be computed from the generating function. Additionally, using the identity $x^r = \sum_{j=0}^{k} S(j,r)(x)_j$, where the values $S(j,r)$ are Stirling numbers of the second kind, it follows the equality $\mathbb{E}[(\mathbf{X}_n)^r] = \sum_{j=0}^{r} S(j,r)\mathbb{E}[(\mathbf{X}_n)_j]$, for each value of n. Consequently, ordinary moments can be expressed in terms of the GF we are dealing with.

In this framework, the *Method of Moments* [8] provides a way to assure convergence in law using only the ordinary moments of the sequence of random variables. Even more, this method provides a direct way to calculate the limit law using its moments.

More concretely, the version we use in this thesis is the following one:

Lemma 1.4 (Method of Moments) *Let $(\boldsymbol{X}_n)_{n>0}$ and $\boldsymbol{X}$ be real random variables satisfying:*

(A) there exists $R > 0$ such that $\frac{R^r}{r!}\mathbb{E}\left[\boldsymbol{X}^r\right] \to 0$, as $r \to \infty$,

(B) for all $r \in \mathbb{N}$, $\mathbb{E}\left[\boldsymbol{X}_n^r\right] \to \mathbb{E}\left[\boldsymbol{X}^r\right]$, as $n \to \infty$.

Then $\boldsymbol{X}_n \xrightarrow{d} \boldsymbol{X}$.

Point (A) in Lemma 1.4 implies that the distribution $\mathbf{X}$ is determined by its moments. We need also the following modification of the Method of Moments:

Lemma 1.5 *Let $\mathcal{A}$ be a combinatorial class and let $\boldsymbol{U}_n$ be the random variable associated to a parameter $U_n : \mathcal{A}(n) \to \mathbb{N}$. Let $A(u, z)$ be the corresponding GF. Denote by $\theta : \mathbb{N} \to \mathbb{N}$ a function such that $\theta(n) \to +\infty$ when n tends to ∞. If a random variable $\boldsymbol{X}$ satisfies Condition (A) in Lemma 1.4 and*

(B') for all $r \in \mathbb{N}$, $\dfrac{[z^n]\frac{\partial^r}{\partial u^r}A(u,z)}{\theta(n)^r[z^n]A(1,z)} \to \mathbb{E}\left[\boldsymbol{X}^r\right]$, as $n \to \infty$,

then the rescaled random variables $\boldsymbol{X}_n = \dfrac{\boldsymbol{U}_n}{\theta(n)} \xrightarrow{d} \boldsymbol{X}$.

Proof. We only need to prove that (B') implies (B). Using Equation (1.3) and the relation between factorial moments and ordinary moments (i.e., Stirling numbers as mentioned before), and the fact that θ tends to infinity gives that, for all $r \geq 0$,

$$\frac{[z^n]\frac{\partial^r}{\partial u^r}A(u,z)}{\theta(n)^r[z^n]A(1,z)} = \mathbb{E}\left[\frac{(\mathbf{U}_n)_r}{\theta(n)^r}\right] = \mathbb{E}\left[\frac{\mathbf{U}_n^r}{\theta(n)^r}\right] + o\left(\sum_{k<r}\mathbb{E}\left[\frac{\mathbf{U}_n^k}{\theta(n)^k}\right]\right).$$

Given Condition (B'), a simple induction on r shows that $\dfrac{\mathbf{U}_n^r}{\theta(n)^r}$ has a finite limit for all $r \geq 0$. Thus,

$$\frac{[z^n]\frac{\partial^r}{\partial u^r}C(u,z)}{\theta(n)^r[z^n]C(1,z)} \to_{n\to\infty} \mathbb{E}\left[\mathbf{X}_n^r\right],$$

as claimed. □

1.5 Graph decomposition and connectivity

In this part we introduce the combinatorial decompositions for general families of graphs. We start introducing these decompositions for general graphs in terms of connected graphs, of connected graphs in terms of 2-connected graphs and, at the end, 2-connected graphs in terms of 3-connected graphs.

It is obvious that a graph is a disjoint union of connected graphs. Connected graphs can be decomposed into 2-connected graphs (or *blocks*) via its cut vertices. In Figure 1.7 a decomposition of a connected graph into 2-connected components is shown.

A more complicated decomposition is that of a 2-connected graph in terms of 3-connected graphs. This decomposition was introduced by Tutte [50]. We only introduce the main ideas and the main concepts. Additional details can be found in [11, 25].

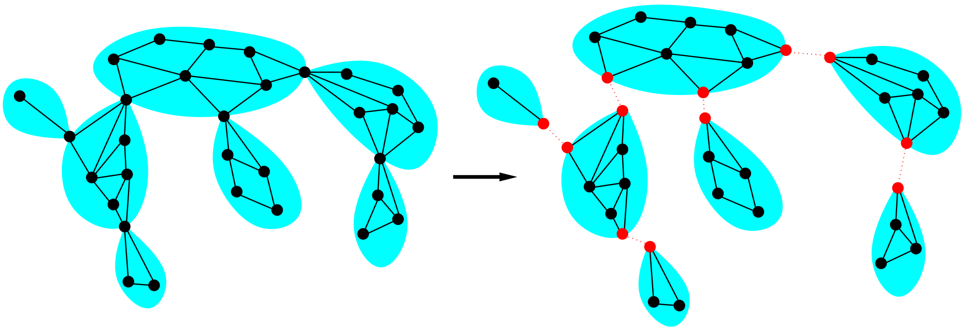


Figure 1.7 A decomposition of a connected graph into 2-connected blocks

Consider a 2-connected graph G, and let x, y be a 2-cut of G. Let $V_1, V_2, \ldots, V_r$ be the vertex sets of the 2-connected components of $G - \{x, y\}$, and denote by G_i the subgraph of G induced by $V_i \cup \{x, y\}$. We say that a 2-cut is *good* if any other 2-cut u, v of G is contained in one of the graphs G_i. These 2-cuts are the ones that help decomposing a 2-connected graph into 3-connected graphs, because they break the graph into graphs with a smaller number of vertices. In other words, they play the role of cut vertices in the decomposition of connected graphs into 2-connected components. The key point in this decomposition is that 2-connected graphs without good 2-cuts are either 3-connected or cycles. In Figure 1.8 a decomposition of a 2-connected graph is shown.

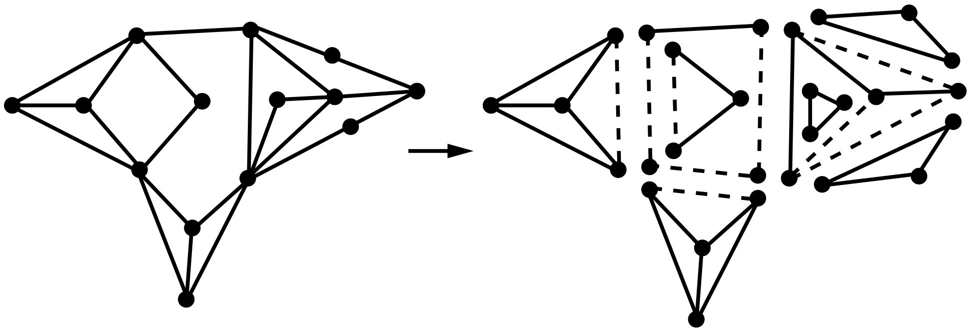


Figure 1.8 A decomposition of a 2-connected graph into 3-connected graphs and cycles

This decomposition can be rephrased in terms of three types of compositions. If we paste different pieces along a 3-connected graph H we have an *h-composition*. If we join subgraphs along a cycle we obtain a *series* composition, and finally if we have a set of subgraphs that share a common 2-cut which is good, we get a *parallel* composition. Observe that when the components are attached, edges that are glued together can be erased. This makes this decomposition more complex than the one for decomposing a connected graph into its blocks, since the blocks share vertices but not edges.

2

Graph classes with given 3-connected components

The first results of this chapter appeared in [28], and a final version in [27]. Consider a family $\mathcal{T}$ of 3-connected graphs of moderate growth, and let $\mathcal{G}$ be the class of graphs whose 3-connected components are graphs in $\mathcal{T}$. We present a general framework for analyzing such graphs classes based on singularity analysis of generating functions, which generalizes previously studied cases such as planar graphs and series-parallel graphs. We provide a general result for the asymptotic number of graphs in $\mathcal{G}$, based on the singularities of the exponential generating function associated to $\mathcal{T}$. We derive limit laws, which are either normal or Poisson, for several basic parameters, including the number of edges, number of blocks and number of components. For the size of the largest block we find a fundamental dichotomy: classes similar to planar graphs have almost surely a unique block of linear size, while classes similar to series-parallel graphs have only sublinear blocks. This dichotomy also applies to the size of the largest 3-connected component. For some classes under study both regimes occur, because of a critical phenomenon as the edge density in the class varies.

2.1 Introduction

Several enumeration problems on planar graphs have been solved recently. It has been shown [26] that the number of labelled planar graphs with n vertices is asymptotically equal to

$$c \cdot n^{-7/2} \cdot \gamma^n n!, \tag{2.1}$$

for suitable constants c and γ. For series-parallel graphs [9], the asymptotic estimate is of the form, again for suitable constants d and δ,

$$d \cdot n^{-5/2} \cdot \delta^n n!. \tag{2.2}$$

As can be seen from the proofs in [9, 26], the difference in the subexponential term comes from a different behaviour of the counting generating functions near their dominant singularities. Related families of labelled graphs have been studied, like outerplanar graphs [9], graphs not containing $K_{3,3}$ as a minor [24], and, more generally, classes of graphs closed under minors [6]. In all cases where asymptotic estimates have been obtained, the subexponential term is systematically either $n^{-7/2}$ or $n^{-5/2}$. The present chapter grows up as an attempt to understand this dichotomy.

A *class* of graphs is a family of labelled graphs which is closed under isomorphism. A class $\mathcal{G}$ is *closed* if the following condition holds: a graph is in $\mathcal{G}$ if and only if its connected, 2-connected and 3-connected components are in $\mathcal{G}$. A closed class is completely determined by its 3-connected members. The basic example is the class of planar graphs, but there are others, specially minor-closed classes whose excluded minors are 3-connected.

In this chapter we present a general framework for enumerating closed classes of graphs. Let $T(x, z)$ be the generating function associated to the family of 3-connected graphs in a closed class $\mathcal{G}$, where x marks vertices and z marks edges, and let g_n be the number of graphs in $\mathcal{G}$ with n vertices. Our first result shows that the asymptotic of g_n depends crucially on the singular behaviour of $T(x, z)$.

For a fixed value of x, let $r(x)$ be the dominant singularity of $T(x, z)$. If $T(x, z)$ has an expansion at $r(x)$ in powers of $Z = \sqrt{1 - z/r(x)}$ with dominant term Z^5, then the estimate for g_n is as in Equation (2.1); if $T(x, z)$ is either analytic everywhere or the dominant term is Z^3, then the pattern is that of Equation (2.2). There are also mixed cases, where 2-connected and connected graphs in $\mathcal{G}$ get different exponents. And there are *critical* cases too, due to the confluence of two sources for the dominant singularity, where a subexponential term $n^{-8/3}$ appears. This is the content of Theorem 2.1, whose proof is based on a careful analysis of singularities.

In Section 2.4, extending the analytic techniques developed for asymptotic enumeration, we analyze random graphs from closed classes of graphs. We show that several basic parameters converge in distribution either to a normal law or to a Poisson law. In particular, the number of edges, number of blocks and number of cut vertices are asymptotically normal with linear mean and variance. This is also the case for the number of special copies of a fixed graph or a fixed block in the class. On the other hand, the number of connected components converges to a discrete Poisson law.

In Section 2.5 we study a key extremal parameter: the size of the largest block, or the largest 2-connected component. And in this case we find a striking difference depending on the class of graphs. For planar graphs there is asymptotically almost surely a block of linear size, and the remaining blocks are of order $O(n^{2/3})$. For series-parallel graphs there is no block of linear size. This also applies more generally to the classes considered in Theorem 2.1. A similar dichotomy occurs when considering the size of the largest 3-connected component. This is proved using the techniques developed by Banderier et al. [3] for analyzing largest components in random maps. For planar graphs we prove the following precise result in Theorem 2.21. If $\mathbf{X}_n$ is the size of the largest block in random planar graphs with n vertices, then

$$\mathbf{p}\left(\{\mathbf{X}_n = \alpha n + x n^{2/3}\}\right) \sim n^{-2/3} c g(cx),$$

where $\alpha \approx 0.95982$ and $c \approx 128.35169$ are well-defined analytic constants, and $g(x)$ is the so called Airy distribution of the map type, which is closely related to a stable law of index 3/2. Moreover, the size of the second largest block is $O(n^{2/3})$. The giant block is uniformly distributed among the planar 2-connected graphs with the same number of vertices, hence according to the results in [5] it has about $2.2629 \cdot 0.95982\, n = 2.172\, n$ edges, with deviations of order $O(n^{1/2})$ (the deviations for the normal law are of order n1/2, but the n2/3 term coming from the Airy distribution dominates). We remark that the size of the largest block has been analyzed too in [42] using different techniques. The main improvement with respect to [42] is that we are able to obtain a precise limit distribution. With respect to the largest 3-connected component in a random planar graph, we show that it has ηn vertices and ζn edges, where $\eta \approx 0.7346$ and $\zeta \approx 1.7921$ are again well-defined.

The picture that emerges for large random planar graphs is the following. Start with a large 3-connected planar graph M (or the skeleton of a polytope in the space if one prefers a more geometric view), and perform the following operations. First edges of M are substituted by small blocks (more precisely, *networks*, to be introduced later in the chapter), giving rise to the giant block L; then small connected graphs are attached to some of the vertices of L, which become cut vertices, giving rise to the largest connected component C. As shown in [35], C contains everything except a few vertices. This model can be made more precise and will be the subject of future research.

An interesting open question is whether there are other parameters besides the size of the largest block (or largest 3-connected component) for which planar graphs and series-parallel graphs differ in a qualitative way. We remark that with respect to the largest component there is no qualitative difference: it contains always everything except a few vertices. This is also true for the degree distribution [15]. If d_k is the probability that a given vertex has degree $k > 0$, then in both cases it can be shown that the d_k decay as $c \cdot n^{\alpha} q^k$, where c, α and q depend on the class under consideration [15].

In Section 2.7 we apply the previous machinery to the analysis of several classes of graphs, including planar graphs and series-parallel graphs. Whenever the generating function $T(x, z)$ can be computed explicitly, we obtain precise asymptotic estimates for the number of graphs g_n, and limit laws for the main parameters. In particular we determine the asymptotic probability of a random graph being connected, the constant κ such that the expected number of edges is asymptotically κn, and several other fundamental constants.

Our techniques allow also to study graphs with a given density, or average degree. To fix ideas, let $g_{n,\lfloor\mu n\rfloor}$ be the number of planar graphs with n vertices and $\lfloor\mu n\rfloor$ edges: μ is the edge density and 2μ is the average degree. For $\mu \in (1,3)$, a precise estimate for $g_{n,\lfloor\mu n\rfloor}$ can be obtained using a local limit theorem [26]. And parameters like the number of components or the number of blocks can be analyzed too when the edge density varies. It turns out that the family of planar graphs with density $\mu \in (1,3)$ shares the main characteristics of planar graphs. This is also the case for series-parallel graphs, where $\mu \in (1,2)$ since maximal graphs in this class have only $2n-3$ edges. In Section 2.8 we show examples of critical phenomena by a suitable choice of the family $\mathcal{T}$ of 3-connected graphs. In the associated closed class $\mathcal{G}$, graphs below a critical density μ_0 behave like series-parallel graphs, and above μ_0 they behave like planar graphs, or conversely. We even have examples with more than one critical value.

2.2 Preliminaries

Generating functions are of the exponential type, unless we say explicitly the contrary. The partial derivatives of $A(x,y)$ are written $A_x(x,y)$ and $A_y(x,y)$. In some cases the derivative with respect to x is written $A'(x,y)$. The second derivatives are written $A_{xx}(x,y)$, and so on.

The decomposition of a graph into connected components, and of a connected graph into blocks (2-connected components) are well known, and the main ideas are shown in Section 1.5 in Chapter 1. We also need the decomposition of a 2-connected graph decomposes into 3-connected components [50]. A 2-connected graph is built by series and parallel compositions and 3-connected graphs in which each edge has been substituted by a block.

A class of labelled graphs $\mathcal{G}$ is *closed* if a graph G is in $\mathcal{G}$ if and only if the connected, 2-connected and 3-connected components of G are in $\mathcal{G}$. A closed class is completely determined by the family $\mathcal{T}$ of its 3-connected members. Let g_n be the number of graphs in $\mathcal{G}$ with n vertices, and let $g_{n,k}$ be the number of graphs with n vertices and k edges. We define similarly c_n, b_n, t_n for the number of connected, 2-connected and 3-connected graphs, respectively, as well as the corresponding $c_{n,k}, b_{n,k}, t_{n,k}$. We introduce the EGFs

$$G(x,y) = \sum_{n,k} g_{n,k}\, y^k \frac{x^n}{n!},$$

and similarly for $C(x,y)$ and $B(x,y)$. When $y=1$ we recover the univariate EGFs

$$B(x) = \sum b_n \frac{x^n}{n!}, \qquad C(x) = \sum c_n \frac{x^n}{n!}, \qquad G(x) = \sum g_n \frac{x^n}{n!}.$$

Recall that in Section 1.5 we show how to decompose a general graph into connected components, and a connected graph into 2-connected graphs. The following equations reflect the decomposition into connected components and 2-connected components:

$$G(x,y) = \exp(C(x,y)), \qquad xC'(x,y) = x\exp\left(B'(xC'(x,y),y)\right), \tag{2.3}$$

In the first decomposition, one must notice that a general graph is simply a *set* of labelled connected graphs, hence the equation $G(x,y) = \exp(C(x,y))$. The second decomposition is more involved. The EGF $xC'(x,y)$ is related to the family of connected graphs with a vertex pointed. Then, the second equation in (2.3) says that a connected graph with a vertex pointed is obtained from a set of pointed 2-connected graphs (where we erase the root), in which we substitute each vertex by a connected graph with a vertex pointed (pointed vertices let us paste graphs recursively). We also define

$$T(x,z) = \sum_{n,k} t_{n,k}\, z^n \frac{x^n}{n!},$$

where the only difference is that the variable which marks edges is now z. This convention is useful and will be maintained throughout the chapter.

A *network* is a graph with two distinguished vertices, called *poles*, such that the graph obtained by adding an edge between the two poles is 2-connected. Moreover, the two poles are not labelled. Networks are the key technical device for encoding the decomposition of 2-connected graphs into 3-connected components. Let $D(x, y)$ be the GF associated to networks, where again x and y mark vertices and edges, respectively. Then $D = D(x, y)$ satisfies (see [5], who draws on [48, 51])

$$\frac{2}{x^2}T_z(x, D) - \log\left(\frac{1+D}{1+y}\right) + \frac{xD^2}{1+xD} = 0, \tag{2.4}$$

and $B(x, y)$ is related to $D(x, y)$ through

$$B_y(x, y) = \frac{x^2}{2}\left(\frac{1+D(x, y)}{1+y}\right), \tag{2.5}$$

For future reference, we set

$$\Phi(x, z) = \frac{2}{x^2}\,T_z(x, z) - \log\left(\frac{1+z}{1+y}\right) + \frac{xz^2}{1+xz}, \tag{2.6}$$

so that Equation (2.4) is written in the form $\Phi(x, D) = 0$, for a given value of y. By integrating (2.5) using the techniques developed in [26], we obtain an explicit expression for $B(x, y)$ in terms of $D(x, y)$ and $T(x, z)$ (see the first part of the proof of Lemma 5 in [26]).

$$\begin{aligned} B(x, y) \quad = \quad & T(x, D(x, y)) - \frac{1}{2}xD(x, y) + \frac{1}{2}\log(1 + xD(x, y)) + \\ & \frac{x^2}{2}\left(D(x, y) + \frac{1}{2}D(x, y)^2 + (1 + D(x, y))\log\left(\frac{1+y}{1+D(x, y)}\right)\right). \end{aligned} \tag{2.7}$$

This relation is valid for every closed defined in terms of 3-connected graphs, and can be proved in a more combinatorial way [11].

We assume that for a fixed value of x, $T(x, z)$ has a unique dominant singularity $r(x)$, and that there is a singular expansion near $r(x)$ of the form

$$T(x, z) = \sum_{n \geq n_0} T_n(x)\left(1 - \frac{z}{r(x)}\right)^{n/\kappa}, \tag{2.8}$$

where n_0 is an integer, possibly negative, and the functions $t_n(x)$ and $r(x)$ are analytic. The *singular exponent* of T is $\alpha = n/\kappa$, where n is the smallest integer such that $\alpha \notin \{0, 1, 2, \dots\}$. This is a rather general assumption, as it includes singularities coming from algebraic and meromorphic functions.

The case when $\mathcal{T}$ is empty (there are no 3-connected graphs) gives rise to the class of series-parallel graphs. It is shown in [9] that, for a fixed value $y = y_0$, $D(x, y_0)$ has a unique dominant singularity $R(y_0)$. This is also true for arbitrary $\mathcal{T}$, since adding 3-connected graphs can only increase the number of networks.

2.3 Asymptotic enumeration

Throughout the rest of the chapter we assume that $\mathcal{T}$ is a family of 3-connected graphs whose GF $T(x, z)$ satisfies the requirements described in Section 2.2. We assume that a singular expansion like (2.8) holds, and we let $r(x)$ be the dominant singularity of $T(x, z)$, and α the singular exponent.

Our main result gives precise asymptotic estimates for g_n, c_n, b_n depending on the singularities of $T(x, z)$. Cases (1) and (2) in the next statement can be considered as generic, although (1) and (2.1) are those encountered in 'natural' classes of graphs. The two situations in case (3) come from critical conditions, when two possible sources of singularities coincide. This is the reason for the unusual exponent $-8/3$, which comes from a singularity of cubic-root type instead of the familiar square-root type.

Theorem 2.1 *Let $\mathcal{G}$ be a closed family of graphs, and let $T(x,z)$ be the GF of the family of 3-connected graphs in $\mathcal{G}$. In all cases b, c, g, R, ρ are explicit positive constants and $\rho < R$.*

(1) If $T_z(x,z)$ is either analytic or has singular exponent $\alpha < 1$, then

$$b_n \sim b\, n^{-5/2} R^{-n} n!, \qquad c_n \sim c\, n^{-5/2} \rho^{-n} n!, \qquad g_n \sim g\, n^{-5/2} \rho^{-n} n!$$

(2) If $T_z(x,z)$ has singular exponent $\alpha = 3/2$, then one of the following holds:

(2.1) $b_n \sim b\, n^{-7/2} R^{-n} n!, \qquad c_n \sim c\, n^{-7/2} \rho^{-n} n!, \qquad g_n \sim g\, n^{-7/2} \rho^{-n} n!$

(2.2) $b_n \sim b\, n^{-7/2} R^{-n} n!, \qquad c_n \sim c\, n^{-5/2} \rho^{-n} n!, \qquad g_n \sim g\, n^{-5/2} \rho^{-n} n!$

(2.3) $b_n \sim b\, n^{-5/2} R^{-n} n!, \qquad c_n \sim c\, n^{-5/2} \rho^{-n} n!, \qquad g_n \sim g\, n^{-5/2} \rho^{-n} n!$

(3) If $T_z(x,z)$ has singular exponent $\alpha = 3/2$, and in addition a critical condition is satisfied, one of the following holds:

(3.1) $b_n \sim b\, n^{-8/3} R^{-n} n!, \qquad c_n \sim c\, n^{-5/2} \rho^{-n} n!, \qquad g_n \sim g\, n^{-5/2} \rho^{-n} n!$

(3.2) $b_n \sim b\, n^{-7/2} R^{-n} n!, \qquad c_n \sim c\, n^{-8/3} \rho^{-n} n!, \qquad g_n \sim g\, n^{-8/3} \rho^{-n} n!$

Application of the Transfer Theorem for singularity analysis (recall Theorem 1.2 in Chapter 1), the previous theorem is a direct application of the following analytic result for $y = 1$. We prove it for arbitrary values of $y = y_0$, since this has important consequences later on.

Theorem 2.2 *Let $\mathcal{G}$ be a closed family of graphs, and let $T(x,z)$ be the GF of the family of 3-connected graphs in $\mathcal{G}$.*

For a fixed $y = y_0$, let $R = R(y_0)$ be the dominant singularity of $D(x, y_0)$, and $D_0 = D(x_0)$.

(1) If $T_z(x,z)$ is either analytic or has singular exponent $\alpha < 1$ at (R, D_0), then $B(x,y_0)$, $C(x,y_0)$ and $G(x,y_0)$ have singular exponent $3/2$.

(2) If $T_z(x,z)$ has singular exponent $\alpha = 3/2$ at (R, D_0), then one of the following holds:

(2.1) $B(x,y_0), C(x,y_0)$ and $G(x,y_0)$ have singular exponent $5/2$.

(2.2) $B(x,y_0)$ has singular exponent $5/2$, and $C(x,y_0), G(x,y_0)$ have singular exponent $3/2$.

(2.3) $B(x,y_0), C(x,y_0)$ and $G(x,y_0)$ have singular exponent $3/2$.

(3) If $T_z(x,z)$ has singular exponent $\alpha = 3/2$ at (R, D_0), and in addition a critical condition is satisfied for the singularities of either $B(x,y)$ or $C(x,y)$, then one of the following holds:

(3.1) $B(x,y_0)$ has singular exponent $5/3$, and $C(x,y_0), G(x,y_0)$ have singular exponent $3/2$.

(3.2) $B(x,y_0)$ has singular exponent $5/2$, and $C(x,y_0), G(x,y_0)$ have singular exponent $5/3$.

The rest of the section is devoted to the proof of Theorem 2.2, which implies Theorem 2.1. First we study the singularities of $B(x,y)$, which is the most technical part. Then we study the singularities of $C(x,y)$ and $G(x,y)$, which are always of the same type since $G(x,y) = \exp(C(x,y))$.

2.3.1 Singularity analysis of $B(x, y)$

From now on, we assume that $y = y_0$ is a fixed value, and let $D(x) = D(x, y_0)$. Recall from Equation (2.6) that $D(x)$ satisfies $\Phi(x, D(x)) = 0$, where

$$\Phi(x,z) = \frac{2}{x^2}T_z(x,z) - \log\left(\frac{1+z}{1+y_0}\right) + \frac{xz^2}{1+xz}.$$

Since a 3-connected graph has at least four vertices, $T(x,z)$ is $O(x^4)$. If follows that $D(0) = y_0$ and $\Phi_z(0, D(0)) = -1/(1+y_0) < 0$. Then, the Implicit Function Theorem implies that $D(x)$ is analytic at $x = 0$.

Lemma 2.3 *With the previous assumptions, $D(x)$ has a finite singularity $R = R(y)$, and $D(R)$ is also finite.*

Proof. We first show that $D(x)$ has a finite singularity. Consider the family of networks without 3-connected components, which corresponds to series-parallel networks, and let $D_\emptyset(x, y)$ be the associated GF. It is shown in [9] that the radius of convergence $R_\emptyset(y_0)$ of $D_\emptyset(x, y_0)$ is finite for all $y > 0$. Since the set of networks enumerated by $D(x, y_0)$ contains the networks without 3-connected components, it follows that $D_\emptyset(x, y) \le D(x, y)$ and $D(x)$ has a finite singularity $R(y_0) \le R_\emptyset(y_0)$.

Next we show that $D(x)$ is finite at its dominant singularity $R = R(y_0)$. Since R is the smallest singularity and $\Phi_z(0, D(0)) < 0$, we have $\Phi_z(x, D(x)) < 0$ for $0 \le x < R$. We also have $\Phi_{zz}(x, z) > 0$ for $x, z > 0$. Indeed, the first summand in Φ is a series with positive coefficients, and all its derivatives are positive; the other two terms have with second derivatives $1/(1+z)^2$ and $2x/(1+xz)^3$, which are also positive. As a consequence, $\Phi_z(x, D(x))$ is an increasing function and $\lim_{x\to R^-} \Phi(x, D(x))$ exists and is finite. It follows that $D(R)$ cannot go to infinity, as claimed. □

Since R is the smallest singularity of $D(x)$, $\Phi(x, z)$ is analytic for all $x < R$ along the curve defined by $\Phi(x, D(x)) = 0$. For $x, z > 0$ it is clear that Φ is analytic at (x, z) if and only if $T(x, z)$ is also analytic. Thus $T(x, z)$ is also analytic along the curve $\Phi(x, D(x)) = 0$ for $x < R$. As a consequence, the singularity R can only have two possible sources:

(a) A branch-point (R, D_0) when solving $\Phi(x, z) = 0$, that is, Φ and Φ_z vanish at (R, D_0).

(b) $T(x, z)$ becomes singular at (R, D_0), so that $\Phi(x, z)$ is also singular.

Case (a) corresponds to case (1) in Theorem 2.2. For case (b) we assume that the singular exponent of $T(x, z)$ at the dominant singularity is $5/2$, which corresponds to families of 3-connected graphs coming for 3-connected planar maps, and related families of graphs. The typical situation is case (2.1) in Theorem 2.2, but (2.2) and (2.3) are also possible. It is also possible to have a critical situation, where (a) and (b) both hold, and this leads to case (3.1): this is treated at the end of this subsection. Finally, a confluence of singularities may also arise when solving equation

$$xC'(x, y) = x\exp\left(B'(xC'(x, y), y)\right),$$

and we are in case (3.2). This is treated at the end of Section 2.3.2.

Φ has a branch-point at (R, D_0)

We assume that $\Phi_z(R, D_0) = 0$ and that Φ is analytic at (R, D_0). We have seen that $\Phi_{zz}(x, z) > 0$ for $x, z > 0$. Under these conditions, $D(x)$ admits a singular expansion near R of the form

$$D(x) = D_0 + D_1X + D_2X^2 + D_3X^3 + O(X^4), \tag{2.9}$$

where $X = \sqrt{1 - x/R}$, and $D_1 = -\sqrt{2R\Phi_x(R, D_0)/\Phi_{zz}(R, D_0)}$ (see [19]). We remark that R and the D_i's depend implicitly on y_0.

Proposition 2.4 *Consider the singular expansion (2.9). Then $D_1 < 0$ is given by*

$$D_1 = -\left(\frac{2RT_{xz} - 4T_z + \dfrac{R^3 D_0^2}{(1+RD_0)^2}}{\dfrac{R^2}{2(1+D_0)^2} + \dfrac{R^3}{(1+RD_0)^3} + T_{zzz}} \right)^{1/2},$$

where the partial derivatives of T are evaluated at (R, D_0).

Proof. We plug the expansion (2.9) inside (2.6) and extract work out the undetermined coefficients D_i. The expression for D_1 follows from a direct computation of Φ_x and Φ_{zz}, and evaluating at (R, D_0). To show that D_1 does not vanish, notice that

$$2RT_{xz} - 4T_z = R^3 \frac{\partial}{\partial x}\left(\frac{2}{x^2} T_z(x,z) \right).$$

This is positive since $2/x^2 T_z$ is a series with positive coefficients. Since $R, D_0 > 0$, the remaining term in the numerator inside the square root is clearly positive, and so is the denominator. Hence $D_1 < 0$. □

From the singular expansion of $D(x)$ and the explicit expression (2.7) of $B(x, y_0)$ in terms of $D(x, y_0)$, it is clear that $B(x) = B(x, y_0)$ also admits a singular expansion at the same singularity R of the form

$$B(x) = B_0 + B_1 X + B_2 X^2 + B_3 X^3 + O(X^4). \tag{2.10}$$

The next result shows that the singular exponent of $B(x)$ is $3/2$, as claimed.

Proposition 2.5 *Consider the singular expansion (2.10). Then $B_1 = 0$ and $B_3 > 0$ is given by*

$$B_3 = \frac{1}{3}\left(4T_z - 2RT_{xz} - \frac{R^3 D_0^2}{(1+RD_0)^2} \right) D_1, \tag{2.11}$$

where the partial derivatives of T are evaluated in (R, D_0).

Proof. We plug the singular expansion (2.9) of $D(x)$ into Equation (2.7) and work out the undetermined coefficients B_i. One can check that $B_1 = 2R^2\Phi(R, D_0)D_1$, which vanishes because $\Phi(x, D(x)) = 0$.

When computing B_3, it turns out that the values D_2 and D_3 are irrelevant because they appear in a term which contains a factor Φ_z, which by definition vanishes at (R, D_0). This observation gives directly Equation (2.11). The fact $B_3 \neq 0$ follows from applying the same argument as in the proof of Proposition 2.4, that is, $4T_z - 2RT_{xz} < 0$. Then $B_3 > 0$ since it is the product of two negative numbers. □

Φ is singular at (R, D_0)

In this case we assume that $T(x, z)$ is singular at (R, D_0) and that $\Phi_z(R, D_0) < 0$. The situation where both $T(x, z)$ is singular and $\Phi_z(R, D_0) = 0$ is treated in the next subsection.

Lemma 2.6 *The function T_{zz} is bounded at the singular point (R, D_0).*

Proof. By differentiating Equation (2.6) with respect to z we obtain

$$\Phi_z(x,z) = \frac{2}{x^2} T_{zz}(x,z) - \frac{1}{1+z} - \frac{1}{(1+xz)^2} + 1.$$

Since $\Phi_z(R, D_0) < 0$, we have

$$\frac{2}{R^2} T_{zz}(R, D_0) < \frac{1}{1+D_0} + \frac{1}{(1+RD_0)^2} - 1 < 1.$$

Hence $T_{zz}(R, D_0) < R^2/2$. □

Let us consider now the singular expansions of Φ and T in terms of $Z = \sqrt{1 - z/r(x)}$, where $r(x)$ is the dominant singularity. Note that, by Equation (2.6), Φ and T_z have the same singular behaviour. By Lemma 2.6, the singular exponent α of the dominant singular term Z^α of T_{zz} must be greater than 0 and, consequently, the singular exponent of T_z and Φ is greater than 1. As discussed above, we only study the case where the singular exponent of $T(x, z)$ is 5/2 (equivalently, the singular exponent of $\Phi(x, z)$ is 3/2), which corresponds to several families of 3-connected graphs arising from maps. That is, we assume that T has a singular expansion of the form

$$T(x, z) = T_0(x) + T_2(x)Z^2 + T_4(x)Z^4 + T_5(x)Z^5 + O(Z^6),$$

where $Z = \sqrt{1 - z/r(x)}$, and the functions $r(x)$ and $T_i(x)$ are analytic in a neighborhood of R. Notice that $r(R) = D_0$. Since we are assuming that the singular exponent is 5/2, we have that $T_5(R) \neq 0$.

We introduce now the Taylor expansion of the coefficients $T_i(x)$ at R. However, since we aim at computing the singular expansions of $D(x)$ and $B(x)$ at R, we expand in even powers of $X = \sqrt{1 - x/R}$:

$$\begin{aligned} T(x, z) \;=\; & T_{0,0} + T_{0,2}X^2 + O(X^4) + \\ & \big(T_{2,0} + T_{2,2}X^2 + O(X^4)\big) \cdot Z^2 + \\ & \big(T_{4,0} + T_{4,2}X^2 + O(X^4)\big) \cdot Z^4 + \\ & \big(T_{5,0} + T_{5,2}X^2 + O(X^4)\big) \cdot Z^5 + O(Z^6). \end{aligned} \tag{2.12}$$

Notice that $T_{5,0} = T_5(R) \neq 0$. Similarly, we also consider the expansion of Φ given by

$$\begin{aligned} \Phi(x, z) \;=\; & \Phi_{0,0} + \Phi_{0,2}X^2 + O(X^4) + \\ & \big(\Phi_{2,0} + \Phi_{2,2}X^2 + O(X^4)\big) \cdot Z^2 + \\ & \big(\Phi_{3,0} + \Phi_{3,2}X^2 + O(X^4)\big) \cdot Z^3 + O(Z^4), \end{aligned} \tag{2.13}$$

where $\Phi_{2,0} \neq 0$ because $\Phi_z(R, D_0) < 0$.

Proposition 2.7 *The function $D(x)$ admits the following singular expansion*

$$D(x) = D_0 + D_2X^2 + D_3X^3 + O(X^4),$$

where $X = \sqrt{1 - x/R}$. Moreover,

$$D_2 = D_0\frac{P}{Q} - Rr', \qquad D_3 = -\frac{5T_{5,0}(-P)^{3/2}}{R^2\, Q^{5/2}} > 0,$$

where r' is the evaluation of the derivative $r'(x)$ at $x = R$, and $P < 0$ and $Q > 0$ are given by

$$\begin{aligned} P = \Phi_{0,2} &= -\frac{4T_{2,0} + 2T_{2,2}}{R^2D_0} - \frac{2T_{2,0}r'}{RD_0^2} + \frac{Rr'}{1 + D_0} - \frac{RD_0(D_0 + (2 + RD_0)Rr')}{(1 + RD_0)^2}, \\ Q = \Phi_{2,0} &= -\frac{4T_{4,0}}{R^2D_0} + \frac{D_0}{1 + D_0} - \frac{2RD_0^2}{1 + RD_0} + \frac{R^2D_0^3}{(1 + RD_0)^2}. \end{aligned}$$

Proof. We consider Equation (2.13) as a power series $\Phi(X, Z)$, where $X = \sqrt{1 - x/R}$ and $Z = \sqrt{1 - z/D_0}$. We look for a solution $Z(X)$ such that $\Phi(X, Z(X)) = 0$; we also impose $Z(0) = 0$, since $\Phi_{0,0} = \Phi(R, D_0) = 0$. Define $D(x)$ as

$$D(x) = r(x)(1 - Z(X)^2),$$

which satisfies $\Phi(x, D(x)) = 0$ and $D(R) = D_0$. By indeterminate coefficients we obtain

$$Z(X) = \pm\sqrt{\frac{-\Phi_{0,2}}{\Phi_{2,0}}}X + \frac{\Phi_{3,0}\,\Phi_{0,2}}{2\,{\Phi_{2,0}}^2}X^2 + O(X^3),$$

where the sign of the coefficient in X is determined later. Now we use this expression and the Taylor series of the analytic function $r(x)$ at $x = R$ to obtain the following singular expansion for $D(x)$:

$$D(x) = D_0 + \left(D_0 \frac{\Phi_{0,2}}{\Phi_{2,0}} - Rr' \right) X^2 \pm D_0 \frac{(-\Phi_{0,2})^{3/2}\, \Phi_{3,0}}{{\Phi_{2,0}}^{5/2}} X^3 + O(X^4).$$

Observe in particular that the coefficient of X vanishes. We define $P = \Phi_{0,2}$ and $Q = \Phi_{2,0}$. The fact that $P < 0$ and $Q > 0$ follows from the relations

$$\Phi_z = \frac{-1}{D_0} \Phi_{2,0}, \qquad \Phi_x = \frac{-1}{R} \Phi_{0,2} + \frac{r'}{D_0^2} \Phi_{2,0},$$

that are obtained by differentiating Equation (2.13). We have $\Phi_z < 0$ by assumption, and $\Phi_x > 0$ following the proof of Proposition 2.4.

The coefficient D_3 must have positive sign, since $D''(x)$ is a positive function and its singular expansion is $D_{xx}(x) = 3D_3(4R^2)^{-1}X^{-1} + O(1)$. The coefficients $\Phi_{i,j}$ in Equation (2.13) are easily expressed in term of the $T_{i,j}$, and a simple computation gives the result as claimed. □

Proposition 2.8 *The function $B(x)$ admits the following singular expansion*

$$B(x) = B_0 + B_2X^2 + B_4X^4 + B_5X^5 + O(X^6),$$

where $X = \sqrt{1 - x/R}$. Moreover,

$$\begin{aligned}
B_0 &= \frac{R^2}{2}\left(D_0 + \frac{1}{2}D_0^2\right) - \frac{1}{2}RD_0 + \frac{1}{2}\log\left(1 + RD_0\right) - \frac{1}{2}(1 + D_0)\frac{R^3 D_0^2}{1 + RD_0} + \\
&\quad T_{0,0} + \frac{1 + D_0}{D_0} T_{2,0}, \\
B_2 &= \frac{R^2 D_0 (D_0^2 R - 2)}{2(1 + RD_0)} + T_{0,2} - \left(2\frac{1 + D_0}{D_0} + \frac{Rr'}{D_0}\right) T_{2,0}, \\
B_4 &= \left(T_{0,4} + \frac{2R^3 D_0^2 - R^4 D_0^4 + 2R^2 D_0}{4(1 + RD_0)^2}\right) + \left(\frac{1 + D_0 + r''}{D_0}\right) T_{2,0} + \frac{P^2}{Q}\frac{R^2 D_0}{4} + \\
&\quad \left(\frac{2R}{D_0}T_{2,0} + \frac{R^4 D_0^2}{2(1 + RD_0)^2}\right) r' + \frac{R^4}{4}\left(\frac{D_0}{1 + D_0} - \frac{1}{(1 + RD_0)^2}\right)(r')^2, \\
B_5 &= T_{5,0}\left(-\frac{P}{Q}\right)^{5/2} < 0,
\end{aligned}$$

where P and Q are as in Proposition 2.7, and r' and r'' are the derivatives of $r(x)$ evaluated at $x = R$.

Proof. Our starting point is Equation (2.7) relating functions D, B and T. We replace T by the singular expansion in Equation (2.12), D by the singular expansion given in Proposition 2.7, and we set $x = X^2(1 - R)$. The expressions for B_i follow by indeterminate coefficients.

When performing these computations we observe that the coefficients B_1 and B_3 vanish identically, and that several simplifications occur in the remaining expressions. □

Φ has a branch-point and $T(x, z)$ is singular at (R, D_0)

This is the first critical situation, and corresponds to case (3.1) in Theorem 2.1. To study this case we proceed exactly as in the case where Φ is singular at (R, D_0) (Section 2.3.1), except that now $\Phi_z(R, D_0) = 0$. It is easy to check that Lemma 2.6 still applies (with the bound $T_{zz}(R, D_0) \leq R^2/2$). As done in the previous section, we only take into consideration families of graphs where the singular exponent of $T(x, z)$ is $5/2$ (equivalently, the singular exponent of $\Phi(x, z)$ is $3/2$). Equations (2.12) and (2.13) still hold, except that now $\Phi_{2,0} = 0$ because of the branch point at (R, D_0). This missing term is crucial, as we make clear in the following analogous of Proposition 2.7.

Proposition 2.9 *The function $D(x)$ admits the following singular expansion*

$$D(x) = D_0 + D_{4/3}X^{4/3} + O(X^2),$$

where $X = \sqrt{1 - x/R}$ and

$$D_{4/3} = -D_0 \left(\frac{-\Phi_{0,2}}{\Phi_{3,0}} \right)^{2/3}.$$

Proof. As in the proof of Proposition 2.7, we consider a solution $Z(X)$ of the functional equation $\Phi(X, Z(X)) = 0$, and define $D(x)$ as $r(x)(1 - Z(X)^2)$. However, the singular development of $\Phi(x, z)$ is now

$$\begin{aligned} \Phi(x,z) \quad = \quad & \Phi_{0,2}X^2 + O(X^4) + \\ & \left(\Phi_{2,2}X^2 + O(X^4)\right) \cdot Z^2 + \\ & \left(\Phi_{3,0} + \Phi_{3,2}X^2 + O(X^4)\right) \cdot Z^3 + O(Z^4), \end{aligned}$$

so that the singular development of $Z(X)$ starts with a term $X^{2/3}$. Indeed, we obtain that

$$Z(X) = \left(\frac{-\Phi_{0,2}}{\Phi_{3,0}} \right)^{2/3} X^{2/3} + O(X^{4/3}).$$

To obtain the actual development of $D(x)$ we just need to use the equalities $D(x) = r(x)(1-Z(X)^2)$ and $r(R) = D_0$. □

Note that $X = \sqrt{1 - x/R}$. Consequently the previous result implies that the singular exponent of $D(x)$ is 2/3. By using the explicit integration of $B_y(x, y)$ of Equation (2.7), one can check that the singular exponent of $B(x)$ is 5/3 (the first non-analytic term of $B(x)$ that does not vanish is $X^{10/3}$). This implies that the subexponential term in the asymptotic of b_n is $n^{-8/3}$, as claimed.

2.3.2 Singularity analysis of $C(x, y)$ and $G(x, y)$

The results in this section are relatively straightforward to obtain, and the discussion follows essentially the same lines as in Section 4 of [26] and Section 3 of [9]. Let $F(x) = xC'(x)$, which is the GF of rooted connected graphs. We know that $F(x) = x \exp(B'(F(x))$. Then $\psi(u) = u \exp(-B'(u))$ is the functional inverse of $F(x)$. Denote by ρ the dominant singularity of F. As for 2-connected graphs, there are two possible sources for the singularity:

1. There exists $\tau \in (0, R)$ (necessarily unique) such that $\psi'(\tau) = 0$. By the Inverse Function Theorem ψ ceases to be invertible at τ, and $\rho = \psi(\tau)$.

2. We have $\psi'(u) \neq 0$ for all $u \in (0, R)$, and there is no obstacle to the analyticity of the inverse function. Then $\rho = \psi(R)$.

The critical case $\psi'(R) = 0$ is discussed at the end of this subsection.

Condition $\psi'(\tau) = 0$ is equivalent to $B''(\tau) = 1/\tau$. Since $B''(u)$ is increasing (the series $B(u)$ has positive coefficients) and $1/u$ is decreasing, we are in case (1) if $B''(R) > 1/R$, and in case (2) if $B''(R) < 1/R$. As we have already discussed, series-parallel graphs correspond to case (1) and planar graphs to case (2). In particular, if B has singular exponent 3/2, like for series-parallel graphs, the function $B''(u)$ goes to infinity when u tends to R, so there is always a solution $\tau < R$ satisfying $B''(\tau) = 1/\tau$. This explains why in Theorem 2.1 there is no case where b_n has sub-exponential growth $n^{-5/2}$ and c_n has $n^{-7/2}$.

Proposition 2.10 *The value $S = RB''(R)$ determines the type of singularity of $C(x)$ and $G(x)$ as follows:*

(1) If $S > 1$, then $C(x)$ and $G(x)$ admit the singular expansions

$$\begin{aligned} C(x) &= C_0 + C_2X^2 + C_3X^3 + O(X^4), \\ G(x) &= G_0 + G_2X^2 + G_3X^3 + O(X^4), \end{aligned}$$

where $X = \sqrt{1 - x/\rho}$, $\rho = \psi(\tau)$, and τ is the unique solution to $\tau B''(\tau) = 1$. We have

$$C_0 = \tau(1 + \log\rho - \log\tau) + B(\tau), \qquad C_2 = -\tau,$$

$$C_3 = \frac{3}{2}\sqrt{\frac{2\rho\exp\left(B'(\rho)\right)}{\tau B'''(\tau) - \tau B''(\tau)^2 + 2B''(\tau)}},$$

$$G_0 = e^{C_0}, \qquad G_2 = C_2 e^{C_0}, \qquad G_3 = C_3 e^{C_0}.$$

(2) If $S < 1$, then $C(x)$ and $G(x)$ admit the singular expansions

$$\begin{aligned} C(x) &= C_0 + C_2X^2 + C_4X^4 + C_5X^5 + O(X^6), \\ G(x) &= G_0 + G_2X^2 + G_4X^4 + G_5X^5 + O(X^6), \end{aligned}$$

where $X = \sqrt{1 - x/\rho}$, $\rho = \psi(R)$. We have

$$\begin{aligned} &C_0 = \tau(1 + \log\rho - \log R) + B_0, && C_2 = -R, \\ &C_4 = -\frac{RB_4}{2B_4 - R}, && C_5 = B_5\left(1 - \frac{2B_4}{R}\right)^{-5/2}, \\ &G_0 = e^{C_0}, && G_2 = C_2e^{C_0}, \\ &G_4 = \left(C_4 + \frac{1}{2}{C_2}^2\right)e^{C_0}, && G_5 = C_5e^{C_0}, \end{aligned}$$

where B_0, B_4 and B_5 are as in Proposition 2.8.

Proof. The two cases $S > 1$ and $S < 1$ arise from the previous discussion. In case (1) we follow the proof of Theorem 3.6 from [9], and in case (2) the proof of Theorem 1 from [26].

First, we obtain the singular expansion of $F(x) = xC'(x)$ near $x = \rho$. This can be done by indeterminate coefficients in the equality $\psi(F(x)) = x = \rho(1 - X^2)$, with $X = \sqrt{1 - x/\rho}$. The expansion of ψ can be either at $\tau = F(\rho)$ where it is analytic, or at $R = F(\rho)$ where it is singular.

From the singular expansion of $F(x)$ we obtain C_2 and C_3 in case (1), and C_2, C_4 and C_5 in case (2) by direct computation. To obtain C_0, however, it is necessary to compute

$$C(x) = \int_0^x \frac{F(t)}{t}\,dt,$$

and this is done using the integration techniques developed in [9] and [26]. Finally, the coefficients for $G(x)$ are obtained directly from the general relation $G(x) = \exp(C(x))$. □

To conclude this section we consider the critical case where both sources of the dominant singularity ρ coincide, that is, when $\psi'(R) = 0$. In this case ψ is singular at R because R is the singularity of $B(x)$, and at the same time the inverse $F(x)$ is singular at $\rho = \psi(R)$ because of the inverse function theorem. As we have shown before, this can only happen if $B(x)$ has singular exponent $5/2$. Hence the singular development of $\psi(z)$ in terms of $Z = \sqrt{1 - z/R}$ must be

$$\psi(z) = \psi_0 + \psi_2Z^2 + \psi_3Z^3 + O(Z^4),$$

where in addition ψ_2 vanishes due to $\psi'(R) = 0$. Now a similar analysis as that in Section 2.3.1 shows that the singular exponent of $C(x)$ is $5/3$. Indeed, since $\psi(F(x)) = x = \rho(1 - X^2)$, we deduce that the development of $F(x)$ in terms of $X = \sqrt{1 - x/\rho}$ is

$$F(x) = \rho + \left(\frac{-\rho^{5/3}}{\psi_3^{2/3}}\right)X^{4/3} + O(X^2).$$

Thus we obtain, by integration of $F(x) = xC'(x)$, that the singular exponent of $C(x)$ is $5/3$, so that the subexponential term in the asymptotic of c_n is $n^{-8/3}$. Since $G(x) = \exp(C(x))$, the same exponents hold for $G(x)$ and g_n.

2.4 Limit laws

In this section we discuss parameters of random graphs from a closed family whose limit laws do not depend on the singular behaviour of the GFs involved. As we are going to see, only the constants associated to the first two moments depend on the singular exponents.

The parameters we consider are asymptotically either normal or Poisson distributed. The number of edges, number of blocks, number of cut vertices, number of copies of a fixed block, and number of special copies of a fixed subgraph are all normal. On the other hand, the number of connected components is Poisson. The size of the largest connected component (rather, the number of vertices not in the largest component) also follows a discrete limit law. A fundamental extremal parameter, the size of the largest block, is treated in the next section, where it is shown that the asymptotic limit law depends very strongly on the family under consideration.

As in the previous section, let $\mathcal{G}$ be a closed family of graphs. For a fixed value of y, let $\rho(y)$ be the dominant singularity of $C(x,y)$, and let $R(y)$ be that of $B(x,y)$. We write $\rho = \rho(1)$ and $R = R(1)$. Recall that $B'(x,y)$ denotes the derivative with respect to x.

When we speak of cases (1) and (2), we refer to the statement of Proposition 2.10, which are exemplified, respectively, by series-parallel and planar graphs. That is, in case (1) the singular dominant term in $C(x)$ and $G(x)$ is $(1-x/\rho)^{3/2}$, whereas in case (2) it is $(1-x/\rho)^{5/2}$. Recall from the previous section that in case (1) we have $\rho(y) = \tau(y)\exp\left(-B'(\tau(y),y)\right)$, where $\tau(y)B''(\tau(y)) = 1$. In case (2) we have $\rho(y) = R(y)\exp\left(-B'(R(y),y)\right)$.

2.4.1 Number of edges

The number of edges obeys a limit normal law, and the asymptotic expression for the first two moments is always given in terms of the function $\rho(y)$ for connected graphs, and in terms of $R(y)$ for 2-connected graphs.

Theorem 2.11 *The number of edges in a random graph from $\mathcal{G}$ with n vertices is asymptotically normal, and the mean μ_n and variance σ_n^2 satisfy*

$$\mu_n \sim \kappa n, \qquad \sigma_n^2 \sim \lambda n,$$

where

$$\kappa = -\frac{\rho'(1)}{\rho(1)}, \qquad \lambda = -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2.$$

The same is true, with the same constants, for connected *random graphs.*

The number of edges in a random 2-connected graph from $\mathcal{G}$ with n vertices is asymptotically normal, and the mean μ_n and variance σ_n^2 satisfy

$$\mu_n \sim \kappa_2 n, \qquad \sigma_n^2 \sim \lambda_2 n,$$

where

$$\kappa_2 = -\frac{R'(1)}{R(1)}, \qquad \lambda_2 = -\frac{R''(1)}{R(1)} - \frac{R'(1)}{R(1)} + \left(\frac{R'(1)}{R(1)}\right)^2.$$

Proof. The proof is as in [26] and [9], an it is a simple consequence of the Quasi-Powers Theorem (Theorem 1.3). In all cases the derivatives of $\rho(y)$ and $R(y)$ are readily computed, and for a given family of graphs we can compute the constants exactly. □

2.4.2 Number of blocks and cut vertices

Again we have normal limit laws but the asymptotic for the first two moments depends in which case we are. In the next statements we set $\tau = \tau(1)$.

Theorem 2.12 *The number of blocks in a random connected graph from $\mathcal{G}$ with n vertices is asymptotically normal, and the mean μ_n and variance σ_n^2 are linear in n. In case (1) we have*

$$\mu_n \sim \log(\tau/\rho)\, n, \qquad \sigma_n^2 \sim \left(\log(\tau/\rho) - \frac{1}{1+\tau^2 B'''(\tau)}\right) n.$$

In case (2) we have

$$\mu_n \sim \log(R/\rho)\, n, \qquad \sigma_n^2 \sim \log(R/\rho)\, n.$$

The same is true, with the same constants, for arbitrary random graphs.

Proof. The proof for case (2) is as in [26], and is based in an application of the Quasi-Powers Theorem. If $C(x,u)$ is the generating function of connected graphs where now u marks blocks, then we have

$$xC'(x,u) = x \exp\left(uB'(xC'(x,u))\right), \tag{2.14}$$

where derivatives are as usual with respect to x. For fixed u, $\psi(t) = t\exp(-uB'(t))$ is the functional inverse of $xC'(x,u)$. We know that for $u = 1$, $\psi'(t)$ does not vanish, and the same is true for u close to 1 by continuity. The dominant singularity of $C(x,u)$ is at $\sigma(u) = \psi(R) = R\exp(-uB'(R))$, and it is easy to compute the derivatives $\sigma'(1)$ and $\sigma''(1)$ (see [26] for details).

In case (1), Equation (2.14) holds as well, but now the dominant singularity is at $\psi(\tau)$. A routine (but longer) computation gives the constants as claimed. □

Theorem 2.13 *The number of cut vertices in a random connected graph from $\mathcal{G}$ with n vertices is asymptotically normal, and the mean μ_n and variance σ_n^2 are linear in n. In case (1) we have*

$$\mu_n \sim \left(1 - \frac{\rho}{\tau}\right) n, \qquad \sigma_n^2 \sim \sigma^2 = \left(\frac{\rho}{\tau} - \left(\frac{\rho}{\tau}\right)^2 - \left(\frac{\rho}{\tau}\right)^2 \frac{1}{1+\tau^2 B'''(\tau)}\right) n.$$

In case (2) we have

$$\mu_n \sim \left(1 - \frac{\rho}{R}\right) n, \qquad \sigma_n^2 \sim \frac{\rho}{R}\left(1 - \frac{\rho}{R}\right) n.$$

The same is true, with the same constants, for arbitrary random graphs.

Proof. If u marks cut vertices in $C(x,u)$, then we have

$$xC'(x,u) = xu(\exp\left(B'(xC'(x,u))\right) - 1) + x.$$

It follows that, for given u,

$$\psi(t) = \frac{t}{u(\exp\left(B'(t)\right) - 1) + 1}$$

is the inverse function of $xC'(x,u)$. In case (2) the dominant singularity $\sigma(u)$ is at $\psi(R)$. Taking into account that $\rho = R\exp(B'(R))$, the derivatives of σ are easily computed. In case (1) the singularity is at $\psi(\tau(u))$, where $\tau(u)$ is given by $\psi'(\tau(u)) = 0$. In order to compute derivatives, we differentiate $\psi(\tau(u)) = 0$ with respect to u and solve for $\tau'(u)$, and once more in order to get $\tau''(u)$. After several computations and simplifications using `Maple`, we get the values as claimed. □

2.4.3 Number of copies of a subgraph

Let H be a fixed rooted graph from the class $\mathcal{G}$, with vertex set $\{1,\dots,h\}$ and root r. Following [36], we say that H *appears* in G at $W \subset V(G)$ if (a) there is an increasing bijection from $\{1,\dots,h\}$ to W giving an isomorphism between H and the induced subgraph $G[W]$ of G; and (b) there is exactly one edge in G between W and the rest of G, and this edge is incident with the root r.

Thus an appearance of H gives a copy of H in G of a very particular type, since the copy is joined to the rest of the graph through a unique pendant edge. We do not know how to count the number of subgraphs isomorphic to H in a random graph, but we can count very precisely the number of appearances.

Theorem 2.14 *Let H be a fixed rooted connected graph in $\mathcal{G}$ with h vertices. Let $\boldsymbol{X}_n$ denote the number of appearances of H in a random rooted connected graph from $\mathcal{G}$ with n vertices. Then $\boldsymbol{X}_n$ is asymptotically normal and the mean μ_n and variance σ_n^2 satisfy*

$$\mu_n \sim \frac{\rho^h}{h!}\, n, \qquad \sigma_n^2 \sim \rho\, n,$$

Proof. The proof is as in [26], and is based on the Quasi-Powers Theorem. If $f(x,u)$ is the generating function of rooted connected graphs and u counts appearances of H then, up to a simple term that does not affect the asymptotic estimates, we have

$$f(x,u) = x \exp\left(B'(f(x,u)) + (u-1)\frac{x^h}{h!}\right).$$

The dominant singularity is computed through a change of variable, and the rest of the computation is standard; see the proof of Theorem 5 in [26] for details. For this parameter is no difference between cases (1) and (2). □

Now we study appearances of a fixed 2-connected subgraph L from $\mathcal{G}$ in rooted connected graphs. An appearance of L in this case corresponds to a block with a labelling order isomorphic to L. Notice in this case that an appearance can be anywhere in the tree of blocks, not only as a terminal block.

Theorem 2.15 *Let L be a fixed rooted 2-connected graph in $\mathcal{G}$ with $\ell+1$ vertices. Let $\boldsymbol{X}_n$ denote the number of appearances of L in a random connected graph from $\mathcal{G}$ with n vertices. Then $\boldsymbol{X}_n$ is asymptotically normal and the mean μ_n and variance σ_n^2 satisfy*

$$\mu_n \sim \frac{R^\ell}{\ell!}\, n, \qquad \sigma_n^2 \sim \frac{R^\ell}{\ell!}\, n,$$

Proof. If $f(x,u)$ is the generating function of rooted connected graphs and u counts appearances of L, then we have

$$f(x,u) = x \exp\left(B'(f(x,u)) + (u-1)\frac{f(x,u)^\ell}{\ell!}\right).$$

The reason is that each occurrence of L is single out by multiplying by u. Notice that L has $\ell+1$ vertices by the root bears no label. It follows that the inverse of $f(x,u)$ is given by (for a given value of u)

$$\phi(t) = t \exp\left(-B'(t) - (u-1)t^\ell/\ell!\right).$$

The singularity of $\phi(t)$ is equal to R, independently of t. Since for $u=1$ we know that $\phi'(t)$ does not vanish, the same is true for u close to 1. Then the dominant singularity of $f(x,u)$ is given by

$$\sigma(u) = \phi(R) = \rho \cdot \exp(-(u-1)R^\ell/\ell!),$$

since $\rho = R\exp(-B'(R))$. A simple calculation gives

$$\sigma'(1) = -\rho\frac{R^\ell}{\ell!}, \qquad \sigma''(1) = \rho\frac{R^{2\ell}}{\ell!^2}$$

and the results follows easily as in the proof of Theorem 2.11. Again, for this parameter is no difference between cases (1) and (2). □

2.4.4 Number of connected components

Our next parameter, as opposed to the previous one, follows a discrete limit law.

Theorem 2.16 *Let $\boldsymbol{X}_n$ denote the number of connected components in a random graph $\mathcal{G}$ with n vertices. Then $\boldsymbol{X}_n - 1$ is distributed asymptotically as a Poisson law of parameter ν, where $\nu = C(\rho)$.*

As a consequence, the probability that a random graph $\mathcal{G}$ is connected is asymptotically equal to $e^{-\nu}$.

Proof. The proof is as in [26]. The generating function of graphs with exactly k connected components is $C(x)^k/k!$. Taking the k-th power of the singular expansion of $C(x)$, we have $[x^n]C(x)^k \sim k\nu^{k-1}[x^n]C(x)$. Hence the probability that a random graphs has exactly k components is asymptotically

$$\frac{[x^n]C(x)^k/k!}{[x^n]G(x)} \sim \frac{k\nu^{k-1}}{k!}\, e^{-\nu} = \frac{\nu^{k-1}}{(k-1)!}\, e^{-\nu}$$

as was to be proved. □

2.4.5 Size of the largest connected component

Extremal parameters are treated in the next two sections. However, the size of the largest component is easy to analyze and we include it here. The notation $\mathbf{M}_n$ in the next statement, suggesting vertices *missed* by the largest component, is borrowed from [35]. Recall that g_n, c_n are the numbers of graphs and connected graphs, respectively, R is the radius of convergence of $B(x)$, and C_i are the singular coefficients of $C(x)$.

Theorem 2.17 *Let $\boldsymbol{L}_n$ denote the size of the largest connected component in a random graph $\mathcal{G}$ with n vertices, and let $\boldsymbol{M}_n = \boldsymbol{L}_n - n$. Then*

$$\mathbf{p}\left(\{\boldsymbol{M}_n = k\}\right) \sim p_k = p \cdot g_k \frac{\rho^k}{k!},$$

where p is the probability of a random graph being connected. Asymptotically, either $p_k \sim c\, k^{-5/2}$ or $p_k \sim c\, k^{-7/2}$ as $k \to \infty$, depending on the subexponential term in the estimate of g_k.

In addition, we have $\sum p_k = 1$ and $\mathbb{E}\left[\boldsymbol{M}_n\right] \sim \tau$ in case (1) and $\mathbb{E}\left[\boldsymbol{M}_n\right] \sim R$ in case (2). In case (1) the variance $\sigma^2(\boldsymbol{M}_n)$ does not exist and in case (2) we have $\sigma^2(\boldsymbol{M}_n) \sim R + 2C_4$.

Proof. The proof is essentially the same as in [35]. For fixed k, the probability that $\mathbf{M}_n = k$ is equal to

$$\binom{n}{k} \frac{c_{n-k} g_k}{g_n},$$

since there are $\binom{n}{k}$ ways of choosing the labels of the vertices not in the largest component, c_{n-k} ways of choosing the largest component, and g_k ways of choosing the complement. In case (1), given the estimates

$$g_n \sim g \cdot n^{-5/2} \rho^{-n} n!, \qquad c_n \sim c \cdot n^{-5/2} \rho^{-n} n!,$$

the estimate for p_k follows at once (we argue similarly in each subcase of (2)). Observe that $p = \lim c_n/g_n = c/g$.

For the second part of the statement notice that,

$$\sum p_k = p \sum g_k \frac{\rho^k}{k!} = p\, G(\rho) = 1,$$

since from Theorem 2.16 it follows that $p = e^{-C(\rho)} = 1/G(\rho)$. To compute the moments notice that the probability GF is $f(u) = \sum p_k u^k = pG(\rho u)$. Then the expectation is estimated as

$$f'(1) = p\,\rho G'(\rho) = p\,G(\rho)\rho C'(\rho) = \rho C'(\rho),$$

which corresponds to τ in case (1) and R in case (2), since $G(x) = \exp C(x)$. For the variance we compute

$$f''(1) + f'(1) - f'(1)^2 = \rho C'(\rho) + \rho^2 C''(\rho).$$

In case (1) $\lim_{x\to\rho} C''(x) = \infty$, so that the variance does not exist. In case (2) we have $\rho C'(\rho) = R$ and $\rho^2 C''(\rho) = 2C_4$. □

2.5 Largest block and 2-connected core

The problem of estimating the largest block in random maps has been well studied. We recall that a map is a connected planar graph together with a specific embedding in the plane. Moreover, an edge has been oriented and marked as the root edge. Gao and Wormald [22] proved that the largest block in a random map with n edges has almost surely $n/3$ edges, with deviations of order $n^{2/3}$. More precisely, if $\mathbf{X}_n$ is the size of the largest block, then

$$\mathbf{p}\left(\left\{|\mathbf{X}_n - n/3| < \lambda(n)n^{2/3}\right\}\right) \to 1, \qquad \text{as } n \to \infty,$$

where $\lambda(n)$ is any function going to infinity with n. The picture was further clarified by Banderier et al. [3]. They found that the largest block in random maps obeys a continuous limit law, which is called by the authors the 'Airy distribution of the map type', and is closely related to a stable law of index 3/2. As we will see shortly, the Airy distribution also appears in random planar graphs.

A useful technical device is to work with the *2-connected core*, which in the case of maps is the (unique) block containing the root edge. For graphs it is a bit more delicate. Consider a connected graph R rooted at a vertex v. We would like to say that the core of R is the block containing the root, but if v is a cut vertex then there are several blocks containing v and there is no clear way to single out one of them. Another possibility is to say that the 2-connected core is the union of 2-connected components the blocks containing the root, but then the core is not in general a 2-connected graph.

The definition we adopt is the following. If the root is not a cut vertex, then the *core* is the unique block containing the root. Otherwise, we say that the rooted graph is *coreless*. Let $C^\bullet(x, u)$ be the generating function of rooted connected graphs, where the root bears no label, and u marks the size of the 2-connected core. Then we have

$$C^\bullet(x, u) = B'(uxC'(x)) + \exp(B'(xC'(x)) - B'(xC'(x)),$$

where $C(x)$ and $B(x)$ are the GFs for connected and 2-connected graphs, respectively. The first summand corresponds with graphs which have a core, whose size is recorded through variable u, and the second one to coreless graphs. We rewrite the former equation as

$$C^\bullet(x, u) = Q(uH(x)) + Q_L(x),$$

where

$$H(x) = xC'(x), \quad Q(x) = B'(x), \quad Q_L(x) = \exp(B'(xC'(x)) - B'(xC'(x)).$$

With this notation, $Q_L(x)$ enumerates coreless graphs, and $Q(uH(x))$ enumerates graphs with core. The asymptotic probability that a graph is coreless is

$$p_L = \lim_{n\to\infty} \frac{[x^n]Q_L(x)}{[x^n]C'(x)} = 1 - \lim_{n\to\infty} \frac{[x^n]Q(H(x))}{[x^n]C'(x)}.$$

The key point is that graphs with core fit into a composition scheme

$$Q(uH(x)).$$

This has to be understood as follows. A rooted connected graph whose root is not a cut vertex is obtained from a 2-connected graph (the core), replacing each vertex of the core by a rooted connected graph. It is shown in [3] that such a composition scheme leads either to a discrete law or to a continuous law, depending on the nature of the singularities of $Q(x)$ and $H(x)$.

Our analysis for a closed class $\mathcal{G}$ is divided into two cases. If we are in case (1) of Proposition 2.10, we say that the class $\mathcal{G}$ is *series-parallel-like*; in this situation the size of the core follows invariably a discrete law which can be determined precisely in terms of $Q(x)$ and $H(x)$. If we are in case (2) we say that the class $\mathcal{G}$ is *planar-like*. In this situation the size of the core has two modes, a discrete law when the core is small, and a continuous Airy distribution when the core has linear size. Moreover, for planar-like classes, the size of the largest block follows the same Airy distribution and is concentrated around αn for a computable constant α. The critical case, discussed at the end of Section 2.3, is not treated here.

2.5.1 Core of series-parallel-like classes

Recall that in case (1) of Proposition 2.10 we have $H(\rho) = \rho C'(\rho) = \tau$, where τ is the solution to the equation $\tau B''(\tau) = 1$. Since $RB''(R) > 1$ and $uB''(u)$ is an increasing function, we conclude that $H(\rho) < R$. This gives rise to the so called *subcritical* composition scheme. We refer to the exposition in section IX.3 of [19]. The main result we use is Proposition IX.1 from [19], which is the following.

Proposition 2.18 *Consider the composition scheme $Q(uH(x))$. Let R, ρ be the radius of convergence of Q and H, respectively. Assume that Q and H satisfy the subcritical condition $\tau = H(\rho) \leq R$, and that $H(x)$ has a unique singularity at ρ on its disk of convergence with a singular expansion*

$$H(x) = \tau - c_\lambda(1 - z/\rho)^\lambda + o((1 - z/\rho)^\lambda),$$

where $\tau, c_\lambda > 0$ and $0 < \lambda < 1$. Then the size of the Q-core follows a discrete limit law,

$$\lim_{n\to\infty} \frac{[x^n u^k]Q(uH(x))}{[x^n]Q(H(x))} = q_k.$$

The probability generating function $q(u) = \sum q_k u^k$ of the limit distribution is

$$q(u) = \frac{uQ'(\tau u)}{Q'(\tau)}.$$

The previous result applies to our composition scheme $Q(uH(x))$, that is, to the family of rooted connected graphs that have core.

Theorem 2.19 *Let $\mathcal{G}$ be a series-parallel-like class, and let $\boldsymbol{Y}_n$ be the size of the 2-connected core in a random rooted connected graph $\mathcal{G}$ with core and n vertices. Then $\mathbf{p}(\{\boldsymbol{Y}_n = k\})$ tends to a limit q_k as n goes to infinity. The probability generating function $q(u) = \sum q_k u^k$ is given by*

$$q(u) = \tau u B''(u\tau).$$

The estimates of q_k for large k depend on the singular behaviour of $B(x)$ near R as follows, where $X = \sqrt{1 - x/R}$:

(a) If $B(x) = B_0 + B_2X^2 + B_3X^3 + O(X^4)$, then $q_k \sim \dfrac{3B_3}{4R\sqrt{\pi}} k^{-1/2} \left(\dfrac{\tau}{R}\right)^k.$

(b) If $B(x) = B_0 + B_2X^2 + B_4X^4 + B_5X^5 + O(X^6)$, then $q_k \sim -\dfrac{5B_5}{2R\sqrt{\pi}} k^{-3/2} \left(\dfrac{\tau}{R}\right)^k.$

Finally, the probability of a graph being coreless is asymptotically equal to $1 - \rho/\tau$.

Proof. We apply Proposition 2.18 with $Q(x) = B'(x)$ and $H(x) = xC'(x)$. Since $\tau B''(\tau) = 1$, we have

$$q(u) = \frac{uB''(u\tau)}{B''(\tau)} = \tau u B''(u\tau),$$

as claimed. The dominant singularity of $q(u)$ is at $u = R/\tau$. The asymptotic for the tail of the distribution follows from the corresponding singular expansions. In case (a) we have

$$B''(X) = \frac{3B_3}{4R^2}X^{-1} + O(1).$$

In case (b) we have

$$B''(X) = \frac{2B_4}{R^2} + \frac{5B_5}{R^2}X + O(X^2).$$

By applying singularity analysis to $q(u)$, the result follows. We remark that $B_3 > 0$ and $B_5 < 0$, so that the multiplicative constants are in each case positive. □

2.5.2 Core and largest block of planar-like classes

In order to state our main result, we need to introduce the Airy distribution. Its density is given by

$$g(x) = 2e^{-2x^3/3}(x\mathrm{Ai}(x^2) - \mathrm{Ai}'(x^2)), \tag{2.15}$$

where $\mathrm{Ai}(x)$ is the Airy function, a particular solution of the differential equation $y'' - xy = 0$. An explicit series expansion is (see equation (2) in [3])

$$g(x) = \frac{1}{\pi x}\sum_{n\geq 1}(-3^{2/3}x)^n \frac{\Gamma(1+2n/3)}{n!}\sin(-2n\pi/3).$$

A plot of $g(x)$ is shown in Figure 2.1. We remark that the left tail (as $x \to -\infty$) decays polynomially while the right tail (as $x \to +\infty$) decays exponentially.

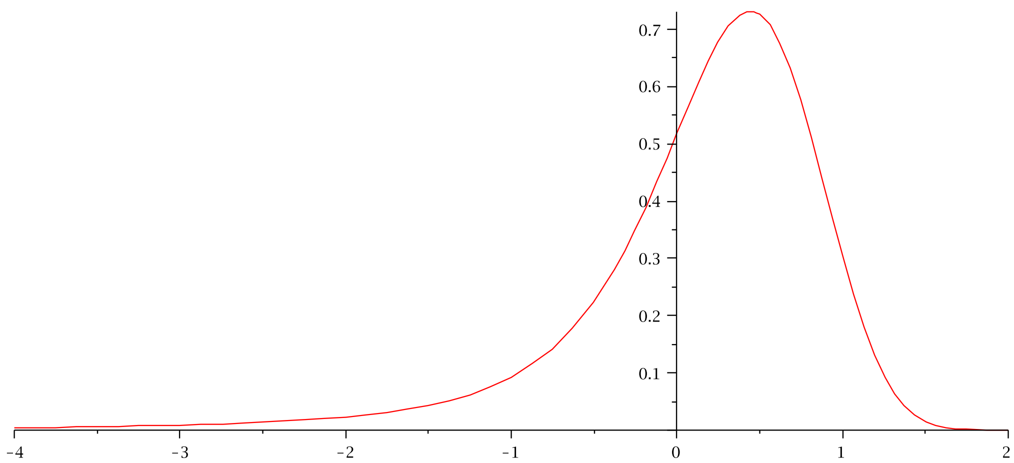


Figure 2.1 The Airy distribution.

We are in case (2) of Proposition 2.10. In this situation we have $\rho = \psi(R)$ and $H(\rho) = R$, which is a *critical* composition scheme. We need Theorem 5 of [3] and the discussion preceding it, which we rephrase in the following proposition.

Proposition 2.20 *Consider the composition scheme $Q(uH(x))$. Let R, ρ be the radius of convergence of Q and H respectively. Assume that Q and H satisfy the critical condition $H(\rho) = R$, and that $H(x)$ and $Q(z)$ have a unique singularity at ρ and R in their respective discs of convergence. Moreover, the singularities of $H(x)$ and $Q(z)$ are of type $3/2$, that is,*

$$\begin{aligned} H(x) &= H_0 + H_2X^2 + H_3X^3 + O(X^4), \\ Q(z) &= Q_0 + Q_2Z^2 + Q_3Z^3 + O(Z^4), \end{aligned}$$

where $X = \sqrt{1 - x/\rho}$, $Z = \sqrt{1 - z/R}$. Let α_0 and M_3 be

$$\alpha_0 = -\frac{H_0}{H_2}, \qquad M_3 = -\frac{Q_2H_3}{R} + Q_3\alpha_0^{-3/2}.$$

Then the asymptotic distribution of the size of the Q-core in $Q(uH(z))$ has two different modes. With probability $p_s = -Q_2H_3/(RM_3)$ the core has size $O(1)$, and with probability $1 - p_s$ the core follows a continuous limit law of the map-Airy type concentrated at $\alpha_0 n$. More precisely, let $\boldsymbol{Y}_n$ be the size of the Q-core of a random element of size n of $Q(uH(z))$.

(a) For fixed k,

$$\mathbf{p}\left(\{\,\boldsymbol{Y}_n = k\}\right) \sim \frac{H_3}{M_3} kR^{k-1}[z^k]Q(z).$$

(b) For $k = \alpha_0 n + xn^{2/3}$ with $x = O(1)$,

$$n^{2/3}\mathbf{p}\left(\{\,\boldsymbol{Y}_n = k\}\right) \sim \frac{Q_3\alpha_0^{-3/2}}{M_3} cg(cx), \quad c = \frac{1}{\alpha_0}\left(\frac{-H_2}{3H_3}\right)^{2/3},$$

where $cg(cx)$ is the map-Airy distribution of parameter c.

In particular, we have $\mathbb{E}\left[\mathbf{X}_n\right] \sim \alpha n$. The parameter c quantifies in some sense the dispersion of the distribution (not the variance, since the second moment does not exist). Note that the asymptotic probability that the core has size $O(1)$ is

$$p_s = \sum_{k=0}^{\infty} \mathbf{p}\left(\{\mathbf{X}_n = k\}\right) \sim \frac{H_3}{M_3}\sum_{k=0}^{\infty} kR^{k-1}[z^k]Q(z) = \frac{H_3}{M_3}Q'(R) = \frac{H_3}{M_3}\left(\frac{-Q_2}{R}\right),$$

and that the asymptotic probability that the core has size $\Theta(n)$ is

$$\frac{Q_3\alpha_0^{-3/2}}{M_3} = 1 - p_s.$$

Now we state the main result in this section. Recall that for a planar-like class of graphs we have

$$B(X) = B_0 + B_2X^2 + B_4X^4 + B_5X^5 + O(X^6),$$

where R is the dominant singularity of $B(x)$ and $X = \sqrt{1 - x/R}$.

Theorem 2.21 *Let $\mathcal{G}$ be a planar-like class, and let X_n be the size of the largest block in a random connected graph $\mathcal{G}$ with n vertices. Then*

$$\mathbf{p}\left(\left\{\boldsymbol{X}_n = \alpha n + xn^{2/3}\right\}\right) \sim n^{-2/3}cg(cx),$$

where

$$\alpha = \frac{R - 2B_4}{R}, \qquad c = \left(\frac{-2R}{15B_5}\right)^{2/3},$$

and $g(x)$ is as in (2.15). Moreover, the size of the second largest block is $O(n^{2/3})$. In particular, for the class of planar graphs we have $\alpha \approx 0.95982$ and $c \approx 128.35169$.

Proof. The composition scheme in our case is $B'(uxC'(x))$. In the notation of the previous proposition, we have $Q(x) = B'(x)$ and $H(x) = xC(x)$.

The size of the core is obtained as a direct application of Proposition 2.20. The exact values for planar graphs have been computed using the known singular expansions for $B(x)$ and $C(x)$ given in the appendix of [26].

For the size of the largest block, one can adapt the arguments from [3], implying that the probability that the core has linear size while not being the largest block tends to 0 exponentially fast. It follows that the distribution of the size of the largest block is exactly the same as the distribution of the core in the linear range. □

The main conclusion is that for planar-like classes of graphs (and in particular for planar graphs) there exists a unique largest block of linear size, whose expected value is asymptotically αn for some computable constant α. The remaining block are of size $O(n^{2/3})$. This is in complete contrast with series-parallel graphs, where we have seen that there are only blocks of sublinear size.

Remark. An observation that we need later, is that if the largest block L has N vertices, the it is uniformly distributed among all the 2-connected graphs in the class. This is because the number of graphs of given size whose largest block is L depends only on the number of vertices of L, and not on its isomorphism type.

We can also analyze the size of the largest block for graphs with a given edge density, or average degree. We have performed the computations for planar graphs, that is, planar graphs with n vertices and $\lfloor \mu n \rfloor$ edges. For $\mu \in (1, 3)$ we find a limit Airy distribution with computable values of the constants $\alpha(\mu)$ and $c(\mu)$. As discussed in [26], we choose a value $y_0 > 0$ depending on μ such that, if we give weight y_0^k to a graph with k edges, then only graphs with n vertices and μn edges have non negligible weight. If $\rho(y)$ is the radius of convergence of $C(x, y)$ as usual, the right choice is the unique positive solution y_0 of

$$-y\rho'(y)/\rho(y) = \mu,$$

Then we work with the generating function $xC(x, y_0)$ instead of $xC'(x)$. Figure 2.2 shows a plot of the main parameter $\alpha(\mu)$ for planar graphs and $\mu \in (1, 3)$.

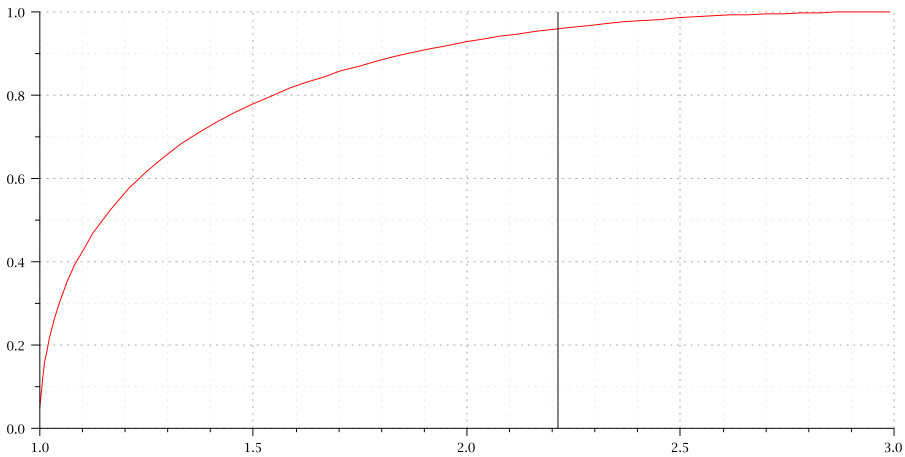


Figure 2.2 Size of largest block for planar graphs with μn edges, $\mu \in (1, 3)$. The ordinate gives the value $\alpha(\mu)$ such that the largest block has size $\sim \alpha(\mu)n$. The value at κ is 0.9598 as in Theorem 2.21.

2.6 Largest 3-connected component

Let us recall that, given a 2-connected graph G, the 3-connected components of G are those 3-connected graphs that are the support of h-networks in the network decomposition of G, as discussed in Section 1.5.

We have seen in Theorem 2.21 that the largest block in a random graph from a planar-like class is almost surely of linear size, and it is unique. In this section we prove a similar result for the largest 3-connected component in random connected graphs with n vertices. Again we obtain a limit Airy law, but the proof is more involved. There are three main technical issues we need to address:

1. We start with a connected graph G. We know from Theorem 2.21 that the largest block L of G is distributed according to an Airy law. We show that the largest 3-connected component T of L is again Airy distributed. Thus we have to *concatenate* two Airy laws, and we show that we obtain another Airy law with computable parameters. Our proof is based on the fact that the sum of two independent stable laws of the same index α (recall that the Airy law corresponds with a particular stable law of index $3/2$) is again an Airy law with computable parameters. In order to illustrate this step, we prove a result of independent interest: given a random planar map with m edges, the size of the largest 3-component is Airy distributed with expected value $n/9$.

2. We also need to analyze the number of edges in the largest block L of a connected graph. The number of vertices of L is Airy distributed with known parameters (recall theorem 2.21). On the other hand, the number of edges in 2-connected graphs with N vertices is asymptotically normally distributed with expected value $\kappa_2 N$ (see Theorem 2.11). Thus we have to study a parameter normally distributed (number of edges) within the largest block, whose size (number of vertices) follows an Airy law. We show that the composition of these two limit laws gives rise to an Airy law for the number of edges in the largest block, again with computable parameters.

3. The analysis of the largest block in random connected graphs is in terms of the number of vertices, but the analysis for the largest 3-connected component of a 2-connected graph is necessarily in terms of the number of edges. Thus we need a way to relate both models. This is done through a technical lemma that shows that two probability distributions on 2-connected graphs with m edges are asymptotically equivalent. This is the content of Lemma 2.27.

Our main result is the following. We state it for planar graphs, since this is the most interesting case and we can give explicitly the parameters, but it holds more generally for planar-like classes of graphs.

Theorem 2.22 *Let $\boldsymbol{X}_n$ be the number of vertices in the largest 3-connected component of a random connected planar graph with n vertices. Then*

$$\mathbf{p}\left(\left\{\boldsymbol{X}_n = \alpha_2 n + x n^{2/3}\right\}\right) \sim n^{-2/3} c_2 g(c_2 x),$$

where $\alpha_2 \approx 0.7346$ and $c_2 \approx 3.14596$ are computable constants. Additionally, the number of edges in the largest 3-connected component of a random connected planar graph with n vertices also follows asymptotically an Airy law with parameters $\alpha_3 \approx 1.7921$ and $c_3 \approx 1.28956$.

The rest of the section is devoted to the proof of the theorem. The next three subsections address the technical points discussed above.

2.6.1 Largest 3-connected component in random planar maps

Recall that a planar map (we say just a map) is a connected planar graph together with a specific embedding in the plane. The size of largest k-components in several families of maps was thoroughly studied in [3]. Denote by $M(z)$, $B(z)$ and $C(z)$ the ordinary GFs associated to maps, 2-connected maps and 3-connected maps, respectively; in all cases, z marks edges. Let $\mathbf{L}_n$ be the random variable, defined over the set of maps with n edges, equal to the size of the largest 2-connected component. Let $\mathbf{T}_m$ be the random variable, defined over the set of 2-connected maps with n edges, equal to the size of the largest 3-connected component.

In [3] it is shown the following result:

Theorem 2.23 *The distribution of both $\boldsymbol{L}_n$ and $\boldsymbol{T}_m$ follows asymptotically an Airy law, namely*

$$\begin{aligned}\mathbf{p}\left(\left\{\boldsymbol{L}_n = a_1 n + x n^{2/3}\right\}\right) &\sim n^{-2/3} c_1 g(c_1 x),\\ \mathbf{p}\left(\left\{\boldsymbol{T}_m = a_2 m + y m^{2/3}\right\}\right) &\sim m^{-2/3} c_2 g(c_2 y),\end{aligned}$$

where $g(z)$ is the map Airy distribution, $a_1 = 1/3$, $c_1 = 3/4^{2/3}$, $a_2 = 1/3$, and $c_2 = 3^{4/3}/4$.

Proof. Here is a sketch of the proof. In both cases, the distribution arises from a critical composition scheme of the form $\frac{3}{2} \circ \frac{3}{2}$. The distribution of $\mathbf{L}_n$ is given by the scheme $B\left(z(1+M(z))^2\right)$, which reflects the fact that a map is obtained by gluing a map at each corner of a 2-connected map. In the second case, the result is obtained from the composition scheme $C\left(B(z)/z - 2\right)$, which reflects the fact that a 2-connected map is obtained by replacing each edge of a 3-connected map by a non-trivial 2-connected map (to complete the picture one must take also into account series and parallel compositions, but these play no role in the analysis of the largest 3-connected component, see [49] for details). □

Let $\mathbf{X}_n$ be the random variable equals to the size of the largest 3-connected component in maps of n vertices. In order to get a a limit law for $\mathbf{X}_n$, we need a more detailed study of stable laws. In particular, Airy laws are particular examples of stable laws of index 3/2. Our main reference is the forthcoming book [40]. The result we need is Proposition 1.17, which appears in [40, Section 1.6]. We rephrase it here in a form convenient for us.

Proposition 2.24 *Let $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ be independent Airy distributions, with density probability functions $c_1 g(c_1 x)$ and $c_2 g(c_2 x)$. Then $\boldsymbol{Y}_1 + \boldsymbol{Y}_2$ follows an Airy distribution with density probability function $cg(cx)$, with $c = \left(c_1^{-3/2} + c_2^{-3/2}\right)^{-2/3}$.*

Proof. We use the notation of [40]. A stable law is characterized by its *stability factor* $\alpha \in [0,2)$, its *skewness* $\beta \in [-1,1]$, its *factor scale* $\gamma > 0$, and its *location parameter* $\delta \in \mathbb{R}$. A stable random variable with this parameters is written in the form $S(\alpha,\beta,\gamma,\delta;1)$ (the constant 1 refers to the type of the parameterization; we only deal with this type). Proposition 1.17 in [40] states that if $S_1 = S(\alpha,\beta_1,\gamma_1,\delta_1;1)$ and $S_2 = S(\alpha,\beta_2,\gamma_2,\delta_2;1)$ are independent random variables, then $S_1 + S_2 = S(\alpha,\beta,\gamma,\delta,1)$, with

$$\beta = \frac{\beta_1\gamma_1^\alpha + \beta_2\gamma_2^\alpha}{\gamma_1^\alpha + \gamma_2^\alpha},\ \gamma^\alpha = \gamma_1^\alpha + \gamma_2^\alpha,\ \delta = \delta_1 + \delta_2. \tag{2.16}$$

Let us identify the Airy distribution with density probability function $cg(cx)$ within the family of stable laws as defined. By definition, the stability factor is equal to 3/2. Additionally, $\beta = -1$: this is the unique value that makes that a stable law decreases exponentially fast (see Section 1.5 of [40]). The value of the location parameter δ coincides with the expectation of the random variable, hence $\delta = 0$ (see Proposition 1.13). Finally, the factor scale can be written in the form γ_0/c, for a suitable value of γ_0, the one which corresponds with the normalized Airy distribution with density $g(x)$. Since $\mathbf{Y}_1 = S(3/2,-1,\gamma_0/c_1,0;1)$ and $\mathbf{Y}_2 = S(3/2,-1,\gamma_0/c_2,0;1)$, the result follows from (2.16). □

Theorem 2.25 *The size $\boldsymbol{X}_n$ of the largest 3-connected component in a random map with n edges follows asymptotically an Airy law of the form*

$$\mathbf{p}\left(\left\{\boldsymbol{X}_n = an + zn^{2/3}\right\}\right) \sim n^{-2/3} cg(cz),$$

where $g(z)$ is the Airy distribution and

$$a = a_1 a_2 = 1/9, \qquad c = \left(\left(\frac{c_1}{a_2} \right)^{-3/2} + c_2^{-3/2} a_1 \right)^{-2/3} \approx 1.71707.$$

Proof. Let us estimate $n^{2/3}\mathbf{p}\left(\left\{\mathbf{X}_n = an + zn^{2/3}\right\}\right)$ for large n. Considering the possible values size of the largest 2-connected component, we obtain

$$n^{2/3}\mathbf{p}\left(\left\{\mathbf{X}_n = an + zn^{2/3}\right\}\right) = n^{2/3} \sum_{m=1}^{\infty} \mathbf{p}\left(\{\mathbf{L}_n = m\}\right) \mathbf{p}\left(\left\{\mathbf{T}_m = an + zn^{2/3}\right\}\right).$$

In the previous equation we have used the fact that the largest 2-connected component is distributed uniformly among all 2-connected maps with the same number of edges; this is because the number of ways a 2-connected map M can be completed to a map of given size depends only on the size of M.

Notice that $\mathbf{X}_n$ and $\mathbf{T}_m$ are *integer* random variables, hence the previous equation should be written in fact as

$$n^{2/3}\mathbf{p}\left(\left\{\mathbf{X}_n = \lfloor an + zn^{2/3} \rfloor\right\}\right) = n^{2/3} \sum_{m=1}^{\infty} \mathbf{p}\left(\{\mathbf{L}_n = m\}\right) \mathbf{p}\left(\left\{\mathbf{T}_m = \lfloor an + zn^{2/3} \rfloor\right\}\right).$$

Let us write $m = a_1 n + xn^{2/3}$. Then $an + zn^{2/3} = a_2 m + ym^{2/3} + o\left(m^{2/3}\right)$, where $y = a_1^{-2/3}(z - a_2 x)$. Observe that when we vary m in one unit, we vary x in $n^{-2/3}$ units. Let $x_0 = (1 - a_1 n)n^{-2/3}$, so that $a_1 n + x_0 n^{2/3} = 1$ is the initial term in the sum. The previous sum can be written in the form

$$n^{2/3} \sum_{x = x_0 + \ell n^{-2/3}} \mathbf{p}\left(\left\{\mathbf{L}_n = a_1 n + xn^{2/3}\right\}\right) \mathbf{p}\left(\left\{\mathbf{T}_m = a_2 m + \alpha_1^{-2/3}(z - a_2 x)m^{2/3}\right\}\right).$$

where the sum is for all values $\ell \geq 0$. From Theorem 2.23 it follows that

$$\begin{aligned} & n^{2/3} \sum_{x = x_0 + \ell n^{-2/3}} \mathbf{p}\left(\left\{\mathbf{L}_n = a_1 n + xn^{2/3}\right\}\right) \mathbf{p}\left(\left\{\mathbf{T}_m = a_2 m + \alpha_1^{-2/3}(z - a_2 x)m^{2/3}\right\}\right) \\ \sim \ & n^{2/3} \sum_{x = x_0 + \ell n^{-2/3}} n^{-2/3} c_1 g(c_1 x)\ m^{-2/3} c_2 g\left(c_2 a_1^{-2/3}(z - a_2 x)\right) \\ \sim \ & \frac{1}{n^{2/3}} \sum_{x = x_0 + \ell n^{-2/3}} c_1 g(c_1 x)\ c_2 a_1^{-2/3} g\left(c_2 a_1^{-2/3}(z - a_2 x)\right). \end{aligned}$$

In the last equality we have used that $m^{-2/3} = (a_1 n)^{-2/3}(1 + o(1))$. Now we approximate by an integral:

$$\begin{aligned} & n^{-2/3} \sum_{x = x_0 + \ell n^{-2/3}} c_1 g(c_1 x)\ c_2 a_1^{-2/3} g\left(c_2 a_1^{-2/3}(z - a_2 x)\right) \\ \sim \ & \int_{-\infty}^{\infty} c_1 g(c_1 x)\ c_2 a_1^{-2/3} g\left(c_2 a_1^{-2/3}(z - a_2 x)\right) dx \end{aligned}$$

The previous estimate holds uniformly for x in a bounded interval. Now we set $a_2 x = u$, and with this change of variables we get

$$\int_{-\infty}^{\infty} \frac{c_1}{a_2} g\left(\frac{c_1}{a_2} u\right)\ c_2 a_1^{-2/3} g\left(c_2 a_1^{-2/3}(z - u)\right) du.$$

This convolution can be interpreted as a sum of stable laws with parameter $3/2$ in the following way. Let $\mathbf{Y}_1$ and $\mathbf{Y}_2$ be independent random variables with densities $\frac{c_1}{a_2} g\left(\frac{c_1}{a_2} u\right)$ and $c_2 a_1^{-2/3} g\left(c_2 a_1^{-2/3} u\right)$, respectively. Then, the previous integral is precisely $\mathbf{p}\left(\{\mathbf{Y}_1 + \mathbf{Y}_2 = z\}\right)$, and the result follows from Proposition 2.24. $\square$

Remark. The previous theorem can be obtained, alternatively, using the machinery developed in [3]. The two composition schemes $B(zM(z)^2)$ and $C(B(z)/z-2)$ can be composed algebraically into a single composition scheme $C(B(zM(z)^2)/z-2$. This is again a critical scheme with exponents 3/2 and an Airy law follows from the general scheme in [3]. The parameters can be computed using the singular expansions of $M(z)$, $B(z)$, $C(z)$ at their dominant singularities which are, respectively, equal to 1/12, 4/27 and 1/4. We have performed the corresponding computations in complete agreement with values obtained in Theorem 2.25. We have chosen the present proof since the same ideas are used later in the case of graphs, where no algebraic composition seems available.

2.6.2 Number of edges in the largest block of a connected graph

As discussed above, we have a limit Airy law $\mathbf{X}_n$ for the number of vertices in the largest block L in a random connected planar graph. In order to analyze the largest 3-connected component of L, we need to express $\mathbf{X}_n$ in terms of the number of edges. This amounts to combine the limit Airy law with a normal limit law, leading to slightly modified Airy law. The precise result is the following.

Theorem 2.26 *Let $\boldsymbol{Z}_n$ be the number of* edges *in the largest block of a random connected planar graph with n vertices. Then*

$$\mathbf{p}\left(\left\{\boldsymbol{Z}_n = \kappa_2\alpha n + zn^{2/3}\right\}\right) \sim n^{-2/3}\frac{c}{\kappa_2}g\left(\frac{c}{\kappa_2}z\right),$$

here α and c are as in Theorem 2.21, and $\kappa_2 \approx 2.26288$ is the constant for the expected number of edges in random 2-connected planar graphs, as in Theorem 2.11.

Proof. Let $\mathbf{X}_n$ be, as in Theorem 2.21, the number of vertices in the largest block. In addition, let $\mathbf{Y}_N$ be the number of edges in a random 2-connected planar graph with N vertices. Then

$$\mathbf{p}\left(\left\{\mathbf{Z}_n = \kappa_2\alpha n + zn^{2/3}\right\}\right) = \sum_{x=x_0+\ell n^{-2/3}} \mathbf{p}\left(\left\{\mathbf{X}_n = \alpha n + xn^{2/3}\right\}\right)\mathbf{p}\left(\left\{\mathbf{Y}_{\alpha n + xn^{2/3}} = \kappa_2\alpha n + zn^{2/3}\right\}\right), \tag{2.17}$$

with the same convention for the index of summation as in the previous section.

Since $\mathbf{Y}_N$ is asymptotically normal (Theorem 2.11)

$$\mathbf{p}\left(\left\{\mathbf{Y}_N = \kappa_2 N + yN^{1/2}\right\}\right) \sim N^{-1/2}h(y),$$

here $h(y)$ is the density of normal law suitably scaled. If we take $N = \alpha n + xn^{2/3}$, then

$$\kappa_2 N = \kappa_2\alpha N + \kappa_2 xn^{2/3}.$$

As a consequence, the significant terms in the sum in (2.17) are concentrated around $\alpha n + (z/\kappa_2)n^{2/3}$, within a window of size $N^{1/2} = \Theta(n^{1/2})$. Thus we can conclude that

$$\mathbf{p}\left(\left\{\mathbf{Z}_n = \kappa_2\alpha n + zn^{2/3}\right\}\right) \sim \frac{1}{\kappa_2}\mathbf{p}\left(\left\{\mathbf{X}_n = \lfloor \alpha n + (z/\kappa_2)n^{2/3}\rfloor\right\}\right) \sim n^{-2/3}\frac{c}{\kappa_2}g\left(\frac{c}{\kappa_2}z\right),$$

where c is the constant in Theorem 2.21. The factor $\frac{1}{\kappa_2}$ in the middle arises since in $\lfloor \kappa_2\alpha n + zn^{2/3}\rfloor$ we have steps of length $n^{-2/3}$, whereas in $\lfloor \alpha n + (z/\kappa_2)n^{2/3}\rfloor$ they are of length $n^{-2/3}/\kappa_2$. □

2.6.3 Probability distributions for 2-connected graphs

In this section we study several probability distributions defined on the set of 2-connected graphs of m edges. The first distribution $\mathbf{X}_n^m$ (in fact, a family of probability distributions, one for each n)

models the appearance of largest blocks with m edges in random connected graphs with n vertices. The second one is a weighted distribution where each 2-connected graph with m edges receives a weight according to the number of vertices, and for which it is easy to obtain an Airy law for the size of the largest 3-connected component. We show that these two distributions are asymptotically equivalent in a suitable range. In particular, it follows that the Airy law of the latter distribution also occurs in the former one. We start by defining precisely both distributions. We use capital letters like $\mathbf{X}_n^m$ and $\mathbf{Y}_m$ to denote random variables whose output is a graph in the corresponding universe of graphs, so that our distributions are associated to these random variables. We find this convention more transparent than defining the associated probability measures.

Let n, m be fixed numbers, and let $\mathcal{C}_n^m$ denote the set of connected graphs on n vertices such that their largest block L has m edges. The probability distribution associated to $\mathbf{X}_n^m$ is the distribution of L in a graph of $\mathcal{C}_n^m$ chosen uniformly at random. That is, if B is a 2-connected graph with m edges, and $\mathcal{C}_n^B \subseteq \mathcal{C}_n^m$ denotes the set of connected graphs that have $L = B$ as the largest block, then

$$\mathbf{p}\left(\{\mathbf{X}_n^m = B\}\right) = \frac{|\mathcal{C}_n^B|}{|\mathcal{C}_n^m|}.$$

Let m be a fixed number. The second probability distribution $\mathbf{Y}_m$ assigns to a 2-connected graph B of m edges and k vertices the probability

$$\mathbf{p}\left(\{\mathbf{Y}_m = B\}\right) = \frac{R^k}{k!} \frac{1}{[y^m]B(R, y)}, \tag{2.18}$$

where R is the radius of convergence of the exponential generating function $B(x) = B(x, 1)$ enumerating 2-connected graphs. It is clear that $[y^m]B(R, y)$ is the right normalization factor.

Now we state precisely what we mean when we say that these two distributions are asymptotically equivalent in a suitable range. In what follows, α and κ_2 are the multiplicative constants of the expected size of the largest block and the expected number of edges in a random connected graph (see Theorems 2.21 and 2.11).

Lemma 2.27 *Fix positive values $\bar{y}, \bar{z} \in \mathbb{R}^+$. For fixed m, let I_n and I_k denote the intervals*

$$\begin{aligned} I_k &= \left[\frac{1}{\kappa_2} m - \bar{z} m^{1/2}, \frac{1}{\kappa_2} m + \bar{z} m^{1/2}\right], \\ I_n &= \left[\frac{1}{\alpha\kappa_2} m - \bar{y} m^{2/3}, \frac{1}{\alpha\kappa_2} m + \bar{y} m^{2/3}\right]. \end{aligned}$$

Then the probability distributions $\mathbf{Y}_m$ and $\mathbf{X}_n^m$, for $n \in I_n$, are asymptotically equal on graphs with $k \in I_k$ vertices, with uniform convergence for both k and n. That is, there exists a function $\epsilon(m)$ with $\lim_{m\to\infty} \epsilon(m) = 0$ such that, for every 2-connected graph B with m edges and $k \in I_k$ vertices, and for every $n \in I_n$, it holds that

$$\left|\frac{\mathbf{p}\left(\{\mathbf{X}_n^m = B\}\right)}{\mathbf{p}\left(\{\mathbf{Y}_m = B\}\right)} - 1\right| < \epsilon(m).$$

Proof. Fix $y \in [-\bar{y}, \bar{y}]$, and let $n = \lfloor \frac{1}{\alpha\kappa_2} m + y m^{2/3} \rfloor \in I_n$. First, we prove that $\mathbf{X}_n^m$ and $\mathbf{Y}_m$ are concentrated on graphs with $k = \frac{1}{\kappa_2} m + O(m^{1/2})$ vertices, that is,

$$\mathbf{p}\left(\left\{\frac{1}{\kappa_2} m - z m^{1/2} \leq |V(\mathbf{X}_n^m)| \leq \frac{1}{\kappa_2} m + z m^{1/2}\right\}\right)$$

goes to 1 when $z, m \to \infty$, and the same is true for $\mathbf{Y}_m$. Then, we show that $\mathbf{X}_n^m$ and $\mathbf{Y}_m$ are asymptotically *proportional* for graphs on $k \in I_k$ vertices. A direct consequence of both facts is that $\mathbf{X}_n^m$ and $\mathbf{Y}_m$ are asymptotically *equal* in I_k, since the previous results are valid for arbitrarily large $\bar{z}$.

We start by considering the probability distribution $\mathbf{Y}_m$. If we add Expression (2.18) over all the $b_{k,m}$ 2-connected graphs with k vertices and m edges, we get

$$\mathbf{p}\left(\{|V(\mathbf{Y}_m)| = k\}\right) = b_{k,m}\frac{R^k}{k!}\frac{1}{[y^m]B(R,y)}.$$

On the one hand, the value $[y^m]B(R,y)$ is a constant that does not depend on k. On the other hand, since the numbers $b_{k,m}$ satisfy a local limit theorem (the proof is the same as in [26]), it follows that the numbers $b_{k,m}R^k/k!$ follow a normal distribution concentrated at $k = 1/\kappa_2$ on a scale $m^{1/2}$, as desired.

We show the same result for $\mathbf{X}_n^m$. Let B be a 2-connected graph with k vertices and m edges. We write the probability that a graph drawn according to $\mathbf{X}_n^m$ is B as a conditional probability on the largest block L_n of a random connected graph of n vertices. In what follows, $v\left(L_n\right)$ and $e\left(L_n\right)$ denote, respectively, the number of vertices and edges of L_n.

$$\mathbf{p}\left(\{\mathbf{X}_n^m = B\}\right) = \mathbf{p}\left(\{L_n = B \mid e(L_n) = m\}\right) = \frac{\mathbf{p}\left(\{L_n = B,\ e(L_n) = m\}\right)}{\mathbf{p}\left(\{e(L_n) = m\}\right)} = \frac{\mathbf{p}\left(\{L_n = B\}\right)}{\mathbf{p}\left(\{e(L_n) = m\}\right)}.$$

Note that in the last equality we drop the condition $e\left(L_n\right) = m$ because it is subsumed by $L_n = B$. The probability that the largest block L_n is B is the same for all 2-connected graphs on k vertices. Hence, if b_k denotes the number of 2-connected graphs on k vertices, we have

$$\mathbf{p}\left(\{\mathbf{X}_n^m = B\}\right) = \frac{1}{b_k}\frac{\mathbf{p}\left(\{v(L_n) = k\}\right)}{\mathbf{p}\left(\{e(L_n) = m\}\right)},$$

If we sum over all the $b_{k,m}$ 2-connected graphs B with k vertices and m edges, we finally get the probability of $\mathbf{X}_n^m$ having k vertices,

$$\mathbf{p}\left(\{|V(\mathbf{X}_n^m)| = k\}\right) = \frac{b_{k,m}}{b_k}\frac{\mathbf{p}\left(\{v(L_n) = k\}\right)}{\mathbf{p}\left(\{e(L_n) = m\}\right)}.$$

For fixed n, m, the numbers $\mathbf{p}\left(\{V(L_n) = k\}\right)$ follow an Airy distribution of scale $n^{2/3}$ concentrated at $k_1 = \alpha n$ (see Section 2.5.2), and the numbers $b_{k,m}/b_k$ are normally distributed around $k_2 = m/\kappa_2$ on a scale $m^{1/2}$. The choice of n makes k_1 and k_2 coincide but for a lower-order term $O(m^{2/3})$; hence, it follows that $\mathbf{p}\left(\{|V(\mathbf{X}_n^m)| = k\}\right)$ is concentrated at $k_2 = m/\kappa_2$ on a scale $m^{1/2}$, as desired.

Now that we have established concentration for both probability distributions, we just need to show that they are asymptotically proportional in the range $k = m/\kappa_2 + O(m^{1/2})$. This is easy to establish by considering asymptotic estimates. Indeed, we have

$$\mathbf{p}\left(\{\mathbf{X}_n^m = B\}\right) = \frac{1}{b_k}\frac{\mathbf{p}\left(\{v(L_n) = k\}\right)}{\mathbf{p}\left(\{e(L_n) = m\}\right)},$$

and since $\mathbf{p}\left(\{v(L_n) = k\}\right)$ is Airy distributed in the range $k = m/\kappa_2 + O(m^{2/3})$ and we have the asymptotic estimate $b_k \sim b \cdot k^{-7/2}R^{-k}k!$, it follows that

$$\mathbf{p}\left(\{\mathbf{X}_n^m = B\}\right) \sim b^{-1} \cdot k^{7/2}\frac{R^k}{k!}\frac{g(x)}{\mathbf{p}\left(\{e(L_n) = m\}\right)},$$

where x is defined as $(k - \alpha n)n^{-2/3}$ and $g(x)$ is the Airy distribution of the appropriate scale factor. Let us compare it with the exact expression for the probability distribution $\mathbf{Y}_m$, that is,

$$\mathbf{p}\left(\{\mathbf{Y}_m = B\}\right) = \frac{R^k}{k!}\frac{1}{[y^m]B(R,y)}.$$

Clearly, both expressions coincide in the high order terms R^k and $1/k!$. The remaining terms are either constants like b, $\mathbf{p}\left(\{e(L_n) = m\}\right)$ and $[y^m]B(R,y)$, or expressions that are asymptotically constant in the range of interest. This is the case for $k^{7/2}$, which is asymptotically equal to

$((1/\kappa_2)m)^{7/2}$. And also for $g(x)$, which is asymptotically equal to $g(y(\alpha\kappa_2)^{2/3})$, since $x = (k - \alpha n)n^{-2/3}$ and $n = \frac{1}{\alpha k_2}m + ym^{2/3}$ implies that

$$\begin{aligned} x &= \left(\frac{1}{k_2}m + O(m^{1/2}) - \frac{\alpha}{\alpha k_2}m + ym^{2/3}\right) n^{-2/3} \\ &= \left(ym^{2/3}\right)\left(\frac{m}{\alpha\kappa_2}\right)^{-2/3} + o(1) \\ &= y(\alpha\kappa_2)^{2/3} + o(1) \end{aligned}$$

in the given range. Hence, both distributions are asymptotically proportional in the given range.

Thus, we have shown the result when n is linked to m by $n = \frac{1}{\alpha\kappa_2}m + ym^{2/3}$, for any y. Clearly, uniformity holds when y is restricted to a compact set of $\mathbb{R}$, like $[-\bar{y}, \bar{y}]$. $\square$

2.6.4 Proof of the main result

In order to prove Theorem 2.22, we have to concatenate two Airy laws. The first one is the number of edges in the largest block, given by Theorem 2.26. The second is the number of edges in the largest 3-connected component of a random 2-connected planar graph with a given number of edges. This is a gain an Airy law produced by the composition scheme $T_z(x, D(x, y))$, which encodes the combinatorial operation of substituting each edge of a 3-connected graph by a network (which is essentially a 2-connected graph rooted at an edge). However, this scheme is relative to the variable y marking edges. In order to have a legal composition scheme we need to take a fixed value of x. The right value is $x = R$, as shown by Lemma 2.27. Indeed, taking $x = R$ amounts to weight a 2-connected graph G with m edges with $R^k/k!$, where k is the number of vertices in G. Thus the relevant composition scheme is precisely $T_z(R, uD(R, y))$, where u marks the size of the 3-connected core. Formally, we can write it as the scheme

$$C(uH(y)), \qquad H(y) = D(R, y), \qquad C(y) = T_z(R, y).$$

The composition scheme $T_z(R, D(R, y))$ is critical with exponents 3/2, and an Airy law appears. In order to compute the parameters we need the expansion of $D(R, y)$ at the dominant singularity $y = 1$, which is of the form

$$D(R, y) = \widetilde{D}_0 + \widetilde{D}_2 Y^2 + \widetilde{D}_3 Y^3 + O(Y^4), \tag{2.19}$$

and $Y = \sqrt{1-y}$. The different $\widetilde{D}_i$ can be obtained in the same way as in Proposition 2.7.

Proposition 2.28 *Let $\boldsymbol{W}_m$ be the number of edges in the largest 3-connected component of a 2-connected planar graph with m edges, weighted with $R^k/k!$, where k is the number of vertices. Then*

$$\mathbf{p}\left(\left\{ \boldsymbol{W}_m = \beta n + zn^{2/3} \right\}\right) \sim n^{-2/3} c_2 g\left(c_2 z\right),$$

where $\beta = -\widetilde{D}_0/\widetilde{D}_2 \approx 0.82513$ and $c_2 = -\widetilde{D}_2/\widetilde{D}_0 \left(-\widetilde{D}_2/3\widetilde{D}_3\right)^{2/3} \approx 2.16648$, and the $\widetilde{D}_i$ are as in Equation (2.19).

Proof. The proof is a direct application of the methods in Theorem 2.21. $\square$

Proof of Theorem 2.22. Recall that $\mathbf{X}_n$ is the number of vertices in the largest 3-connected component of a random connected planar graph with n vertices. This variable arises as the composition of two random variables we have already studied. First we consider $\mathbf{Z}_n$ as in Theorem 2.26, which is the number of edges in the largest block, and then $\mathbf{W}_m$ as in Proposition 2.28.

The main parameter turns out to be $\alpha_2 = \mu\beta(\kappa_2\alpha)$, where

1. α is for the expected number of vertices in the largest block;
2. κ_2 is for the expected number of edges in 2-connected graphs;
3. β is for the expected number of edges in the largest 3-connected component;
4. μ is for the expected number of vertices in 3-connected graphs weighted according to $R^k/k!$ if k is the number of vertices.

The constants in 1. and 3. correspond to Airy laws, and the constants in 2. and 4. to normal laws.

Let $\mathbf{Y}_n$ be the number of edges in the largest 3-connected component of a random connected planar graph with n vertices (observe that our main random variable $\mathbf{X}_n$ is linked directly to $\mathbf{Y}_n$ after extracting a parameter normally distributed like the number of vertices). Then

$$\mathbf{p}\left(\left\{\mathbf{Y}_n = \beta\kappa_2\alpha n + zn^{2/3}\right\}\right) = \sum_{m=1}^{\infty} \mathbf{p}\left(\{\mathbf{Z}_n = m\}\right) \mathbf{p}\left(\left\{\mathbf{W}_m = \beta\kappa_2\alpha n + zn^{2/3}\right\}\right).$$

This convolution can be analyzed in exactly the same way as in the proof of Theorem 2.25, giving rise to a limit Airy law with the parameters as claimed.

Finally, in order to go from $\mathbf{Y}_n$ to $\mathbf{X}_n$ we need only to multiply the main parameter by μ and adjust the scale factor. To compute μ we need the dominant singularity $\tau(x)$ of the generating function $T(x,z)$ of 3-connected planar graphs, for a given value of x (see Section 6 in [15]). Then

$$\mu = -R\tau'(R)/\tau(R).$$

Given that the inverse function $r(z)$ is explicit (see Equation (25) in [15]), the computation is straightforward.

2.7 Minor-closed classes

In this section we apply the machinery developed so far to analyse families of graphs closed under minors. A class of graphs $\mathcal{G}$ is minor-closed if whenever a graph is in $\mathcal{G}$ all its minors are also in $\mathcal{G}$. Given a minor-closed class $\mathcal{G}$, a graph H is an excluded minor for $\mathcal{G}$ if H is not in $\mathcal{G}$ but every proper minor is in $\mathcal{G}$. It is an easy fact that a graph is in $\mathcal{G}$ if and only if it does not contain as a minor any of the excluded minors from $\mathcal{G}$. According to the fundamental theorem of Robertson and Seymour, for every minor-closed class the number of excluded minors is finite [44]. We use the notation $\mathcal{G} = \mathrm{Ex}(H_1, \cdots, H_k)$ if $H_1, \ldots, H_k$ are the excluded minors of $\mathcal{G}$. If all the H_i are 3-connected, then $\mathrm{Ex}(H_1, \cdots, H_k)$ is a closed family. This is because if none of the 3-connected components of a graph G contains a forbidden, the same is true for G itself.

In order to apply our results we must know which connected graphs are in the set $\mathrm{Ex}(H_1, \cdots, H_k)$. There are several results in the literature of this kind. The easiest one is $\mathrm{Ex}(K_4)$, which is the class of series-parallel graphs. Since a graph in this class always contains a vertex of degree at most two, there are no 3-connected graphs. Table 2.1 contains several such results, due to Wagner, Halin and others (see Chapter X in [13]). The proofs make systematic use of Tutte's Wheels Theorem (consult Chapter 3 of [14], for instance): a 3-connected graph can be reduced to a wheel by a sequence of deletions and contractions of edges, while keeping it 3-connected. The 3-connected graphs when excluding W_5 and the triangular prism take longer to describe. For $K_3 \times K_2$ they are: $K_5, K_5^-, \{W_n\}_{n\geq 3}$, and the family G_Δ of graphs obtained from $K_{3,n}$ by adding any number of edges to the part of the bipartition having 3 vertices. For W_5 they are: K_4, K_5, the family G_Δ, the graphs of the octahedron and the cube Q, the graph obtained from Q by contracting one edge, the graph L obtained from $K_{3,3}$ by adding two edge in one of the parts of the bipartition, plus all the 3-connected subgraphs of the former list. Care is needed here for checking that all 3-connected graphs are included and for counting how many labellings each graph has.

Ex. minors	3-connected graphs	Generating function $T(x,z)$
K_4	$\emptyset$	0
W_4	K_4	$z^6x^4/4!$
K_5-e	$K_3, K_{3,3}, K_3\times K_2, \{W_n\}_{n\geq 3}$	$\frac{70}{6!}z^9x^6 - \frac{1}{2}x(\log(1-z^2x)+2z^2x+z^4x^2)$
$K_5, K_{3,3}$	Planar 3-connected	$T_p(x,z)$
$K_{3,3}$	Planar 3-connected, K_5	$T_p(x,z)+z^{10}x^5/5!$
$K_{3,3}^+$	Planar 3-connected, $K_5, K_{3,3}$	$T_p(x,z)+z^{10}x^5/5!+10z^9x^6/6!$

Table 2.1 Classes of graphs defined from one excluded minor. $T_p(x,z)$ is the GF of planar 3-connected graphs.

Once we have the full collection of 3-connected graphs, we have $T(x,z)$ at our disposal. For the family of wheels we have a logarithmic term (see the previous table) and for the family G_Δ it is a simple expression involving $\exp(z^3x)$. We can then apply the machinery developed in this chapter and compute the generating functions $B(x,y)$ and $C(x,y)$. For the last three entries in Table 2.1, the main problem is computing $B(x,y)$ and this was done in [26] and [24]; these correspond to the planar-like case. In the remaining cases $T(x,z)$ is either analytic or has a simple singularity coming from the term $\log(1-xz^2)$, and they correspond to the series-parallel-like case.

In Table 2.2 we present the fundamental constants for the classes under study. For a given class $\mathcal{G}$ they are: the growth constants ρ^{-1} of graphs in $\mathcal{G}$; the growth constant R^{-1} of 2-connected graphs in $\mathcal{G}$; the asymptotic probability p that a random graph in $\mathcal{G}$ is connected; the constant κ such that κn is the asymptotic expected number of edges for graphs in $\mathcal{G}$ with n vertices; the analogous constant κ_2 for 2-connected graphs in $\mathcal{G}$; the constant β such that βn is the asymptotic expected number of blocks for graphs in $\mathcal{G}$ with n vertices; and the constant δ such that δn is the asymptotic expected number of cut vertices for graphs in $\mathcal{G}$ with n vertices. The values in Table 2.2 have been computed with `Maple` using the results in sections 2.3 and 2.4.

Class	ρ^{-1}	R^{-1}	κ	κ_2	β	δ	p
Ex(K_4)	9.0733	7.8123	1.61673	1.71891	0.149374	0.138753	0.88904
Ex(W_4)	11.5437	10.3712	1.76427	1.85432	0.107065	0.101533	0.91305
Ex(W_5)	14.6667	13.5508	1.90239	1.97981	0.0791307	0.0760808	0.93167
Ex(K_5^-)	15.6471	14.5275	1.88351	1.95360	0.0742327	0.0715444	0.93597
Ex$(K_3\times K_2)$	16.2404	15.1284	1.92832	1.9989	0.0709204	0.0684639	0.93832
Planar	27.2269	26.1841	2.21327	2.2629	0.0390518	0.0382991	0.96325
Ex$(K_{3,3})$	27.2293	26.1866	2.21338	2.26299	0.0390483	0.0382957	0.963262
Ex$(K_{3,3}^+)$	27.2295	26.1867	2.21337	2.26298	0.0390481	0.0382956	0.963263

Table 2.2 Constants for a given class of graphs: ρ and R are the radius of convergence of $C(x)$ and $G(x)$, respectively; constants $\kappa, \kappa_2, \beta, \delta$ give, respectively, the first moment of the number of: edges, edges in 2-connected graphs, blocks and cut vertices; p is the probability of connectedness.

2.8 Critical phenomena

We have seen that the estimates for the number of planar graphs with μn edges have the same shape for all values $\mu \in (1,3)$. This is also the case for series-parallel graphs, where $\mu \in (1,2)$ since maximal graphs in this class have only $2n-3$ edges. It is natural to ask if there are classes in which there is a *critical phenomenon*, that is, a different behaviour depending on the edge density. We have not found such phenomenon for 'natural' classes of graphs, in particular those defined in terms of forbidden minors. But we have been able to construct examples of critical phenomena by a suitable choice of the family $\mathcal{T}$ of 3-connected graphs, as we now explain.

Let $\mathcal{T}$ a family of 3-connected graphs whose function $T(x, z)$ has singularity on z of type 5/2. We have seen that the singular types of the associated functions B, C and G depend on the existence of branch-points before T becomes singular. We have obtained families of graphs for which the singular types of B, C and G depend on the particular value of y_0.

Now we have two sources for the main singularity of $B(x, y)$ for a given value of y: either (a) it comes from the singularities of $T(x, z)$; or (b) it comes from a branch point of the equation defining $D(x, y)$. For planar graphs the singularity always comes from case (a), and for series-parallel graphs always from case (b). If there is a value y_0 for which the two sources coalesce, then we get a different singular exponent depending on whether $y < y_0$ or $y > y_0$. The most important consequence in this situation is that there is a critical edge density μ_0, such that below μ_0 the largest block has linear size, and above μ_0 it has sublinear size, or conversely.

Here are some examples.

- If $\mathcal{T}$ is the family of *3-connected cubic planar graphs*, then $B(x, y)$ has singular exponent 5/2 when $y < y_0 \approx 0.07422$, and 3/2 when $y > y_0$. The corresponding critical value for the number of edges is $\mu_0 \approx 1.3172$.
- If $\mathcal{T}$ is the family of *planar triangulations (maximal planar graphs)*, then $B(x, y)$ exponent 3/2 when $y < y_0 \approx 0.4468$, and 5/2 when $y > y_0$. The corresponding critical value for the number of edges is $\mu_0 \approx 1.8755$.
- This example shows that more than one critical value may occur. This is done by adding a single dense graph to the family $\mathcal{T}$ in the last example. Let $\mathcal{T}$ be the family of *triangulations plus the exceptional graph* K_6. Then there are two critical values $y_0 \approx 0.4469$ and $y_1 \approx 108.88$, and the corresponding critical edge densities are $\mu_0 \approx 1.8756$ and $\mu_1 \approx 3.4921$. This last value is close to 7/2; this is the maximal edge density, which is approached by taking many copies of K_6 glued along a common edge. It turns out that $B(x, y)$ has exponent 3/2 when $y < y_0$, 5/2 when $y_0 < y < y_1$, and again 3/2 for $y_1 < y$.

3

Dissections of the projective plane

The results of this chapter appear in [41]. For each non-negative integer $k \geq 2$, we determine the number of ways of dissecting a polygon embedded in the projective plane into n subpolygons with $k+1$ sides each. We also solve the problem when the polygon is dissected into subpolygons of arbitrary size. In each case, the associated generating function is a rational function in the corresponding variable and the generating function of plane polygon dissections. Finally, we obtain asymptotic estimates for the number of dissections of various kinds, and determine probability limit laws for natural parameters associated to triangulations and dissections.

3.1 A problem from Stanley's book

In page 242 of Stanley's book *Enumerative combinatorics*, Volume 2 [47], one can find the following problem:

> Define a *Catalan triangulation* of the Möbius band to be an abstract simplicial complex triangulating the Möbius band that uses no interior vertices, and has vertices labelled $1, 2, \ldots, n$ in order as one traverses the boundary. Let $MB(n)$ be the number of Catalan triangulations of the Möbius band with n vertices. Show that:
>
> $$\sum_{n\geq 0} MB(n)x^n = \frac{x^2\left[(2-5x-4x^2)+(-2+x+2x^2)\sqrt{1-4x}\right]}{(1-4x)(1-4x+2x^2+(1-2x)\sqrt{1-4x})}. \qquad (3.1)$$

This problem is the starting point of this chapter. Our approach is slightly different: let $\mathbb{P}_1$ be the real projective plane, obtained by adding a cross-cap to the sphere (recall Section 1.1.2 of Chapter 1). We fix a polygon Q in $\mathbb{P}_1$, that is, a simple contractible closed curve, in which n points are labelled $1, 2, \ldots, n$ circularly. The topological space obtained by deleting from $\mathbb{P}_1$ the interior of Q (i.e., deleting the contractible 2-dimensional region defined by Q) is a surface with boundary homeomorphic to the Möbius band. This observation links our approach with the problem stated before.

By a *triangulation of a polygon in the projective plane* (or shortly, a *triangulation*) we mean a 2-cell decomposition of the outside of Q into triangles using as vertices only the n labelled points, such that two intersecting triangles meet only in a common vertex or in a common edge (i.e., a simplicial decomposition). The number of triangulations of the Möbius band was first determined by Edelman and Reiner in [16]. We reprove the same result with a different approach, using the Symbolic Method for handling generating functions [19].

There are three possible ways of representing Q and the projective plane: either Q is drawn inside the cross-cap and the edges of a triangulation are outside Q, or the cross-cap is drawn inside Q and the edges of a triangulation are inside Q. Additionally, we can use the representation using the Möbius band (see Figure 3.1 for the triangulation of a pentagon in this three versions). Throughout this chapter we stick to the second representation, that is, the non-boundary edges of a decomposition of Q are drawn inside.

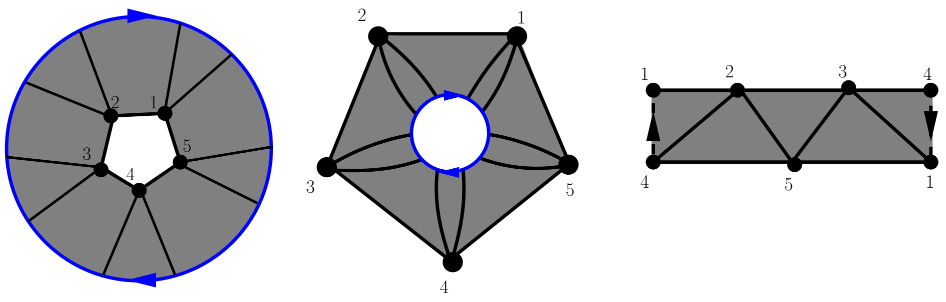


Figure 3.1 Three representations of the unique triangulation of a pentagon in the projective plane.

For a non-boundary edge we have three possibilities: either it crosses the cross-cap, or it leaves the cross-cap to the right or to the left. In principle there are many more possibilities, since an edge can reenter the cross-cap several times, but from a combinatorial point of view there are only these three cases: the proof of this fact can be found in [16] and is based on the fact that the fundamental group of the projective plane is cyclic of order two.

More generally, we are interested in decompositions of a polygon into quadrangles, pentagons, and so on, and also in unrestricted dissections. In each case we require that two cells of a decomposition intersect only at a vertex or at an edge. We believe our proof is more transparent and moreover this approach allows us to solve other related problems which appear difficult to obtain using recurrence equations as in the work of Edelman and Reiner.

The plan for this chapter is the following: in Section 3.2 we present our derivation for computing the number of triangulations, and in Section 3.3 we obtain the number of dissections into $(k+1)$-gons for each value of $k \geq 2$; the cases $k = 3$ an $k = 4$ are somehow exceptional and need a special treatment. In Section 3.4 we compute the number of unrestricted dissections in terms of the number of vertices and the number of regions. In each case our result gives a closed form for the corresponding OGF, which is always a rational function of the independent variable used and the corresponding GF of plane polygon dissection. In Section 3.5 we obtain precise asymptotic estimates for the numbers of polygon dissections of various kinds. Finally, in Section 3.6 we derive limit laws for two parameters of interest: the number of cyclic triangles (to be defined later on) in triangulations; and the number of cells in arbitrary polygon dissections. In the second case we obtain a classical normal law, whereas in the first case the limit law is the *absolute value* of a normal law.

Some work has been done related with this problem. The case of arbitrary maps is treated in [21], and it contains many interesting results. In [23] the authors solve the problem of counting triangulations of the projective plane, and also deal with the first random variable we study in Section 3.6.1.

Notice that in a triangulation the number of triangles equals the number of vertices. This follows from Euler's formula applied over the Möbius band, and and double counting incidences between edges and faces. More generally, in a dissection into $(k+1)$-gons, the number of vertices is $(k-1)n$, where n is the number of $(k+1)$-gons. Finally, in an unrestricted dissection, the number of cells is equal to the number of internal edges. In terms of generating functions, in all this chapter we use the variable x when we refer to vertices, the variable z when we count faces and u when we are dealing with a parameter.

3.2 Triangulations

In this section we rediscover a result from [16]. Our approach is similar in content, but we deal directly with the generating functions involved, thus avoiding working with recurrence relations with one and two indices. The purpose of including a new proof is to exemplify in a simple case the tools we use later.

The expression for the generating function obtained in [16] is written differently from ours (Expression (3.1)), but it can be checked they are in fact the same. The sequence $1, 14, 113, 720, \ldots$ counting triangulations is the one with index $A007817$ in the *On-Line Encyclopedia of Integer Sequences*.

Let $(\mathcal{P}, |\cdot|)$ the combinatorial class of triangulations of the projective plane, where the size measures the number of vertices in a triangulation. By the Euler relation over the projective plane, the number of faces in a triangulation of the projective plane is the same as the number of vertices of the polygon. Consequently, the size also counts the number of vertices of the triangulation.

The following theorem show the exact expression for the generating function associated to $\mathcal{P}$:

Theorem 3.1 *Let $P(n)$ be the number of triangulations of a polygon with n faces in the projective plane, and write $P(z) = \sum_{n\geq 5} P(n)z^n$ .Then*

$$\begin{aligned} P(z) &= \frac{(2 - 9z + 6z^2 + 7z^3 - 2z^4)C(z) - (2 - 7z + z^2 + 5z^3)}{z(1 - 4z)} \qquad (3.2)\\ &= z^5 + 14z^6 + 113z^7 + 720z^8 + 4033z^9 + \cdots \end{aligned}$$

where $C(z)$ is the generating function for triangulations of a disk.

Proof. Let τ be a triangulation of an n-gon in the projective plane, and let $12x$ be the unique triangle of τ that contains the side $\overline{12}$ of the polygon. In principle this triangle can appear as a very involved closed curve in the projective plane, but in fact there are only three possibilities from a combinatorial point of view, as shown in Figure 3.2; the proof that this is indeed the case is again based on the fact that the fundamental group of the projective plane is cyclic of order two.

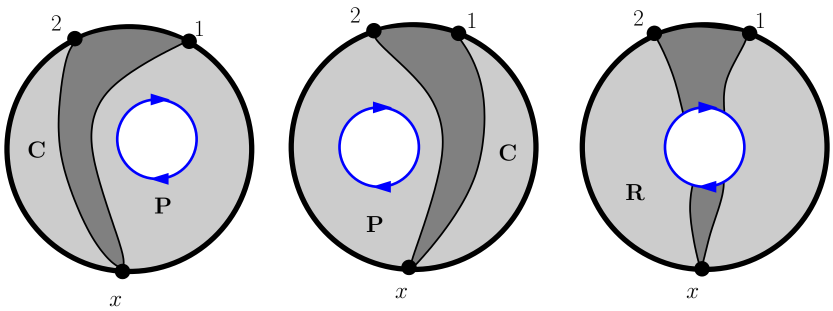


Figure 3.2 Non-equivalent combinatorial possibilities.

The previous decomposition can be specified in the following way:

$$\mathcal{P} = (\mathcal{C} \times \triangle \times \mathcal{P}) \cup (\mathcal{P} \times \triangle \times \mathcal{C}) \cup (\triangle \times \mathcal{R}),$$

where $\mathcal{C}$ corresponds with a plane region, and $\triangle$ is the triangle $12x$. Using the Symbolic Method, this combinatorial condition is translated in the following equation

$$P(z) = z\left(2C(z)P(z) + R(z)\right),$$

where $C(z)$ is the Catalan function (recall that $C(z)$ is defined in Section 1.2.1 of Chapter 1, and it satisfies the implicit equation $C(z) = 1 + zC(z)^2$), $R(z)$ is the GF associated to triangulations of the region $\mathcal{R}$ indicated in Figure 3.2, and the factor z takes account of the root polygon $12x$. Thus we have

$$P(z) = \frac{zR(z)}{1 - 2zC(z)}, \tag{3.3}$$

and it only remains to compute $R(z)$. By a topological "cut and paste" argument, it is clear that region $\mathcal{R}$ is homeomorphic to a polygon with $n+1$ vertices (if there are n vertices on region $\mathcal{R}$) in which two vertices labelled x are identified. This geometric operation is shown in Figure 3.3.

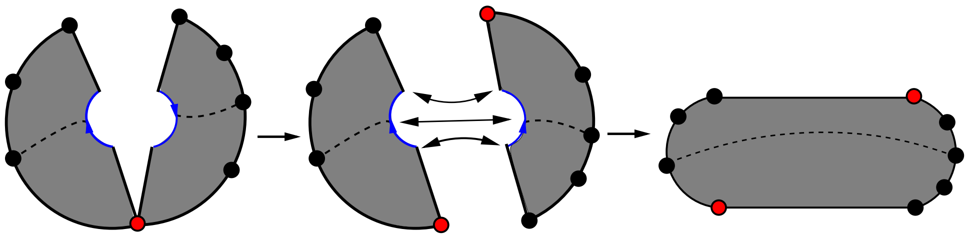


Figure 3.3 The "cut and paste" argument

Triangulations in $\mathcal{R}$ have additional restrictions: it is necessary that in the arc between vertex 1 and the non-adjacent copy of vertex x there is at least one vertex, and similarly in the arc between vertex 2 and the other copy of x (see Figure 3.4), since otherwise we cannot triangulate the polygon in a way compatible with the root triangle $12x$. It follows that region $\mathcal{R}$ must contain at least 6 vertices. Equivalently, 4 triangles are needed for triangulating it.

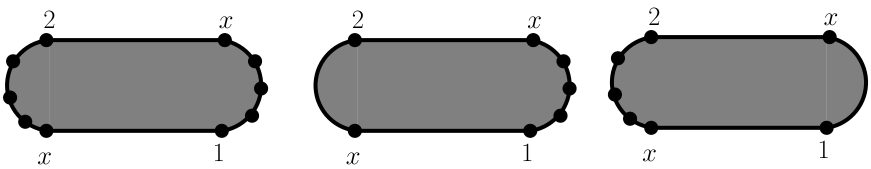


Figure 3.4 The first case can be extended to a valid triangulation; in the two other cases, any triangulation would not be compatible with the root triangle $12x$.

We must find the number of plane triangulations of an $(n+1)$-gon in which two points are identified that give rise to compatible triangulations with the root triangle $12x$ in the projective plane. To this end, we compute the total number of triangulations and subtract, using the principle of inclusion-exclusion, a complete set of forbidden configurations.

Let $C_r(z) = C(z) - \sum_{n=0}^{r-1} C(n)z^n$ be the GF of plane triangulations with at least r triangles; observe that $C_r(z)$ is just a truncation of the Catalan function $C(z)$. The total number of triangulations of region $\mathcal{R}$ is counted by $C_4(z)$ (as noticed before, we need at least four triangles), modified in order to mark the double vertex x. Another way to put this is the following: we have to split $n-3 = (n+1) - 4$ points between the arcs $\widehat{1x}$ and $\widehat{2x}$ in such a way that there is at least one point in each arc. This splitting can be done in $n-3$ different ways (the number of solutions of the equation $x+y = n-3$ with $x, y > 0$). Hence the associated generating function is $C_4^\bullet(z) - 3C_4(z)$. Recall that $C_4^\bullet(z) = zC_4'(z)$.

On the other hand, the forbidden configurations are those shown on the left of Figure 3.5: edges 12, $1x$, $2x$ and xx cannot be used, since each of them gives rise to a triangle sharing exactly two vertices with the root polygon; and no point y can be joined to both copies of x, since we would have two triangles sharing only vertices x and y. The pairwise intersections of these configurations

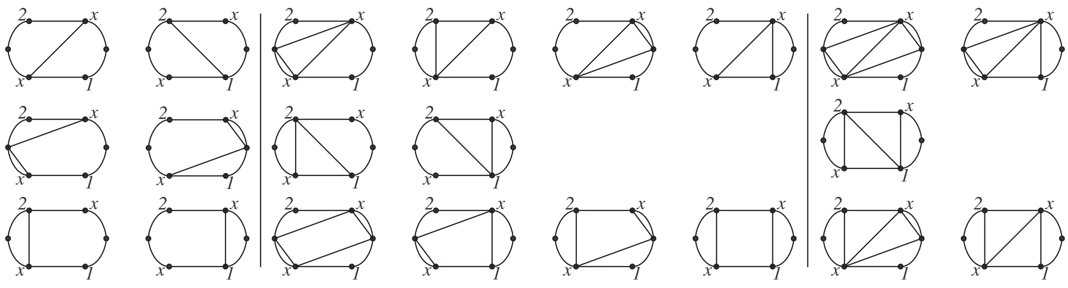


Figure 3.5 From left to right: forbidden configurations, pairwise intersections and triple-wise intersections.

are shown in the middle of the figure, and finally the triple-wise intersections are on the right of the figure.

The GFs associated to each configuration are computed easily: for instance, the presence of edge xx (first configuration) leaves us with two triangulations with at least two triangles each, hence the term $C_2(z)^2$. Similarly, when there is a triangle we add a factor z, and where there is a quadrangle we add a factor $2z^2$ (there are two ways of triangulating a quadrangle). Applying symbolically the principle of inclusion-exclusion, we arrive at

$$\begin{aligned} R(z) &= C_4^\bullet(z) - 3C_4(z) - \big(2C_2(z)^2 + 2(1+C_0(z))C_1(z)C_3(z)\big) + \\ &\quad \big(2z(2+C_0(z))C_1(z)C_2(z) + 2z^2C_1(z)^2(1+C_0(z))^2\big) - \\ &\quad \big(z^2C_1(z)^2 + z^2C_1(z)^2(1+C_0(z))^2\big)\,. \end{aligned}$$

A simple calculation, together with Equation (3.3) and the fact that $C'(z) = C(z)^2(1-2zC(z))^{-1}$, gives the result as claimed in Expression 3.2. □

3.3 Dissections into $(k+1)$-gons

In this section we study dissections of a polygon in the projective plane where all the cells are r-polygons, i.e., polygons with r edges; we call such a decomposition simply an *r-dissection*. The case of $r = 3$ has been treated in the previous section. For r-dissections the essence is the same: we find a combinatorial encoding of the r-dissection, we compute generating functions associated to dissections of some particular regions, we single out forbidden configurations, and we put everything together to obtain the desired generating function.

Let $k > 2$ be an arbitrary but fixed integer. Let $P^{\{k+1\}}(z) = \sum_{n>0} P^{\{k+1\}}(n)z^n$ be the generating function of projective $(k+1)$-dissections, $C_{k+1}(z)$ the GF of planar $(k+1)$-dissections (recall its exact definition in Section 1.2.1, Chapter 1) and $C_{k+1,s}(z) = \sum_{n\geq s} C_{k+1}(n)z^n$. Recall that $C_{k+1}(z)$ satisfies the implicit equation $C_{k+1}(z) = 1 + zC_{k+1}(z)^k$, where z marks cells in the dissection. As opposed to the case of triangulations, in a $(k+1)$-dissection the number of vertices cannot be arbitrary, and is given by the Euler relation over the surface: it must be of the form $(k-1)n$, where n is the number of cells in the dissection. It is necessary also that $n \geq 3$.

The following theorem provides the exact enumeration of these families:

Theorem 3.2 *Let $k > 4$ be fixed. Let $P_{k+1}(n)$ be the number of dissections into n $(k+1)$-gons of a polygon with $(k-1)n$ vertices in the projective plane. Denote by $P^{\{k+1\}}(z) = \sum_{n\geq 3} P_{k+1}(n)z^n$ the associated GF. Then,*

$$P^{\{k+1\}}(z) = \frac{(k-1)(C_{k+1}(z)-1)\,\alpha}{2C_{k+1}(z)^3(C_{k+1}(z) - kC_{k+1}(z) + k)^2},$$

where

$$\begin{aligned}\alpha &= \left(4k^3-12k^2+12k-4\right)C_{k+1}(z)^6+\\&\left(-14k^3+46k^2-38k+6\right)C_{k+1}(z)^5+\left(18k^3-72k^2+42k\right)C_{k+1}(z)^4+\\&\left((k^3-6k^2+5k)z+(-10k^3+68k^2-25k-4)\right)C_{k+1}(z)^3+\\&\left((-k^3+5k^2)z+(2k^3-47k^2+9k+4)\right)C_{k+1}(z)^2+\\&\left(21k^2+2k-2\right)C_{k+1}(z)-(4k^2+2k).\end{aligned}$$

For $k=3$ the expression of $P^{\{4\}}(z)$ is

$$\begin{aligned}P^{\{4\}}(z) &= \frac{128z^2-216z+32}{4-27z}C_4(z)^2+\frac{-105z^2+263z-32}{4-27z}C_4(z)+\frac{18z^2-79z}{4-27z}\\&= z^3+25z^4+348z^5+3703z^6+34240z^7+291485z^8+\cdots,\end{aligned}$$

and for $k=4$ the expression of $P^{\{5\}}(z)$ is

$$\begin{aligned}P^{\{5\}}(z) &= 3\frac{-316z^2+84z}{27-256z}C_5(z)^3+3\frac{-80z^2-1454z+162}{27-256z}C_5(z)^2+\\&\quad 3\frac{64z^2+2574z-261}{27-256z}C_5(z)+3\frac{-1267z+99}{27-256z}\\&= 12z^3+336z^4+5499z^5+73302z^6+880548z^7+9951336z^8+\cdots\end{aligned}$$

In all cases, $C_{k+1}(z)$ is the GF for decompositions of a disk into $(k+1)$-agons.

Proof. Let γ be a dissection into $(k+1)$-gons. As in the case of triangulations, we fix the $(k+1)$-polygon in γ which has the edge 12 as the root, and consider the possible ways in which this polygon crosses the cross-cap. In Figure 3.6 we show the basic cases for 4-dissections: for instance, in the first picture, the cross-cap could be instead on the left or on the right of the subpolygon; and in the third picture, the "leg" to the right of the cross-cap could be instead to the left. The combinatorial definition of region $\mathcal{R}$ is the same as in the case of triangulations. Region $\mathcal{S}$ is similar but now there are no repeated points.

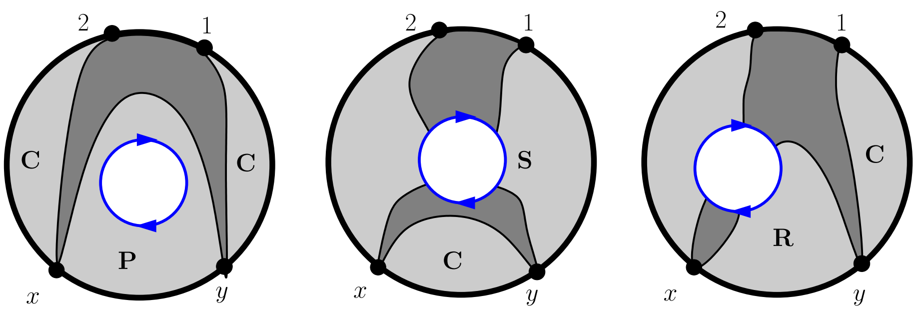


Figure 3.6 Different combinatorial cases for 4-dissections.

This gives the equation

$$\begin{aligned}P^{\{k+1\}}(z) &= zkP^{\{k+1\}}(z)C_{k+1}(z)^{k-1}+(k-1)zC_{k+1}(z)^{k-2}R(z)+\\&\binom{k-1}{2}zS(z)C_{k+1}(z)^{k-2},\end{aligned}$$

where $S(z)$ and $R(z)$ stand for the GFs associated to the number of compatible dissections of regions $\mathcal{S}$ and $\mathcal{R}$, respectively. The term $kP^{\{k+1\}}C_{k+1}(z)^{k-1}$ arises because there are k slots for placing the cross-cap; the term $(k-1)C_{k+1}(z)^{k-2}R(z)$ since there are $k-1$ choices for x; and the last term because there are $\binom{k-1}{2}$ choices for x and y. As before, the factor z indicates the root polygon. Solving for $P^{\{k+1\}}(z)$ we have

$$P^{\{k+1\}}(z) = \frac{zC_{k+1}(z)^{k-2}\left(2(k-1)R(z) + (k^2-3k+2)S(z)\right)}{2(1 - zkC_{k+1}(z)^{k-1})}.$$

Hence, in order to find $P^{\{k+1\}}(z)$ we must compute $S(z)$ and $R(z)$. Denote by $E = E(z)$ the GF of planar $(k+1)-$dissections incompatible with the root polygon (we call them *externally incompatible*); and by $I = I(z)$ the GF of planar $(k+1)-$dissections which are compatible with the root polygon, but *internally incompatible* because of the existence of a repeated point. Both E and I consist of forbidden configurations.

The total number of $(k+1)-$dissections of $\mathcal{S}$ and $\mathcal{R}$ is counted by $(k-1)C^{\bullet}_{k+1,2}(z) - 3C_{k+1,2}(z)$. The argument is the same as in the case of triangulations, with two differences:

1. We take $C_{k+1,2}(z)$ instead of $C_4(z)$ because now the minimum number of polygons needed to dissect the projective plane is 3 instead of 5.
2. We consider the planar relation $(k-1)n + 2 = v$, where v denotes the number of vertices instead of $n + 2 = v$, the latter being a particular case when $k = 2$ (triangulations).

By construction, a forbidden dissection of $\mathcal{S}$ comes from an externally incompatible dissection, so that

$$S(z) = (k-1)C^{\bullet}_{k+1,2}(z) - 3C_{k+1,2}(z) - E.$$

In the case of $\mathcal{R}$, a forbidden dissection comes from either an externally incompatible dissection or an internally incompatible dissection, so that

$$R(z) = (k-1)C^{\bullet}_{k+1,2}(z) - 3C_{k+1,2}(z) - E - I = S(z) - I.$$

We first find the generating function for E. The possible forbidden configurations are those shown in Figure 3.7, but there is an essential difference with Figure 3.5. The incompatibility with the root polygon is produced if any of $\overline{2x}$, $\overline{1y}$, $\overline{12}$ or $\overline{xy}$ are either edges or diagonals of some cell. A solid line indicates that it is an edge of some $(k+1)$-polygon, and a dashed edge that it is a diagonal; and otherwise it is neither an edge nor a diagonal. This means that the configurations are mutually exclusive and there is no need in this case to apply inclusion-exclusion.

Some of the labels correspond in fact to several configurations. For instance, there are four possibilities for U_2, depending on whether the solid edge is $\overline{2x}$ or $\overline{1y}$, and wether the diagonal dashed edge is $\overline{12}$ or $\overline{xy}$. The associated GFs are shown in Table 4.1, together with their multiplicities, that is, the number of times we have to consider the configuration. For instance, the GF for U_2 is $zC_{k+1,1}(z)^2C_{k+1}(z)^{k-2}$: z marks the cell containing $\overline{12}$ as a diagonal; one factor $C_{k+1,1}(z)$ is for the cell containing edge $\overline{2x}$ and the point to the left; and the remaining factor $C_{k+1,1}(z)C_{k+1}(z)^{k-2}$ accounts for the cells arising from the remaining edges of cell c. The multiplicity is 4 since, as we have discussed, there are four different possibilities.

In order to obtain E we only need to sum the corresponding terms. There are however some exceptional cases depending on the value of k.

1. $k = 3$, that is, we are considering dissections into quadrangles. In this case configurations U_1, W_0, W_1 cannot occur, since they imply the existence of a polygon with more than four sides. Hence
$$E = U_0 + 4U_2 + 2U_3 + 2V_1 + 4V_2 + 2V_3 + 2W_2 + 2W_3.$$
2. $k = 4$, In this case configurations U_0 and W_0 cannot occur (they imply the existence of a quadrangle and an hexagon, respectively), hence
$$E = 2U_1 + 4U_2 + 2U_3 + 2V_1 + 4V_2 + 2V_3 + 4W_1 + 2W_2 + 2W_3.$$

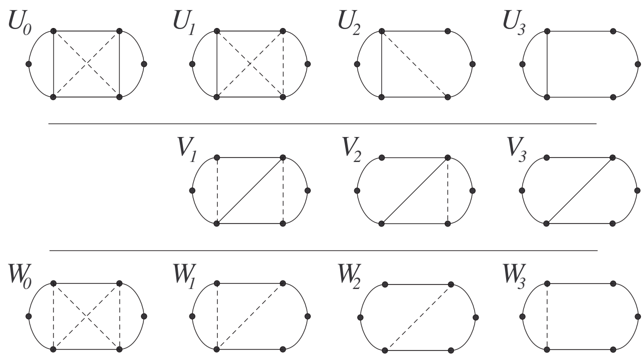


Figure 3.7 Forbidden configurations for an external incompatible planar $(k+1)$-dissection. The labels of the four marked points are shown on the left.

Configuration	GF	Multiplicity
U_0	$zC_{k+1,1}(z)^2$	1
U_1	$zC_{k+1,1}(z)C_{k+1}(z)^{k-2}$	2
U_2	$zC_{k+1,1}(z)^2C_{k+1}(z)^{k-2}$	4
U_3	$zC_{k+1,1}(z)^3C_{k+1}(z)^{k-2}$	2
V_1	$z^2C_{k+1}(z)^{2k-2}$	2
V_2	$z^2C_{k+1,1}(z)C_{k+1}(z)^{2k-2}$	4
V_3	$z^2C_{k+1,1}(z)^2C_{k+1}(z)^{2k-2}$	2
W_0	$zC_{k+1}(z)^{k-1} - z$	1
W_1	$zC_{k+1,1}(z)C_{k+1}(z)^{k-1}$	4
W_2	$zC_{k+1,1}(z)^2C_{k+1}(z)^{k-1}$	2
W_3	$zC_{k+1,1}(z)^2C_{k+1}(z)^{k-1}$	2

Table 3.1 Generating functions and multiplicities of forbidden configurations.

3. $k > 4$. Configuration U_0 does not occur, since it implies the existence of a quadrangle. Hence

$$E = 2U_1 + 4U_2 + 2U_3 + 2V_1 + 4V_2 + 2V_3 + W_0 + 4W_1 + 2W_2 + 2W_3.$$

In all cases, the series $S(z)$ is a polynomial in z and the corresponding planar generating function $C_{k+1}(z)$.

To obtain the GF associated to $R(z)$, we need only to compute I. It corresponds to those "decompositions" which are internally non-compatible; we use quotes to denote that in a planar sense it is a dissection into $(k+1)$-polygons, but when we introduce a repeated point it is not a dissection in the projective plane. A dissection which is internally non-compatible is a planar decomposition in which either:

1. There are two polygons whose intersection is exactly the repeated point and an edge which does not contain the repeated point. We denote them by X_1, X_2, X_3.

2. There are two polygons whose intersection consists of the repeated point and a second point. We denote them by Y_1, Y_2.

These restrictions are summarized in Figure 3.8, and their respective GFs appear in Table 4.2, together with their multiplicities.

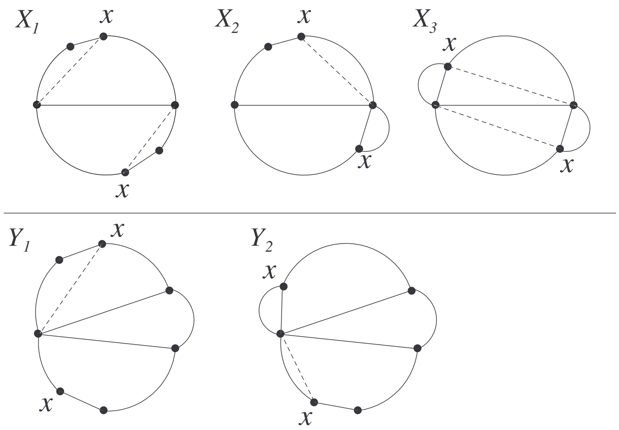


Figure 3.8 Configurations internally incompatible.

Configuration	GF	Multiplicity
X_1	$(k-2)^2 z^2 C_{k+1}(z)^{2k}$	1
X_2	$(k-2) z^2 C_{k+1}(z)^{2k-1} C_{k+1,1}(z)$	2
X_3	$z^2 C_{k+1}(z)^{2k-2} C_{k+1,1}(z)^2$	1
Y_1	$(k-2)(k-1) z^2 C_{k+1}(z)^{2k} C_{k+1,1}(z)$	2
Y_2	$(k-1) z^2 C_{k+1}(z)^{2k-1} C_{k+1,1}(z)^2$	2

Table 3.2 Forbidden configurations for a compatible planar k-decomposition.

The choice of these configurations does not depend on the value of k, and we obtain

$$I = X_1 + X_2 + X_3 + Y_1 + Y_2,$$

which is also a polynomial in z and $C_{k+1}(z)$. Observe that the configurations in Figure 3.8 are disjoint from the ones in Figure 3.7, so that there is no need for inclusion-exclusion.

Having computed both $S(z)$ and $R(z)$, we can write down in an explicit way the generating function $P^{\{k+1\}}(z)$, and a routine computation using $C_{k+1}(z) = 1 + zC_{k+1}(z)^k$ for simplifying the final expressions proves the claim. □

3.4 Unrestricted dissections

In this section we consider dissections of a polygon in the projective plane into cells of any size (at least three). We count them according to the number of vertices (variable x) and to the number of cells (variable u) in the dissection. Again we demand that two intersecting cells meet only in a common vertex or in a common edge (simplicial condition). Later we study this bivariate generating function from a probabilistic point of view (see Section 3.6).

In this case we can apply the previous ideas to obtain the exact enumeration.

More concretely, we have the following theorem:

Theorem 3.3 *Let $P^D_m(n)$ be the number of dissections into m cells of a polygon with n vertices in the projective plane, and let $D(u,x)$ be the generating function for plane dissections (recall Equation (1.2) in Section 1.2.1). Then*

$$\begin{aligned} P^D(u,x) &= \frac{\alpha+\beta D(u,x)}{\gamma} = u^5x^5 + (u^3+6u^4+18u^5+14u^6)x^6 + \\ &\quad (7u^3+56u^4+182u^5+245u^6+113u^7)x^7 + \cdots \end{aligned}$$

where

$$\begin{aligned} \alpha &= -x^9 + \left(u^2+9u+7\right)x^8 - \\ &\quad \left(5u^3+29u^2+43u+19\right)x^7 + \\ &\quad \left(5u^4+32u^3+73u^2+70u+24\right)x^6 - \\ &\quad \left(5u^4+25u^3+46u^2+37u+11\right)x^5 - \\ &\quad \left(u^3+7u^2+11u+5\right)x^4 + \left(7u^2+14u+7\right)x^3 - (2u+2)\,x^2, \end{aligned}$$

$$\begin{aligned} \beta &= (u+1)\,x^7 - \left(u^3+9u^2+15u+7\right)x^6 + \\ &\quad \left(4u^4+26u^3+59u^2+56u+19\right)x^5 - \\ &\quad \left(2u^5+19u^4+65u^3+105u^2+81u+24\right)x^4 + \\ &\quad \left(7u^4+29u^3+48u^2+37u+11\right)x^3 + \left(6u^3+17u^2+16u+5\right)x^2 - \\ &\quad \left(9u^2+16u+7\right)x + (2u+2), \end{aligned}$$

$$\gamma = u^2x^3\left(-4xu+x^2-2x+1\right).$$

Proof. Recall the definition of $D(u,x)$, stated in Section 1.2.1. In particular, $D(u,x)=\sum_n d_n(u)x^n$, where $d_n(u)$ is a polynomial of degree $n-2$ (for a given number of vertices, triangulations maximize the number of faces). We set $D_r(u,x)=\sum_{i\geq r} d_i(u)x^i$. As a general rule it is convenient to work with $D(u,x)/x$, since we work with sequences of consecutive planar dissections, and each vertex is counted twice in the sequence. Hence let $N(u,x)=D(u,x)/x$. As before $S(u,x)$ and $R(u,x)$ are the GFs of marked planar dissections without and with a repeated point, respectively.

We fix the root polygon containing edge $\overline{12}$ and we argue as follows. Assume first that the root polygon r does not cross the cross-cap. If r has k edges, then it determines $(k-2)$ planar dissections and one projective dissection, and there are $k-1$ ways of choosing the region containing the cross-cap. This contributes with a summand of the form $u(k-1)P^D(u,x)N(u,x)^{k-2}$, where the multiplicative term u is referred to the root polygon. Summing up we obtain

$$uP^D(u,x)\left(2N(u,x)+3N(u,x)^2+\dots\right) = uP^D(u,x)\frac{2N(u,x)-N(u,x)^2}{(1-N(u,x))^2}.$$

If the cross-cap is crossed then we distinguish three kinds of edges in r: those found before crossing the cross-cap, those found after first crossing the cross-cap, and finally those found after crossing the cross-cap a second time. This decomposition gives rise to three sequences of planar dissections, drawn as zones defined by dashed lines in Figure 3.9. Each grey zone must contain at least one vertex, and those at the bottom of the first three pictures contain at least two vertices. The final equation is

$$\begin{aligned} P^D(u,x) &= uP^D(u,x)\frac{2N(u,x)-N(u,x)^2}{(1-N(u,x))^2} + \frac{u}{x}\frac{1}{(1-N(u,x))^2}R(u,x) + \\ &\quad \frac{u}{x}\left(\frac{N(u,x)}{1-N(u,x)} + 2\left(\frac{N(u,x)}{1-N(u,x)}\right)^2 + \left(\frac{N(u,x)}{1-N(u,x)}\right)^3\right)S(u,x). \end{aligned} \tag{3.4}$$

The first term is the one obtained above. The second term corresponds to the three cases where we have region $\mathcal{R}$, and the final term to the case where we have region $\mathcal{S}$. The series $N(u,x)/(1-N(u,x))$ is the GF for a non-empty sequence of planar dissections. Solving for $P^D(u,x)$, and writing $N(u,x) = D(u,x)/x$, we obtain

$$P^D(u,x) = \frac{ux\,(D(u,x)S(u,x) + xR(u,x) - D(u,x)R(u,x))}{(D(u,x)-x)\,(-(1+u)D(u,x)^2 + 2(1+u)xD(u,x) - x^2)}.$$

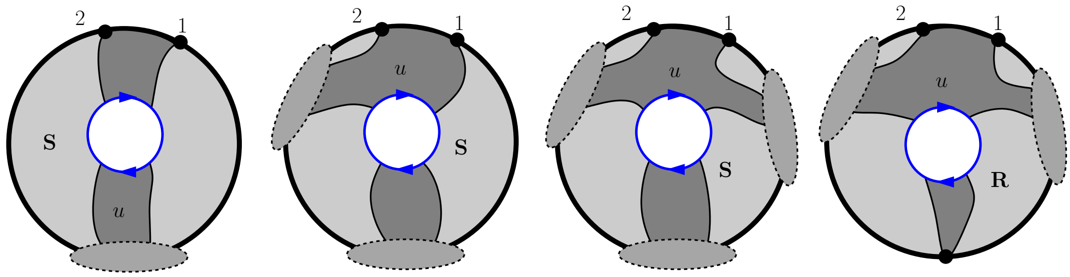


Figure 3.9 Possible cases where the polygon crosses the cross-cap. The GF associated to the grey zones is always $(1 - D(u,x)/x)^{-1}$.

In order to compute $S(u,x)$ and $R(u,x)$, we proceed as in the case of k-dissections. Define E and I as in the previous section. We count the number of marked planar dissections and we subtract E to obtain S or $E+I$ to obtain R.

To count marked planar dissections we proceed as follows. If n is the number of vertices, we have to distribute $n-4$ points into two non-empty sets. This can be done in $n-5$ ways; the minimum number of vertices is 6, hence we must consider $D_6(u,x)$. As a conclusion, the corresponding GF is $D_6^\bullet(u,x) - 5D_6(u,x)$. Then

$$S(u,x) = D_6^\bullet(u,x) - 5D_6(u,x) - E, \qquad R(u,x) = D_6^\bullet(u,x) - 5D_6(u,x) - E - I = S - I.$$

We consider the computation of S and R separately.

Computing S.
In this case, we count marked planar dissections and subtract those which are externally incompatible, that is, we only consider E.

To compute E we use again exclusion-inclusion. Basic configurations are shown in Figure 3.10, and their combinatorial specifications are shown in Table 3.3; the difference now is that a diagonal can be either an edge of a face (which introduces the factor u) or a diagonal in a face (which introduces the factor $1/u$).

For instance, the GF associated to V_0 must have four terms. In the case with edges $\overline{bd}$ and $\overline{bc}$ (first row, first column in V_0, Figure 3.10) the four possible configurations are those showed in Figure 3.11. The corresponding GFs are, from left to right, $u/x(D_4(u,x)D_3(u,x))$, $1/x(D_4(u,x)D_3(u,x))$, $1/x(D_4(u,x)D_3(u,x))$ and $1/(xu)(D_4(u,x)D_3(u,x))$. In all cases we consider two planar dissections ($D_4(u,x)$ on the left and $D_3(u,x)$ on the right) which can be pasted together using the diagonals $\overline{bd}$ and $\overline{bc}$. There is always a term x in the denominator because in all cases we are counting the point b twice. Adding these four terms we obtain the expression in Table 3.3.

In conclusion, we obtain

$$\begin{aligned} S(u,x) &= D_6^\bullet(u,x) - 5D_6(u,x) - E \\ &= D_6^\bullet(u,x) - 5D_6(u,x) - 2U_0 - 2U_1 + 4V_0 + V_1 - 2W_0. \end{aligned} \tag{3.5}$$

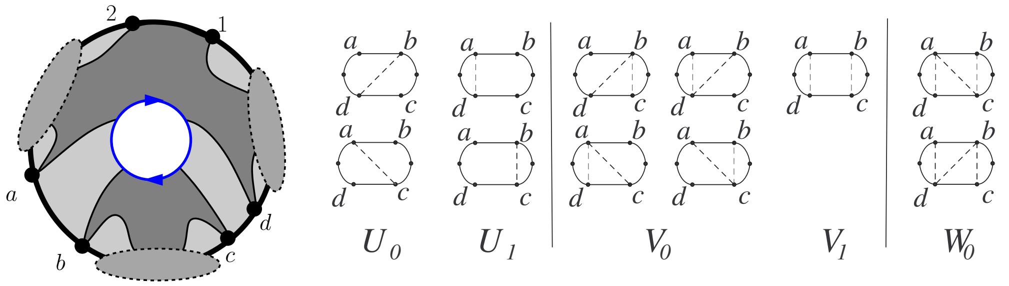


Figure 3.10 Forbidden dissections for an external incompatible planar dissection.

Configuration	GF	Multiplicity
U_0	$\frac{1}{x^2} D_3(u,x) D_5(u,x) \left(1+\frac{1}{u}\right)$	2
U_1	$\frac{1}{x^2} D_4(u,x)^2 \left(1+\frac{1}{u}\right)$	2
V_0	$\frac{u}{x}\left(1+\frac{1}{u}\right)^2 D_3(u,x) D_4(u,x)$	4
V_1	$u\left(1+\frac{1}{u}\right)^2 (u+2u^2) D_3^2(u,x)$	1
W_0	$u^2\left(1+\frac{1}{u}\right)^3 D_3^2(u,x)$	2

Table 3.3 Forbidden configurations for external incompatible planar dissections.

Computing R.
In this case we have identified points, hence we count marked planar dissections and we subtract those which are externally and internally incompatible. The ones which are externally incompatible where counted by S, and this has been done already. We must compute now the number of internally incompatible dissections which are externally compatible with the root polygon. We have

$$R = D_6^\bullet(u,x) - 5D_6(u,x) - E - I = S - I.$$

To describe the possible internal incompatible configurations, we define three fundamental blocks and show how to build from them all possible forbidden configurations. See Figure 3.12, where dashed lines must be taken either as edges or diagonals.

Dissections that are internally incompatible appear because there is one point (denoted by x) that is a vertex of two polygons in the dissection. Denote this two polygons by Q_1 and Q_2. Because we are dealing with cellular decompositions, $Q_1 \cap Q_2$ must be a cell. The existence of the double

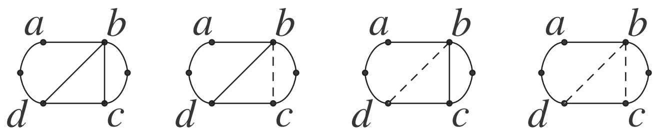


Figure 3.11 A particular case of external incompatible dissection. Continuous lines denote edges, and dashed lines denote diagonals.

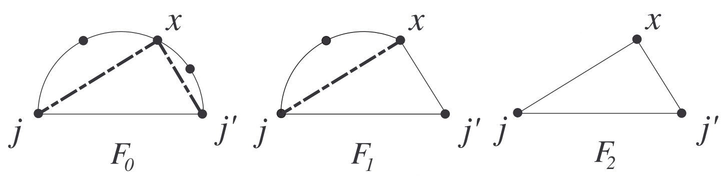


Figure 3.12 Blocks used to construct all possible forbidden configurations.

Configuration	GF
F_0	$\frac{1}{x}(u+2+\frac{1}{u})D_3(u,x)^2$
F_1	$x(1+u)D_3(u,x)$
F_2	ux^3

Table 3.4 Translation of the previous configurations into generating functions.

point x produces internally incompatible configurations in two different ways:

1. The ones such that $Q_1 \cap Q_2$ is $\{x, j\}$, where j is another vertex.
2. The ones such that $Q_1 \cup Q_2 = \{x\} \cup \{jj'\}$.

That is, $Q_1 \cap Q_2$ can be the union of two points or the union of an edge and a third point, and in either case it is not a cell. The two cases are shown in Figure 3.13, and the corresponding GFs are shown in Table 3.5. In G_0 and G_1 a factor 2 appears because we can choose the point j in the two sides of the polygon.

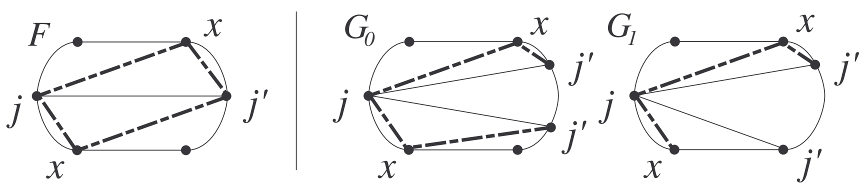


Figure 3.13 Blocks used to construct all possible forbidden configurations.

The final expression for S is

$$R(u,x) = S(u,x) - F - G_0 - G_1. \tag{3.6}$$

It only remains to substitute Expressions (3.6) and (3.5) into the main Equation (3.4), and a routine computation gives the result as claimed. $\square$

Notice that the sequence $[u^k x^k]P^D(u,x)$ is precisely the sequence of triangulations obtained before. This is because, for a fixed number of vertices, unrestricted dissections having a larger number of cells are triangulations.

Configuration	GF	Multiplicity
F	$\frac{1}{x^2}(F_0+F_1)^2$	1
G_0	$(F_0+F_1)^2 D_3(u,x)\frac{1}{x^4}$	2
G_1	$(F_0+F_1)(F_1+F_2)D_3(u,x)\frac{1}{x^4}$	2

Table 3.5 Translation of the previous configurations into GFs.

3.5 Asymptotic enumeration

In this section we obtain precise asymptotic expressions for the number of polygon dissections studied before. It turns out that they are of the form $c\cdot\rho^{-n}(1+O(n^{-1}))$, where c is a constant, and ρ is the radius of convergence of the corresponding generating function for a disk. In the case of the disk, it was shown in [17] that the estimates were always of the form $c\cdot n^{-3/2}\rho^{-n}(1+O(n^{-1}))$. Thus we can say that 0 is the universal exponent for dissections in the projective plane, whereas in the plane it is $-3/2$. As we show in Chapter 5, this exponent is intimate related to the Euler characteristic of the surface we are dealing with.

In order to obtain estimates for coefficients of GF defined implicitly, we follow the approach in the book of Flajolet and Sedgewick [19]. For completeness, we include the following result:

Proposition 3.4 *Suppose that the generating function $F(z)$ satisfies an equation of the form $F(t)=t\phi\left(F(t)\right)$, and assume that:*

1. *$\phi(0)\neq 0$, $\phi''(z)\neq 0$.*
2. *$\phi(u)$ is analytic at $u=0$, and has an expansion with positive coefficients.*
3. *Let R be the radius of convergence of ϕ. There exists a unique positive real solution $0<\tau<R$ of the equation $\phi(\tau)-\tau\phi'(\tau)=0$.*

Then, the radius of convergence of $F(t)$ is equal to $\rho=\tau/\phi(\tau)$, and the singular expansion of $F(t)$ at $t=\rho$ is of the form

$$F(t) \ =_{z\to\rho}\ \tau-\sqrt{\frac{2\phi(\tau)}{\phi''(\tau)}}\,(1-t/\rho)^{1/2}+O\left((1-t/\rho)^{1/2}\right).$$

Generating functions we have obtained in the previous sections are all of the form

$$P^{\Delta}(t)=\frac{1}{Q(t)}\sum_{j=0}^{r}Q_j(t)C_{\Delta}(t)^j,$$

where $Q_j(t)$ and $Q(t)$ are polynomials. We assume that t is used to count regions for triangulations and decompositions in $(k+1)$-agons, and is used to count vertices in unrestricted dissections. In all these cases, $C_{\Delta}(t)$ is an algebraic function that verifies a relation of the form $C_{\Delta}(t)=t\phi(C_{\Delta}(t))$, and the smallest real root of $Q(t)$ is precisely equal to the radius of convergence of $C_{\Delta}(t)$. Hence, the singular expansion of $P^{\Delta}(t)$ at $t=\rho$ is of the from

$$P^{\Delta}(t) \ =_{z\to\rho}\ \sum_{j=r}^{\infty}a_j\,(1-t/\rho)^{j/2},$$

where the integer r (possibly negative) depends on the given generating function. To make calculations easier, we write $Z=\sqrt{1-t/\rho}$ and make the corresponding change of variables. Then we expand the function that is obtained at $Z=0$, obtaining a development of the form

$$P^{\Delta}(Z)=\sum_{j=s}^{\infty}a_jZ^j,$$

where the integer s can be a negative. In all GFs we have obtained the value of s is equal to -2. Performing this calculations for all types of dissections we have studied, we obtain the following singular expansions:

- Triangulations: $P(Z) = \frac{1}{4}Z^{-2} + O(Z^{-1})$.
- Quadrangulations: $P^{\{4\}}(Z) = \frac{2}{4}Z^{-2} + O(Z^{-1})$.
- Decomposition into 5-gons: $P^{\{5\}}(Z) = \frac{3}{4}Z^{-2} + O(Z^{-1})$.
- Decomposition into k-agons, $k > 5$: $P^{\{k+1\}}(Z) = \frac{k-1}{4}Z^{-2} + O(Z^{-1})$.
- Unrestricted dissections: $P^{D}(Z) = \frac{1}{4}Z^{-2} + O(Z^{-1})$.

Observe that the constant term in the first four cases corresponds with $\frac{k-1}{4}$. The corresponding result is summarized in the following theorem.

Theorem 3.5 *For fixed $k \geq 2$, the number of dissections of a polygon in the projective plane with $(k-1)n$ vertices (equivalently, into $(k+1)$-gons) is asymptotically equal to*

$$\frac{k-1}{4}\left(\frac{k^k}{(k-1)^{k-1}}\right)^n.$$

Proof. The Generating function $C_{k+1}(z)$ verifies the equation $C_{k+1}(z) = 1 + zC_{k+1}(z)^k$. The radius of convergence of $C_{k+1}(z)$ is $(k-1)^{k-1}/k^k$. This is a well-known result and is obtained by solving the characteristic system of equations

$$\Phi(y,z) = 0, \qquad \frac{\partial}{\partial y}\Phi(y,z) = 0,$$

Observe that $\Phi(y,z) = y - zy^k$ is the polynomial equation satisfied by $C_k(z)$. Transfer Theorem (Theorem 1.2) applies and we obtain the estimate as claimed. □

Theorem 3.6 *The number of dissections of a polygon in the projective plane with n vertices is asymptotically equal to*

$$\frac{1}{4}\left(3+2\sqrt{2}\right)^n (1 + O(n^{-1})).$$

Proof. The equation satisfied by the generating function $D(x) = D(1,x)$ of plane unrestricted dissections is obtained from Equation (1.2) writing $u = 1$:

$$2D(x)^2 - x(1+x)D(x) + x^3 = 0.$$

From these previous equation, we obtain that the corresponding radius of convergence is $3 - 2\sqrt{2}$, and the result follows again from Theorem 1.2. □

Table 3.6 summarizes the results in this section, together with the corresponding results for plane dissections taken from [17].

3.6 Limit laws

There are two statistical parameters we study in this section: the number of "cyclic" triangles in a triangulation (defined below), and the number of cells in an arbitrary unrestricted dissection. In the first case we obtain as a limit the absolute value of a normal law with expected value of order $\sqrt{n}$ and variance of order n. In the second case we obtain a normal limit law with linear expected value and variance.

Class	Sphere	Projective plane
Dissections into $(k+1)$-gons	$\sqrt{\frac{k}{2\pi(k-1)}}\, n^{-3/2} \left(\frac{k^k}{(k-1)^{k-1}}\right)^n$	$\frac{k-1}{4}\left(\frac{k^k}{(k-1)^{k-1}}\right)^n$
Unrestricted dissections	$\frac{\sqrt{-140+99\sqrt{2}}}{4\sqrt{\pi}}\, n^{-3/2}(3+2\sqrt{2})^n$	$\frac{1}{4}(3+2\sqrt{2})^n$

Table 3.6 Asymptotic estimates for dissections of a disk and the projective plane.

3.6.1 Cyclic triangles in triangulations

Let τ be a polygon triangulation in the projective plane, and let τ^* be the dual graph, whose vertices are the triangles of τ and edges are pairs of triangles sharing a diagonal. Consequently τ^* is a connected unicyclic graph; that is, a graph with a unique cycle. This is most easily seen by considering τ as a triangulation of the Möbius band where the vertices are on the boundary: the triangles corresponding to the unique cycle of τ^* are those whose deletion disconnect the Möbius band.

We say that a polygon triangulation of the projective plane is *cyclic* if its dual graph is a cycle. In the following lemma we get the exact enumeration of all cyclic triangulations of the projective plane:

Lemma 3.7 *The number of cyclic polygon triangulations of the projective plane with n vertices is equal to $2^{n-1} - n^2 + 2n$.*

Proof. Let us analyze the proof of Theorem 3.1 and see how we can obtain a cyclic triangulation. First of all, the unique triangle $12x$ containing edge $\overline{12}$ has to cross the cross-cap, otherwise there would be non cyclic triangles. In the sequel we refer to the left picture in Figure 3.4, corresponding to a valid triangulation. Notice that x varies between 4 and $n-1$.

For a triangulation to be cyclic, all diagonals must join a point on the left with a point on the right, forming a zigzag pattern. This can be done in $\binom{n-1}{x-2}$ ways. However, we have to subtract the configurations giving raise to a non valid triangulation (not simplicial). These are those containing one of the edges $\overline{12}$, $\overline{xx}$, $\overline{(x-1)x}$ or $\overline{x(x+1)}$. A simple computation shows that for each x there are exactly n forbidden configurations. The total number of cyclic triangulations is thus

$$\sum_{x=4}^{n-3} \binom{n-1}{x-2} - n(n-4) = 2^{n-1} - n^2 + 2n,$$

which proves the claimed result. □

It is easy from the previous lemma to obtain the GF of cyclic triangulations: let $M(z) = \sum_{n\geq 5}(2^{n-1} - n^2 + 2n)z^n$. Then

$$M(z) = \sum_{n\geq 5} s_n z^n = \frac{1}{2}\sum_{n\geq 5} 2^n z^n - \sum_{n\geq 5} n^2 z^n + 2\sum_{n\geq 5} n z^n.$$

Summing up these three expressions, we get the following closed form for $M(z)$:

$$M(z) = \frac{z^5\left(1+3z-2z^2\right)}{(1-2z)\left(1-z\right)^3}.$$

Given a triangulation τ, let $\alpha(\tau)$ be the length of the unique cycle in τ^*. We are interested in the distribution of the parameter α among all triangulations of size n. Let $P(k,n)$ be the number of triangulations with n vertices and $\alpha = k$, and let

$$P(u,z) = \sum_{k,n\geq 0} P(k,n) u^k z^n.$$

Lemma 3.8 *The generating functions $P(u,z)$ and $M(z)$ are related through the equation*

$$u\frac{\partial}{\partial u}P(u,z) = z\frac{\partial}{\partial z}M(uzC(z)). \tag{3.7}$$

Proof. Let τ be a triangulation of a polygon Q, and let σ be the union of all triangles in τ that belong to the unique cycle of τ^*. Then σ is a cyclic triangulation of Q with $k=\alpha(\tau)$ vertices, and τ is obtained from σ by gluing planar triangulations to the boundary edges of Q.

Equation (3.7) expresses in two different ways the GF of triangulations with vertices labeled $1,2,\ldots,n$ in circular order and one cyclic triangle marked. In the left term the triangle is marked by means of the $\partial/\partial u$ operator. In the right term, the triangle is marked using the $\partial/\partial z$ operator. The substitution $z \to uzC(z)$ means that to the outer edge of each cyclic triangle we glue a plane triangulation. $\square$

From this equation we obtain an alternative proof for the enumeration of triangulations of a polygon in the projective plane. We must integrate the previous expression, taking care of the initial conditions. The result is

$$P(u,z) = \left(1+\frac{zC'(z)}{C(z)}\right)M(uzC(z)). \tag{3.8}$$

The first terms are

$$P(u,z) = u^5z^5 + (6u^5+8u^6)z^6 + (28u^5+56u^6+29u^7)z^7 + \ldots.$$

Setting $u=1$, we recover the series for polygon triangulations in the projective plane stated in Section 3.2.

Let $\mathbf{X}_n$ be the discrete random variable on the set of all triangulations of a polygon with n vertices in the projective plane, defined by $\mathbf{X}_n(\tau)=\alpha(\tau)$. In Figure 3.14 this unique cycle is shown for a concrete example.

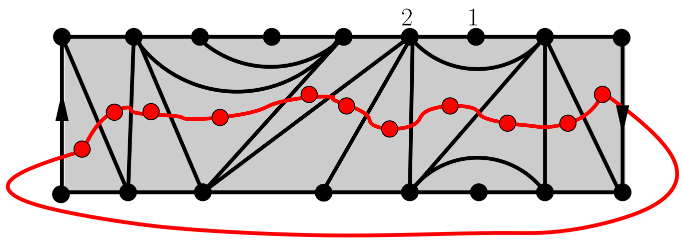


Figure 3.14 The cycle in a triangulation of the Möbius band.

Our next result gives the limit law for the normalized variable $\mathbf{X}_n/\sqrt{n}$. We denote by $\mathcal{N}(0,1)$ the standard normal law with zero mean and unit variance.

Theorem 3.9 *Let $\boldsymbol{Y} = \sqrt{2}|\boldsymbol{Z}|$, where $\boldsymbol{Z} \sim \mathcal{N}(0,1)$. Then $\boldsymbol{X}_n/\sqrt{n} \xrightarrow{d} \boldsymbol{Y}$.*

Proof. The proof is based on the Method of Moments as it is stated in Lemma 1.5. Let $\mathbf{Y}_n$ be the sequence of random variables $\mathbf{Y}_n = \mathbf{X}_n/\sqrt{n}$. We show that $\mathbf{Y}$ characterized by its moments. We also show that for each value of r, the r-moment $\mathbb{E}[\mathbf{Y}_n^r]$ converge to $\mathbb{E}[\mathbf{Y}^r]$ as $n\to\infty$. It follows then using the criteria for the Method of the Moments that $\mathbf{Y}_n \xrightarrow{d} \mathbf{Y}$.

The density probability function of $\mathbf{Y}$ is

$$f_{\mathbf{Y}}(t) = \frac{1}{\sqrt{\pi}}e^{-t^2/4}\mathbb{I}_{[0,\infty[}(t),$$

where $\mathbb{I}_{[0,\infty[}(t)$ is the characteristic function of the set $[0,\infty[$. Hence the r-th moment is

$$\mathbb{E}\left[\mathbf{Y}^r\right] = \int_{-\infty}^{\infty} u^r f(t)dt = \frac{2^r}{\sqrt{\pi}}\,\Gamma\left(\frac{k+1}{2}\right) < 2^r r!.$$

The latter inequality implies that $\mathbf{Y}$ is determined by its moments. In order to compute moments from the generating function $P(u,z)$, we compute first the r-th factorial moment. Disregarding the terms in Equation (3.8) which are analytic, we obtain that the singular expansion of $P(u,z)$ at $(u,z) = (1, 1/4)$ is

$$P(u,z) \sim_{(u,z)\to(1,1/4)} \frac{1}{4}\frac{1}{\sqrt{1-4z}}\frac{1}{1-u\left(1-\sqrt{1-4z}\right)}.$$

From this expression it follows that

$$\sum_{n\geq 0} P(n)\mathbb{E}\left[(\mathbf{X}_n)_r\right] = \left.\frac{\partial^r}{\partial u^r}P(u,z)\right|_{u=1} \sim \frac{1}{4}\frac{r!}{(1-4z)^{1+r/2}}.$$

Extracting coefficients and using singularity analysis we obtain the following expression

$$P(n)\mathbb{E}\left[(\mathbf{X}_n)_r\right] \sim \frac{1}{4}\frac{r!\cdot n^{r/2}}{\Gamma\left(1+\frac{r}{2}\right)}\,4^n.$$

Using the estimate $P(n) \sim 4^{n-1}$ obtained in Theorem 3.5, we deduce that

$$\mathbb{E}\left[(\mathbf{X}_n)_r\right] \sim \frac{r!\cdot n^{r/2}}{\Gamma\left(1+\frac{r}{2}\right)}.$$

The same estimate holds for the ordinary moment $\mathbb{E}\left[\mathbf{X}_n^r\right]$, since $\mathbb{E}\left[\mathbf{X}_n^r\right] \sim \mathbb{E}\left[(\mathbf{X}_n)_r\right]$. Finally, using Gauss multiplication formula (see [1] for additional properties of the Gamma function) $\Gamma(r)\,\Gamma(1/2+r) = \sqrt{2\pi}2^{1/2-2r}\Gamma(2r)$, the result follows. □

A direct consequence of the previous result are the following particular cases:

Corollary 3.10 *The first two moments of* $\boldsymbol{X}_n$ *are, asymptotically,*

$$\mathbb{E}\left[\boldsymbol{X}_n\right] \;\sim\; \frac{2}{\sqrt{\pi}}\sqrt{n}, \qquad \sigma^2(\boldsymbol{X}_n) \;\sim\; \left(2-\frac{4}{\pi}\right)n.$$

Proof. These are the cases $k=1$ and $k=2$ in the previous proof. □

In the work [23] the values with are obtained in the previous corollary are also deduced. Our contribution here is the explicit form of the whole density probability function.

3.6.2 Cells in dissections

Let $\mathbf{U}_n$ be the number of cells in a dissection of a polygon in the projective plane with n vertices. Intuitively, one should expect that $\mathbf{U}_n$ behaves very much like in the planar case, and this is indeed true. The first asymptotic term for the mean and variance is the same as for planar dissections (see [17]). However, there is a difference in the second term. Taking additional terms in the computations from the planar case in [17], the expected value is shown to be asymptotically

$$\frac{\sqrt{2}}{2}n - \left(\frac{3\sqrt{2}}{4} - \frac{1}{8}\right) + O(n^{-1}),$$

where one should notice that $\frac{3\sqrt{2}}{4} - \frac{1}{8}$ is positive. If we compare it with the result in the next theorem, we see that the expected number of cells in projective dissections is larger than in plane projections just by an additive constant.

Theorem 3.11 $\boldsymbol{U}_n$ *is asymptotically normal and*

$$\mathbb{E}\left[\boldsymbol{U}_n\right] \sim \frac{\sqrt{2}}{2}n + A\frac{1}{\sqrt{n}} + O(n^{-1}), \qquad \sigma^2(\boldsymbol{U}_n) \sim \frac{\sqrt{2}}{8}n + B\sqrt{n} + O(1),$$

where $A = \frac{1}{16\sqrt{\pi}}\sqrt{4+3\sqrt{2}}(1089\sqrt{2}-1536) \approx 0.4129$ *and* $B = \frac{1}{128\sqrt{\pi}}\sqrt{4+3\sqrt{2}}(3820\sqrt{2}-6015) \approx -7.7535$.

Proof. From Theorem 3.3, the bivariate GF for dissections is

$$P^D(u,x) = \frac{\alpha + \beta D(u,x)}{\gamma},$$

where $D(u,x)$ satisfies Equation (1.2). Hence $P^D(u,x)$ is an algebraic function and its defining equation can be computed directly using resultants. If follows that $P^D(u,x)$ satisfies the quadratic equation

$$aP^D(u,x)^2 + bP^D(u,x) + c = 0,$$

where

$$\begin{aligned} a &= u^2x^6 - \left(8u^3+4u^2\right)x^5 + \\ &\quad \left(16u^4+16u^3+6u^2\right)x^4 - \left(8u^3+4u^2\right)x^3 + u^2x^2, \end{aligned}$$

$$\begin{aligned} b &= x^{10} - \left(u^2+14u+10\right)x^9 + \\ &\quad \left(10u^3+79u^2+109u+43\right)x^8 - \\ &\quad \left(32u^4+211u^3+423u^2+348u+103\right)x^7 + \\ &\quad \left(32u^5+232u^4+648u^3+875u^2+574u+147\right)x^6 - \\ &\quad \left(48u^5+272u^4+648u^3+790u^2+485u+119\right)x^5 + \\ &\quad \left(32u^4+126u^3+186u^2+129u+35\right)x^4 + \left(75u^3+146u^2+106u+27\right)x^3 - \\ &\quad \left(72u^2+92u+32\right)x^2 + (21u+13)x - 2, \end{aligned}$$

$$\begin{aligned} c &= \left(u^4+u^3\right)x^9 - \\ &\quad \left(u^6+9u^5+13u^4+3u^3\right)x^8 + \left(4u^7+22u^6+36u^5+13u^4+u^3\right)x^7 + \\ &\quad \left(7u^6+23u^5+12u^4+2u^3\right)x^6 + 2x^5u^5. \end{aligned}$$

Under this assumptions, we can apply the Quasi-Powers Theorem, stated in Theorem 1.3. In order to compute the expected value and the variance it is enough (see [19]) to estimate $\partial P^D(1,x)/\partial u$ and $\partial^2 P^D(1,x)/\partial u^2$. Since we have an explicit expression for $P^D(u,x)$, this can be achieved by singularity analysis as in the previous section. The necessary singular expansions are

$$\begin{aligned} \frac{\partial P^D(1,x)}{\partial u} &= \frac{\sqrt{2}}{8}X^{-4} + \sqrt{4+3\sqrt{2}}\left(\frac{61}{16} - \frac{23}{8}\sqrt{2}\right)X^{-3} - \frac{\sqrt{2}}{8}X^{-2} + O(X^{-1}), \\ \frac{\partial^2 P^D(1,x)}{\partial u^2} &= \frac{1}{4}X^{-6} - \sqrt{4+3\sqrt{2}}\left(\frac{69}{16} - \frac{183}{64}\sqrt{2}\right)X^{-5} - \left(\frac{3}{8} + \frac{3}{32}\sqrt{2}\right)X^{-4} + \\ &\quad \sqrt{4+3\sqrt{2}}\left(\frac{4063}{512} - \frac{587}{128}\sqrt{2}\right)X^{-3} + O(X^{-2}) \end{aligned}$$

where $X = \sqrt{1-x/\rho}$ and $\rho = 3-2\sqrt{2}$. Using singularity analysis we estimate the coefficient of x^n in the two series above, and find the asymptotic for $n \to \infty$. The computations are routine using `Maple`. Together with Theorem 3.6, we obtain the result as claimed. □

3.7 Concluding remarks

Symbolic methods lead in many cases to easy derivations of generating function equations. This observation applies with special strength here. Despite we are not dealing with planarity, we know how to reduce the problem to a planar one. Planarity is the fundamental ingredient in this chapter, and the distinguishable character of vertices entails strong decomposition properties. As a result, the generating functions obtained are all algebraic. Additionally, their singularities correspond with singularities of the corresponding planar decompositions. This shows that the exponential behavior of coefficients is the same as in the planar case, but not the subexponential behavior.

Something that must be noticed is that despite generating functions we obtain are complex (they depend on long polynomial expressions), asymptotic expressions are extremely simple. We show in Chapter 5 the reason of this simplification.

4

Dissections of the cylinder

This chapter includes the results of [45]. We obtain the number of ways of dissecting a pair of disjoint polygons embedded in the sphere into n subpolygons with $k+1$ sides each, where k is an arbitrary natural integer. We also obtain the enumeration in the case of unrestricted dissections. We apply singularity analysis over the previous expressions in order to obtain asymptotic estimates, and we determine probability limit laws for the size of the core in dissections and the distribution of vertices in triangulations. In this last part, the key tool is the Laplace transform.

4.1 Introduction: a general problem and a composition scheme

Consider the following problem: let $\mathbb{S}$ be a surface with boundary. Which is the number of simplicial decompositions of $\mathbb{S}$ with the restriction that all vertices lie on the boundary of $\mathbb{S}$? Observe that if $\mathbb{S}$ is the sphere and the number of connected components of $\partial\mathbb{S}$ is 1, the problem reduces to the enumeration of triangulations of a plane polygon. More generally, if the number of connected components of $\mathbb{S}$ is 1, the problem (with the terminology introduced in Chapter 3) consists of counting the number of triangulations of a polygon embedded in $\overline{\mathbb{S}}$. In particular, in Chapter 3 we have studied in detail the case when $\overline{\mathbb{S}}$ is the projective plane. In this chapter we treat the next step in this problem: instead of taking a single polygon, we consider a pair of polygons in the simplest surface, the sphere. In other words, we consider the surface obtained from the sphere by deleting a pair of disjoints disks. Consequently we consider simplicial decompositions of the cylinder, with the restriction that all vertices lie on the boundary.

One may try to use the same strategy used to count simplicial decompositions of the projective plane (combinatorial surgery arguments and inclusion-exclusion arguments over GFs). Unfortunately, the number of cases that must be treated growths notably, and computations for general families becomes extremely involved. The key point is based on exploiting the composition scheme introduced in Equation (3.8). The fundamental idea below this equation is that the dual of triangulations of the projective plane are extremely simple, and this is also true for the cylinder. The aim of this chapter consists of exploiting this second point of view in order to obtain the enumeration for simplicial decompositions, dissections into polygons with a fixed degree, and unrestricted dissections. In all cases we require that two cells of a decomposition intersect only at a vertex or at an edge (condition of dissection).

Notice that in a triangulation the number of triangles equals the number of vertices. This follows from Euler's formula applied to the cylinder, and double counting incidences between edges and faces. More generally, in a dissection into $(k+1)$-gons, the number of vertices is $(k-1)n$, where n is the number of $(k+1)$-gons.

The plan for this chapter is the following: in Section 4.2 we introduce technical lemmas which are used in the forthcoming sections. We continue studying triangulations in Section 4.3. In order to obtain the enumeration of more complex families, we need to study a combinatorial class which is introduced in Section 4.4. Ideas used in Section 4.3 are refined in Section 4.5 and Section 4.6 in order to obtain the GFs for dissections into polygons and unrestricted dissections. In Section 4.7

we study the asymptotic enumeration of these families. We use the Laplace transform and the Method of Moments to study the distribution of limit laws which are related to the families which have been studied in Section 4.8. In Section 4.9 we present results without proof about a different kind of dissections of the cylinder (this was done in [23] for the special case of triangulations). In our approach this result is a simple consequence of the framework developed in the previous sections.

Let us introduce the terminology used in the rest of this chapter. Let $\mathcal{H}$ be a cylinder. Let $\mathbb{S}^1_1$ and $\mathbb{S}^1_2$ be the disjoint circles which are the connected components of the boundary of $\mathcal{H}$. We represent graphically a cylinder drawing $\mathbb{S}^1_2$ inside $\mathbb{S}^1_1$: the cylinder is the region defined by this pair of circles. We say that $\mathbb{S}^1_1$ is the *external circle* of $\mathcal{H}$ and $\mathbb{S}^1_2$ the *internal circle* of $\mathcal{H}$. Vertices on the external circle are called *external vertices* and vertices on the internal circle are called *internal vertices*. We label external vertices with $1, 2, \dots$ in counter clockwise order; similarly for internal vertices, but using labels $1', 2', \dots$. Observe that with the convention that the labelling is in counter clockwise order, the previous condition is the same as marking an internal and an external vertex. Let e be an edge whose end-vertices belong to the boundary of $\mathcal{H}$. We say that e is an *ordinary edge* if its two end-vertices belong to the same circle (either $\mathbb{S}^1_1$ or $\mathbb{S}^1_2$). In particular, we say that e is a *boundary edge* if it is an ordinary edge between two consecutive vertices. We call e a *transversal edge* if it joins an internal vertex with an external vertex.

The terminology used for GFs is similar to the one used in the previous chapter: variable x is used when we are referring to vertices, variable z when we count faces and u when we are dealing with a certain parameter. We use variable x for internal vertices and variable y for external vertices if we need to make distinguish between them.

4.2 Integration lemmas

In order to obtain the GFs that appear in this chapter, we need to differenciate with respect to a set of variables. Later we integrate the resulting expression respect to a set of variables which is disjoint with the first set. The aim of this section is to get a simplified version of these formulas. In all the cases studied, the main tool is the chain rule. To simplify notation we write $(x_1, x_2, \dots, x_r) = \mathbf{x}_r$, $(u_1, u_2, \dots, u_r) = \mathbf{u}_r$ and $(X_1, X_2, \dots, X_r) = \mathbf{X}_r$. Additionally, the operator $\frac{\partial^r}{\partial x_1 \dots \partial x_r}$ is written in the simplified version $\frac{\partial^r}{\partial \mathbf{x}_r}$. The same can be said if we change $\mathbf{x}_r$ by the remaining vectors.

Lemma 4.1 *Consider the power series $H(\boldsymbol{u}_r, \boldsymbol{x}_r) = H(u_1, \dots, u_r, x_1, x_2, \dots, x_r)$, defined in terms of a function M by the formal equation:*

$$u_1 \dots u_r \frac{\partial^r}{\partial \boldsymbol{u}_r} H(\boldsymbol{u}_r, \boldsymbol{x}_r) = x_1 \dots x_r \frac{\partial^r}{\partial \boldsymbol{x}_r} M(u_1 h_1(x_1), \dots, u_r h_r(x_r)), \tag{4.1}$$

for some functions h_i, $i = 1, \dots, r$. Additionally, H and M satisfy the set of initial conditions $H(0, u_2, \dots, u_r, \boldsymbol{x}_r) = H(u_1, 0, \dots, u_r, \boldsymbol{x}_r) = \dots = H(u_1, u_2, \dots, 0, \boldsymbol{x}_r) = 0$, and $M(0, u_2, \dots, u_r) = M(u_1, 0, \dots, u_r) = \dots = M(u_1, u_2, \dots, 0) = 0$. Then,

$$H(\boldsymbol{u}_r, \boldsymbol{x}_r) = \left(\prod_{i=1}^{r} \frac{x_i}{h_i(x_i)} \frac{\partial}{\partial x_i} h_i(x_i) \right) M(u_1 h_1(x_1), \dots, u_r h_r(x_r)).$$

Proof. We start by developing the right-hand side of Equation (4.1). Denote by $\frac{\partial M}{\partial X_i}$ the derivative of M respect its ith variable. We can express Equation (4.1) in the following way:

$$u_1 \dots u_r \frac{\partial^r}{\partial \mathbf{u}_r} H(\mathbf{u}_r, \mathbf{x}_r) = \left(\prod_{i=1}^{r} x_i \frac{\partial}{\partial x_i} u_i h_i(x_i) \right) \frac{\partial^r}{\partial \mathbf{X}_r} M(\mathbf{X}_r) \bigg|_{X_i = u_i h_i(x_i)}.$$

Cancelling $u_1 \dots u_r$, and integrating with respect to u_r, we obtain the identity

$$\frac{\partial^{r-1}}{\partial \mathbf{u}_{r-1}} H(\mathbf{u}_r, \mathbf{x}_r) + F_r(\mathbf{u}_{r-1}, \mathbf{x}_r) = \left(\prod_{i=1}^{r} x_i \frac{\partial h_i(x_i)}{\partial x_i} \right) \frac{\partial^{r-1}}{\partial \mathbf{X}_{r-1}} \left(\int_0^{u_r} \frac{\partial M}{\partial X_r} \bigg|_{X_r = s_r h_r(x_r)} ds_r \right) \bigg|_{X_i = u_i h_i(x_i)},$$

for a certain function F_r. Applying the change of variables $X_r = s_r h_r(x_r)$, $dX_r = h_r(x_r)ds_r$, the integral in the right hand side can be written in the following way:

$$\frac{1}{h_r(x_r)}\left(M\left(u_1h_1(x_1),\ldots,u_rh_r(x_r)\right) - M\left(u_1h_1(x_1),\ldots,u_{r-1}h_{r-1}(x_{r-1}),0\right)\right).$$

Observe that $M(u_1h_1(x_1),\ldots,u_{r-1}h_{r-1}(x_{r-1}),0)$ is equal to 0 by the initial conditions of M. That is, $M(u_1,u_2,\ldots,u_{r-1},0) = 0$. Finally, writing $u_r = 0$ we conclude that F_r is identically equal to 0. To finish the proof, we apply the same argument inductively over the rest of the variables. □

The next lemma is a variation of the previous one, and is used in order to get compact expressions in Sections 4.5 and 4.6. The proof is another application of the chain rule.

Lemma 4.2 *Let $f_{i,j}(r,s)$ be a set of functions in terms of r and s, and consider the (possibly infinite) vector whose components are $f_{i,j}(r,s)z_{i,j}$. We denote this vector by $\mathbf{F}(r,s)\boldsymbol{Z}$, and the vector whose components are $z_{i,j}$ by $\boldsymbol{Z}$. Let $H(\mathbf{F}(r,s)\boldsymbol{Z},x,y)$ be defined by the formal equation*

$$rs\frac{\partial^2}{\partial r\partial s}H(\mathbf{F}(r,s)\boldsymbol{Z},x,y) = xy\frac{\partial^2}{\partial x\partial y}M(\boldsymbol{Z},rg_1(x),sg_2(y)), \tag{4.2}$$

where M, g_1, g_2 are given functions with initial conditions $M(\boldsymbol{Z},0,y) = M(\boldsymbol{Z},x,0) = 0$, and $H(\mathbf{F}(0,s)\boldsymbol{Z},x,y) = H(\mathbf{F}(r,0)\boldsymbol{Z},x,y) = 0$. Then, $H(\mathbf{F}(r,s)\boldsymbol{Z},x,y)$ can be written in the form

$$H(\mathbf{F}(r,s)\boldsymbol{Z},x,y) = \frac{xy}{g_1(x)g_2(y)}\frac{\partial g_1(x)}{\partial x}\frac{\partial g_2(y)}{\partial y}M(\boldsymbol{Z},rg_1(x),sg_2(y)). \tag{4.3}$$

Proof. We omit part of the details. The proof is obtained by the same the arguments as in the proof of Lemma 4.1. Denote by M_{12} the second derivative of M with respect to the last two variables. Equation (4.2) can be written then in the form:

$$\frac{\partial^2}{\partial r\partial s}H(\mathbf{F}(r,s)\mathbf{Z},x,y) = xy\frac{\partial g_1(x)}{\partial x}\frac{\partial g_2(y)}{\partial y}M_{12}(\mathbf{Z},rg_1(x),sg_2(y)).$$

Integrating with respect to r gives the equation

$$\begin{aligned}&\frac{\partial}{\partial s}H(\mathbf{F}(r,s)\mathbf{Z},x,y) - \frac{\partial}{\partial s}H(\mathbf{F}(0,s)\mathbf{Z},x,y)\\ =\ &\frac{xy}{g_1(x)}\frac{\partial g_1(x)}{\partial x}\frac{\partial g_2(y)}{\partial y}\left(M_2(\mathbf{Z},rg_1(x),sg_2(y)) - M_2(\mathbf{Z},0,sg_2(y))\right).\end{aligned}$$

Applying the initial conditions for M and H we obtain

$$\frac{\partial}{\partial s}H(\mathbf{F}(r,s)\mathbf{Z},x,y) = \frac{xy}{g_1(x)}\frac{\partial g_1(x)}{\partial x}\frac{\partial g_2(y)}{\partial y}\left(M_2(\mathbf{Z},rg_1(x),sg_2(y))\right).$$

The same argument applied on the variable s gives rise to Equation (4.3). □

4.3 Simplicial decompositions

In this section we obtain the enumeration for simplicial decompositions. A *triangulation* of the cylinder is a simplicial decomposition of $\mathcal{H}$, such that all the vertices lie on the boundary of $\mathcal{H}$. As in the case of the projective plane, a decomposition into triangles may not verify the simplicial condition. In Figure 4.1 a triangulation of the cylinder and a decomposition into triangles which is not a simplicial decomposition are shown.

Minimal triangulations of $\mathcal{H}$ have 6 vertices (3 on every circle), or 6 triangles (using Euler's relation); in fact, in a triangulation the number of vertices on each circle is at least 3. Hence, a triangulation

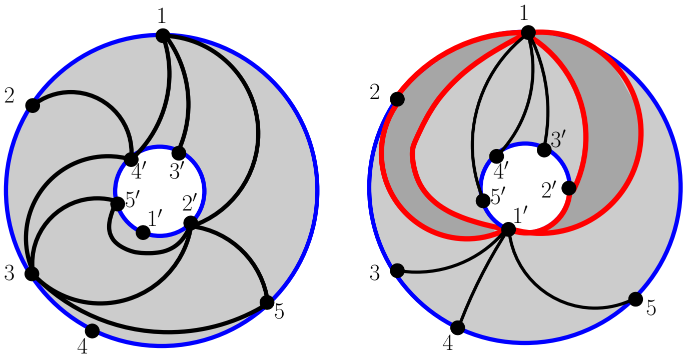


Figure 4.1 The first decomposition is simplicial, but the second is not: the intersection of the triangles $\triangle 121'$ and $\triangle 11'2'$ is the set $\{1, 1'\}$, which is not a simplex.

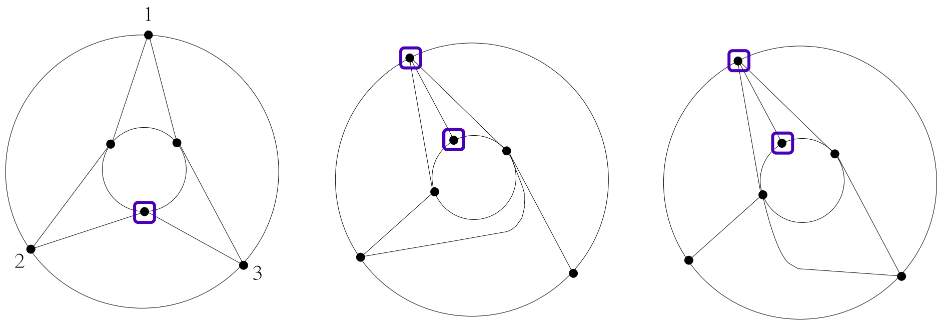


Figure 4.2 Minimal simplicial decompositions of the cylinder. The first one is totally symmetric, so there are 3 ways to label the vertex which is marked. In the second and third case, we need to label vertices that are shown. So we can label them in $3 \times 3 = 9$ ways in each case.

has at least 6 triangles. These minimal triangulations are shown in Figure 4.2. There are 21 different minimal simplicial decompositions.

We classify triangles according to their edges. A triangle has either three ordinary edges or two transversal edges. We call this triangles *ordinary triangles* and *cyclic triangles*, respectively. We say that a triangulation is *cyclic* if it is composed only by cyclic triangles. Let $M(x, y)$ the generating function associated to the family of cyclic triangulations, where x and y mark internal and external vertices, respectively. In the next lemma we count the number of cyclic triangulations, obtaining an explicit expression for $M(x, y)$.

Lemma 4.3 *The GF associated to cyclic triangulations of the cylinder is*

$$\begin{aligned} M(x,y) &= 2\,\frac{x^2y^3(y^2-3y+3)}{(y-1)^3} + \frac{(2y^3-4y^2+2y-1)y}{(x-1)(y-1)^2} + \frac{y}{x+y-1} - \\ &\quad \frac{y(y-1)}{(x+y-1)^2} + \frac{y(-1+4y^3-8y^2+4y)}{(x-1)^2(y-1)^2} + 2\frac{y^2}{(x-1)^3} \\ &= 21x^3y^3 + (48x^4y^3 + 48x^3y^4) + (90x^5y^3 + 124x^4y^4 + 90x^3y^5) + \dots, \end{aligned} \tag{4.4}$$

where x, y mark the number of internal and external vertices, respectively.

Proof. Fix n labelled vertices on the internal circle and m vertices on the external circle. Denote them by $1', 2', \ldots, n'$ and $1, 2, \ldots, m$, respectively. Consider the edge $\overline{12}$, and let x' be the third vertex in the triangle containing $\overline{12}$, which is an internal vertex (all triangles are cyclic). The topological space that results from $\mathcal{H}$ by removing the interior of $\triangle 12x'$ and the edge $\overline{12}$ without erasing vertices 1 and 2 is a rectangle with two extremal vertices identified. Its four corners are vertices 1, 2 and two copies of x' (see Figure 4.3). The points in the internal circle lie on the side defined by $\overline{x'x'}$, and points on the external circle lie on the side defined by the vertices 1 and 2, as shown in Figure 4.3.

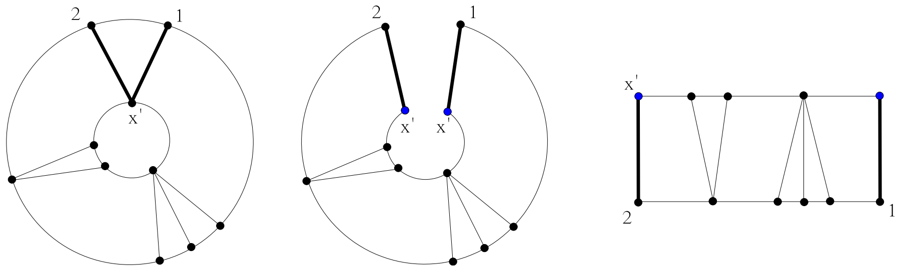


Figure 4.3 The topological space after erasing the interior of $\triangle 12x'$.

The problem has been translated into a problem of enumeration on a rectangle, with some restrictions. There are $m \geq 3$ vertices on the edge defined by the extremal vertices $1, 2$, and $n + 1 \geq 4$ vertices on the edge defined by the two copies of x', including in both cases the corners. We can suppose that $x' = 1'$ and, by symmetry, multiply by n the resulting number of triangulations. We associate to every triangulation a binary word of length $m + n - 1$ (the total number of vertices minus 2) as follows: if the corresponding triangle has its boundary edge on the upper boundary, we write a 0, otherwise we write a 1. Then the binary word is obtained by writing this sequence from left to right. It is clear that every binary word has exactly m zeroes, because this is the number of boundary edges in the upper boundary. This is shown in the left hand side of Figure 4.4. Conversely, we can always construct a triangulation from a binary word of this type. As a conclusion, there is a bijection between the number of triangulations of the rectangle, and the number of binary words of length $m + n - 1$ with n zeros. This number is $\binom{m+n-1}{n}$.

Let us count now the number of forbidden triangulations (i.e. the ones which do not satisfy the simplicial condition). A forbidden decomposition intro triangles appears because we introduce a multiple edge of the form $\overline{1'a}$. This can be done in m ways, depending on the choice of a. An example is shown in the right hand side of Figure 4.4.

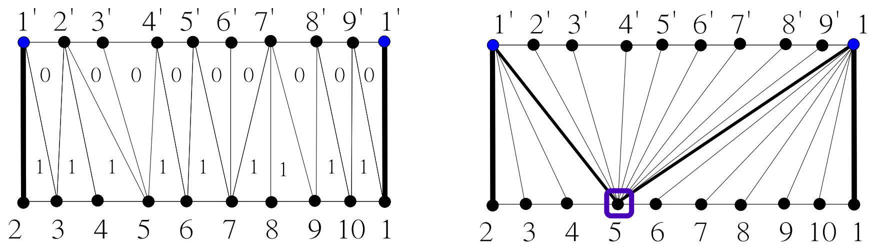


Figure 4.4 On the left, a valid triangulation of the band, with the corresponding encoding. On the right, a forbidden triangulation.

Thus we obtain the number $n\left(\binom{m+n-1}{n} - m\right)$, which gives the following GF

$$M(x, y) = \sum_{n,m \geq 3} \left(n\binom{m+n-1}{n} - nm \right) x^n y^m.$$

Writing $n = k - m$ (k is the number of triangles in the triangulation) we have

$$M(x, y) = \sum_{k \geq 6} x^k \sum_{m=3}^{k-3} \left(n\binom{k-1}{n} - n(k-n) \right) \left(\frac{y}{x}\right)^m.$$

A simple computation gives (4.4) as claimed. □

Let $H(u, x, y)$ be the GF associated to triangulations of the cylinder, where x marks internal vertices, y external vertices, and u cyclic triangles. The following theorem gives $H(u, x, y)$ in terms of $M(x, y)$:

Theorem 4.4 *The generating function for triangulations of a cylinder is*

$$H(u, x, y) = \frac{1 - xC(x)}{1 - 2xC(x)} \frac{1 - yC(y)}{1 - 2yC(y)} M\left(uxC(x), uyC(y)\right), \tag{4.5}$$

where $C(x)$ is the Catalan function, $M(x, y)$ is the rational function defined in Lemma 4.3, x, y mark internal and external vertices, and u marks cyclic triangles.

Proof. Denote by $H_0(w, v, x, y)$ the GF of triangulations of $\mathcal{H}$, where w is an additional variable associated to cyclic triangles with exactly one external vertex. A similar definition is stated for v, but referred to the internal circle. We call these cyclic triangles of *type* w and of *type* v, respectively. In particular, we want to obtain an expression for $H(u, x, y) = H_0(u, u, x, y)$. Consider the class of triangulations of $\mathcal{H}$, where a triangle of type w and a triangle of type v are pointed, whose GF is

$$wv \frac{\partial}{\partial v} \frac{\partial}{\partial w} H_0(w, v, x, y). \tag{4.6}$$

This GF can be obtained in another way. Consider a cyclic triangulation and attach to each boundary edge a planar triangulation (i.e. write $x \leftarrow wxC(x)$ and $y \leftarrow vyC(y)$ on $M(x, y)$). After this substitution, mark an internal and an external vertex. An example is shown in the right hand side of Figure 4.5.

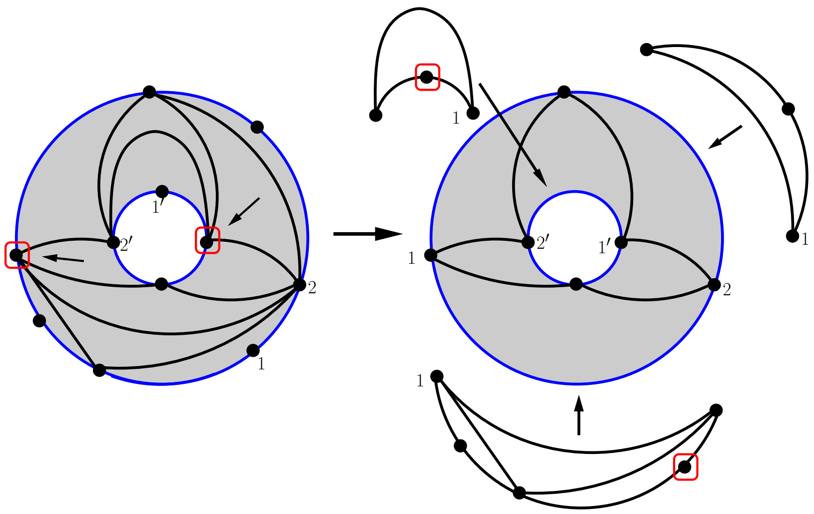


Figure 4.5 Equality between labelled classes. Labellings are only shown in the external circle.

This construction leads to the expression

$$xy \frac{\partial}{\partial x} \frac{\partial}{\partial y} M(wxT(x), vyT(y)). \tag{4.7}$$

Observe that the initial enumeration on the cyclic triangulation shows which triangles of type w and v are pointed, and the labelling of vertices made in the second step provides the final cyclic labelling on the boundaries. Consequently, we obtain also the generating function for the class of triangulations where a pair of cyclic triangles (of type w and v) are pointed. If we match Equation (4.6) and Equation (4.7) we obtain

$$vw\frac{\partial}{\partial v}\frac{\partial}{\partial w}H_0(w,v,x,y) = xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}M(wxT(x), vyT(y)). \tag{4.8}$$

We apply Lemma 4.1 for $r=2$. We are under the hypothesis of the lemma because every triangulation has triangles of type w and triangles of type v; in other words, $H_0(0,v,x,y) = H_0(u,0,x,y) = 0$, and $M(0,y) = M(x,0) = 0$. We obtain the expression

$$H_0(w,v,x,y) = \frac{1-xC(x)}{1-2xC(x)}\frac{1-yC(y)}{1-2yC(y)}M(wxC(x), vyC(y)),$$

from which we obtain the GF (4.5) writing $w=v=u$ (i.e., $H(u,x,y) = H_0(u,u,x,y)$). □

The GF for the number of triangulations of the cylinder in terms of the number of triangles is derived directly from the previous theorem:

Corollary 4.5 *The GF of triangulations of a cylinder, counted by the number of triangles, is*

$$\begin{aligned} H(z) &= \frac{-8z^5+18z^4-52z^3+20z^2+2z-1}{z\,(1-4z)^2}C(z) + \frac{8z^5-2z^4+33z^3-20z^2-z+1}{z\,(1-4z)^2} \\ &= 21z^6+264z^7+2134z^8+14108z^9+\ldots, \end{aligned} \tag{4.9}$$

where $C(z)$ is the Catalan function.

Proof. Write $x=y=z$ and $u=v=1$ in the function obtained in Theorem 4.4. Recall also that the number of triangles in a simplicial decomposition of the cylinder is equal to the total number of vertices. □

4.4 Fundamental cyclic dissections

In order to obtain formulas for general families of dissections, we need to deal with a special class of decompositions. This is the goal of this section. The arguments are quite similar (but more involved) to those in Section 4.3.

We say that a quadrangle with two vertices in each boundary of the cylinder is called a *fundamental quadrangle*. A *fundamental cyclic* dissection is a cellular decomposition of the cylinder into fundamental quadrangles and cyclic triangles, where points on the internal circle *are not labelled*, and two cells intersect only at a vertex or at an edge. To obtain the corresponding GF, we use the variable Z for fundamental quadrangles, Y for cyclic triangles such that their basis lies on the internal circle, and X for cyclic triangles whose basis lies in the external circle. An example of a fundamental cyclic dissection is shown in Figure 4.6.

Lemma 4.6 *The GF of the family of fundamental cyclic dissections is*

$$J(X,Y,Z) = ZJ_1(X,Y,Z) + XJ_2(X,Y,Z) = ZJ_1 + XJ_2,$$

where

$$\begin{aligned} J_1 &= \frac{1}{1-X-Y-Z} - \frac{1}{1-Y} - \frac{X+Z}{(1-Y)^2} - \frac{1}{1-X} - \frac{Z+Y}{(1-X)^2} + \\ &\quad 1+Y+Z+X+2YX, \\ J_2 &= J_1(X,Y,Z) - \frac{(Z+Y)^2}{(1-X)^3} + Y^2 + 2ZY + 3XY^2 - \\ &\quad \frac{X^2}{(1-X)^2}\frac{Y^3}{1-Y} - \frac{2X^2}{1-X}\frac{Y^3}{1-Y}. \end{aligned} \tag{4.10}$$

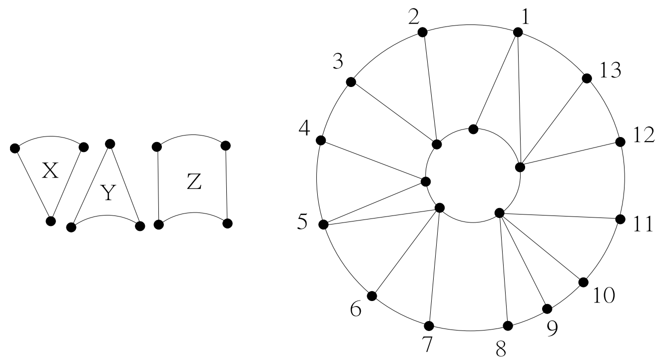


Figure 4.6 On the left, a pair of cyclic triangles and a fundamental quadrangle. On the right, a fundamental cyclic dissection of the cylinder; internal vertices are not labelled.

Proof. We obtain a combinatorial decomposition of $\mathcal{H}$ in terms of the polygon which contains the edge $\overline{12}$. This edge is contained either on a fundamental quadrangle (of type Z) or on a cyclic triangle (of type X). We consider these two cases separately.

First, suppose that $\overline{12}$ belongs to a fundamental quadrangle. Cutting $\mathcal{H}$ through the pair of transversal edges of this quadrangle, we obtain a rectangle. Points on the top of the rectangle correspond to points from the internal circle, and points on the bottom to points from the external circle. In this case, the number of dissections of the cylinder is equal to the number of dissections of this rectangle into cyclic triangles and fundamental quadrangles. This contribution is encapsulated in the generating function $ZJ_1(X,Y,Z)$, where $J_1(X,Y,Z)$ is the GF for the number of dissections into transversal triangles and fundamental quadrangles of the rectangle, and the factor Z corresponds to the root quadrangle. In order to obtain the enumeration, we count the total number of decompositions (dissections or not) and we substract the forbidden ones. In this particular case, the forbidden configurations are those where:

1. the number of vertices on the upper boundary of the rectangle is smaller than three;

2. the number of vertices on the lower boundary of the rectangle is smaller than three.

These configurations are forbidden since a dissection of the dissection of the cylinder has at least three internal vertices and three external vertices.

An arbitrary decomposition of the rectangle is simply a sequence of elements in the set $\{X,Y,Z\}$. From this sequence, we substract those which do not satisfy the conditions listed above. This argument uses the inclusion-exclusion method on GFs. The possible cases are shown in Figure 4.7. On the left of this figure, an arbitrary decomposition is shown. On the right, forbidden conditions. Below, pairs of forbidden conditions. A shaded polygon represents either a fundamental quadrangle or a cyclic triangle, and a white polygon is a sequence of fundamental quadrangles and cyclic triangles.

Table 4.1 shows the translation between these combinatorial specifications and the corresponding GFs.

Summing all the previous contributions, we obtain the first GF stated in Equation (4.10):

$$\begin{aligned} J_1(X,Y,Z) &= \frac{1}{1-X-Y-Z} - \frac{1}{1-Y} - \frac{X+Z}{(1-Y)^2} - \frac{1}{1-X} - \frac{Z+Y}{(1-X)^2} + \\ &\quad 1+Y+Z+X+2YX. \end{aligned}$$

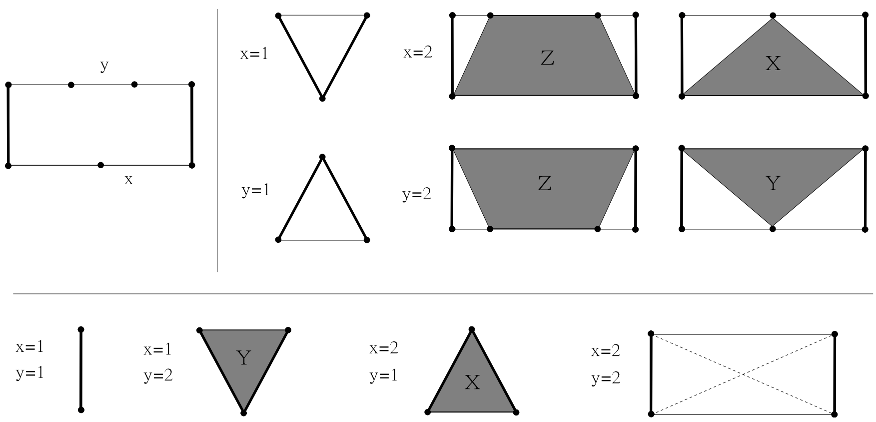


Figure 4.7 Forbidden configurations if the root is a fundamental quadrangle.

x	y	Sign	Structure	GF
		$+$	$\mathrm{Seq}(\{X,Y,Z\})$	$1/(1-X-Y-Z)$
1		$-$	$\mathrm{Seq}(\{Y\})$	$1/(1-Y)$
2		$-$	$\mathrm{Seq}(\{Y\}) \times \{Z,X\} \times \mathrm{Seq}(\{Y\})$	$(X+Z)/(1-Y)^2$
	1	$-$	$\mathrm{Seq}(\{X\})$	$1/(1-X)$
	2	$-$	$\mathrm{Seq}(\{X\}) \times \{Z,Y\} \times \mathrm{Seq}(\{X\})$	$(Y+Z)/(1-X)^2$
1	1	$+$	$\emptyset$	1
1	2	$+$	$\{Y\}$	Y
2	1	$+$	$\{X\}$	X
2	2	$+$	$\{Z\} \cup \{X\} \times \{Y\} \cup \{Y\} \times \{X\}$	$Z+2XY$

Table 4.1 Translation into GFs of the restrictions introduced in Figure 4.7.

Let us consider now the case where $\overline{12}$ belongs to a cyclic triangle. We cut the cylinder through its transversal edges. We obtain also a rectangle, but with two corners identified (recall the construction for cyclic triangulations). Dissections of $\mathcal{H}$ correspond to dissections of the rectangle with some restrictions (to be discussed later). In terms of GFs, this is encapsulated into $X\,J_2(X,Y,Z)$, where $J_2(X,Y,Z)$ is the GF of dissections of the rectangle into transversal triangles and fundamental quadrangles, with two points identified, and where the factor X encodes the root triangle. As in the previous case, we count the total number of decompositions (dissections or not), and substract those which are not dissections. In this case, forbidden decompositions are:

1. the number of vertices on the upper boundary is greater or equal than four. Observe that when we erase a triangle of type X, we are doubling a point on the internal circle;

2. the number of vertices on the lower boundary is greater or equal than three;

3. two cells of the decomposition do not intersect only at a vertex or at an edge.

Observe that all the forbidden cases in the previous situation (i.e., edge $\overline{12}$ belongs to a fundamental quadrangle) are also forbidden here. So we must substract from $J_1(X,Y,Z)$ configurations where $y = 3$ and the forbidden cases which appear as a consequence of the double point. Those are shown in Figure 4.8. In the first row, shaded polygons represent either cyclic triangles (Y) or fundamental quadrangles (Z). In the second row, the possible configurations for $x = 2$, $y = 3$ are shown. The third row shows forbidden decompositions coming from the condition of incompatibility with the initial transversal triangle. Table 4.2 translates these conditions int GFs (recall that the upper boundary corresponds with the external circle, and the lower boundary corresponds with the internal circle).

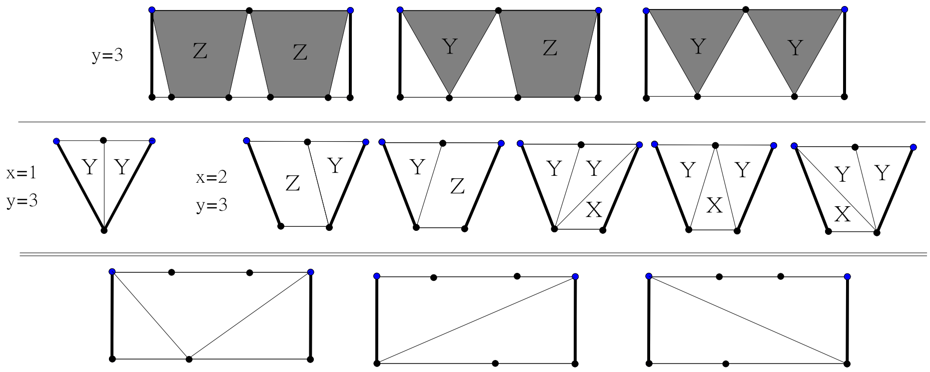


Figure 4.8 Forbidden configurations when the root is a fundamental quadrangle.

x	y	Sign	Structure	GF
	3	-	$\mathrm{Seq}\left(\{X\}\right)^3 \times \{Z, Y\}^2$	$(Z+Y)^2/(1-X)^3$
1	3	$+$	$\{Y\} \times \{Y\}$	Y^2
2	3	$+$	shown in Figure 4.8	$2YZ + 3XY^2$
		$-$	$\mathrm{Seq}\left(\{X\}\right)^2 \times \{Y\}^3 \times \mathrm{Seq}\left(\{Y\}\right)$	$X^2Y^3/((1-X)^2(1-Y))$
		$-$	$\{Y\}^2 \times \mathrm{Seq}\left(\{Y\}\right) \times \{X\}^2 \times \mathrm{Seq}\left(\{X\}\right)$	$X^2Y^3/((1-X)(1-Y))$
		$-$	$\{Y\}^2 \times \mathrm{Seq}\left(\{Y\}\right) \times \{X\}^2 \times \mathrm{Seq}\left(\{X\}\right)$	$X^2Y^3/((1-X)(1-Y))$

Table 4.2 Translation into GFs of the restrictions introduced in Figure 4.8.

Summing up each contribution, we obtain the second term in Equation (4.10):

$$\begin{aligned} J_2(X,Y,Z) &= J_1(X,Y,Z) - \frac{(Z+Y)^2}{(1-X)^3} + Y^2 + 2ZY + 3XY^2 - \\ &\quad \frac{X^2}{(1-X)^2}\frac{Y^3}{1-Y} - \frac{2X^2}{1-X}\frac{Y^3}{1-Y}. \end{aligned}$$

To conclude, we only need to add these contributions, obtaining the term $ZJ_1 + XJ_2$ as claimed. $\square$

4.5 Dissections into r-agons.

In this section we study cellular decompositions of the cylinder. We deal with the case where every cell has a constant degree. We say that a decomposition of the cylinder is a *r-dissection* (or r-agon dissection) if it is a cellular decomposition of the cylinder, where every cell has exactly r edges, all vertices belong either to the internal circle or to the external circle, and two cells intersect only at a vertex or at an edge. The case $r = 3$ corresponds to triangulations of the cylinder, which has been studied in Section 4.3. This implies that there are at least 3 vertices on every circle of the boundary. In what follows we write $r = k + 1$.

Just as we have done for triangulations, we start with cyclic families. A polygon is said to be *cyclic* if it has exactly two transversal edges. A *cyclic* $(k+1)-$*agon* dissection of the cylinder is a dissection where all polygons are cyclic. It is clear that a cyclic polygon can be obtained from either a cyclic triangle or a fundamental quadrangle adding vertices on its boundary. We use the variable u for cyclic polygons which come from a cyclic triangle of type X, and v for transversal polygons which arise from a cyclic triangle of type Y. We denote by z_s a cyclic polygon with $k+1$ vertices which arises from a fundamental quadrangle adding s vertices on the external circle. Consequently, z_s has $s+2$ vertices on the external circle and $k+1-s-2 = k-s-1$ vertices on the internal circle. As an example, in Figure 4.9 all possible transversal hexagons (with the corresponding variable which codifies them) are shown.

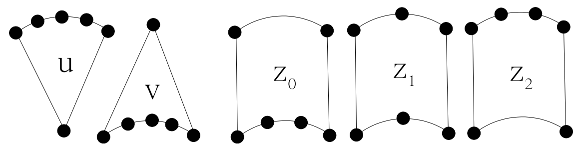


Figure 4.9 Transversal hexagons. The first and the second one are generated from triangles of type X and Y. Polygons z_0, z_1 and z_2 are generated from fundamental quadrangles (letter Z).

We start by computing the generating function for cyclic $(k+1)$-agon dissections. The main point is the use of the family of fundamental cyclic decompositions.

Proposition 4.7 *Let $K_1^{\{k+1\}}(u, v, z_0, \ldots, z_{k-3}, x, y)$ and $K_2^{\{k+1\}}(u, v, z_0, \ldots, z_{k-3}, x, y)$ be the GFs defined by the relation*

$$\begin{aligned} K_1^{\{k+1\}} &= xyJ_1\left(y^{k-1}u, x^{k-1}v, xy\left(\sum_{s=0}^{k-3} z_s y^s x^{k-3-s}\right)\right), \\ K_2^{\{k+1\}} &= yJ_2\left(y^{k-1}u, x^{k-1}v, xy\left(\sum_{s=0}^{k-3} z_s y^s x^{k-3-s}\right)\right), \end{aligned}$$

where J_1 and J_2 are defined in Lemma 4.6. Then, the GF of cyclic $(k+1)-$agon decompositions of a cylinder is

$$M_0^{\{k+1\}}(u, v, z_0, \ldots, z_{k-3}, x, y) = x\frac{\partial}{\partial x}\left(\left(\sum_{s=0}^{k-3}(s+1)z_s y^s x^{k-3-s}\right)K_1^{\{k+1\}} + u(k-1)y^{k-2}K_2^{\{k+1\}}\right),$$

where x marks internal vertices and y marks external vertices.

Proof. Fundamental cyclic dissections are counted using the variables X, Y and Z. We must substitute each polygon of a fundamental cyclic dissection by a transversal polygon, and count then the resulting number of vertices. We also need to put a mark on the internal circle, because we consider labelled cyclic dissections. In other words, to pass from fundamental cyclic dissections to cyclic dissections into $(k+1)-$agons we work in three consecutive steps:

1. Substitution of every cyclic triangle and every fundamental quadrangle by a cyclic $(k+1)-$agon, counted all of them by their number of vertices. This is done using the GFs J_1 and J_2 (i.e., rectangles with vertices on its edges), which do not take into account the root polygon of the final dissection.

2. Pointing process on the external circle. In this case we add new vertices.

3. Pointing process on the internal circle. Recall that the internal circle of a fundamental cyclic decomposition is not labelled.

Let us start with the first point. Consider the families which are counted by J_1 and J_2. We must substitute each variable X, Y and Z in order to deal with vertices. The process of substitution is made writing $X = y \cdot y^{k-2}u$, $Y = x \cdot x^{k-2}v$ and $Z = xy \cdot \sum_{s=0}^{k-3} z_s y^s x^{k-3-s}$ on J_1 and J_2. Observe that we construct a cyclic $(k+1)$-agon from a cyclic triangle of type u adding $k-2$ external vertices, hence the term y^{k-2}. Additionally, an extra factor y appears in order to take into account one of the two vertices of the initial triangle of type u (in all cases we consider vertices in counter clockwise order around the external circle). Similar arguments are made for Y. In the case of Z, we must distribute $k-3$ vertices between $\mathbb{S}_1^1$ and $\mathbb{S}_2^1$, hence the term $\sum_{s=0}^{k-3} z_s y^s x^{k-3-s}$. In all these cases we take into account a pair of vertices on each boundary (hence, the factor xy). Finally, there is a single internal vertex (in family J_1) or a pair of vertices on each boundary (in family J_2) which are not counted. Consequently we multiply J_1 and J_2 by xy and y, respectively. Taking these vertices we recover the cylinder from the rectangle.

The next step consists in marking an external vertex to get the cyclic labelling on the external circle. In other words, we need to decide which vertex is the vertex whose label is "1". This vertex is induced by the root polygon in the fundamental cyclic dissection (which has $\overline{12}$ as an edge). If the root polygon in the fundamental cyclic dissection is a fundamental quadrangle, and we are adding s vertices on the external boundary, we have $(s+1)$ ways of choosing the vertex whose label is 1, and we obtain the sum $\sum_{s=0}^{k-3}(s+1)z_s y^s x^{k-3-s}$. Observe that we are taking into account extremal vertices of the root polygon.

Otherwise, we have a cyclic triangle of type u, so we must add $k-2$ vertices on the external boundary, and we have and additional term of the form: $u(k-1)y^{k-2}$ (we have $k-1$ possibilities for choosing vertex 1).

To conclude, we apply the pointing operator $x\frac{\partial}{\partial x}$ to obtain the vertex which is labelled with $1'$. Since we are considering labellings in counter clockwise order, the position of vertex $1'$ determines the remaining labels on the internal circle. $\square$

To obtain the final enumeration of the family of $(k+1)$-agon dissections, we only need a simplified version of the GF stated in the previous proposition. This is what is shown in the following corollary:

Corollary 4.8 *Let $L_1^{\{k+1\}}(u,x,y)$ and $L_2^{\{k+1\}}(u,x,y)$ be defined by*

$$\begin{aligned} L_1^{\{k+1\}}(u,x,y) &= K_1^{\{k+1\}}(u,u,\ldots,u,x,y), \\ L_2^{\{k+1\}}(u,x,y) &= K_2^{\{k+1\}}(u,u,\ldots,u,x,y), \end{aligned}$$

where $K_1^{\{k+1\}}$ and $K_2^{\{k+1\}}$ are defined in Proposition 4.7. Then, the GF of cyclic $(k+1)$-agon dissections of a cylinder is

$$M_1^{\{k+1\}}(u,x,y) = ux\frac{\partial}{\partial x}\left(\left(\frac{\partial}{\partial y}\frac{x^{k-1}-y^{k-1}}{x-y}\right)L_1^{\{k+1\}}(u,x,y) + u(k-1)y^{k-2}L_2^{\{k+1\}}(u,x,y)\right).$$

where x, y mark internal and external vertices, and u marks the total number of cyclic polygons.

Proof. Write $v = z_0 = z_1 = \cdots = z_{k-3} = u$ in the equation stated in Proposition 4.7. □

Finally, setting $y = x$ we obtain the GF in terms of the total number of vertices. In this case, the expression is explicit:

Corollary 4.9 *The GF of cyclic $(k+1)$-agon decompositions of a cylinder is*

$$M^{\{k+1\}}(u,x) = \frac{(k-1)f_k(u,x)}{12(1-kux^{k-1})^2(1-ux^{k-1})^4} \tag{4.11}$$

where x marks vertices and u marks transversal polygons, and $f_k(u,x)$ is

$$\begin{aligned} f_k(u,x) \quad = \quad & \left(2x^{3k-3}(4k-3)(k-2)^3\right)u^3 - \\ & \left(x^{4k-4}(k-2)^2(5k^3-43k^2-42k+66)\right)u^4 - \\ & \left(4x^{5k-5}(k-2)(6k^4+11k^3-86k^2-13k+72)\right)u^5 + \\ & \left(2x^{6k-6}(22k^5-122k^4-83k^3+677k^2-206k-258)\right)u^6 - \\ & \left(2x^{7k-7}(10k^5-166k^4+243k^3+426k^2-392k-96)\right)u^7 + \\ & \left(x^{8k-8}(5k^5-215k^4+636k^3-20k^2-360k-24)\right)u^8 + \\ & \left(4x^{9k-9}k(13k^3-62k^2+36k+12)\right)u^9 + \left(24x^{10k-10}(k-1)k^2\right)u^{10}. \end{aligned}$$

Proof. Write $y = x$ in the result of Corollary 4.8, and after simplifying the expressions (with `Maple`), we get the result. □

Notice that $M^{\{k+1\}}(u,x)$ is a function of x^{k-1}. This is because, from Euler's relation, the total number of vertices is a multiple of $k-1$. To conlcude this section, we obtain a closed formula for the number of $(k+1)$-agon dissections of the cylinder in terms of the GF for cyclic dissections.

Theorem 4.10 *The GF of $(k+1)$-agon dissections of a cylinder is*

$$\begin{aligned} H^{\{k+1\}}(u,x,y) \quad = \quad & \frac{1-(xC_{k+1}(x^{k-1}))^{k-1}}{1-k(xC_{k+1}(x^{k-1}))^{k-1}}\,\frac{1-(yC_{k+1}(y^{k-1}))^{k-1}}{1-k(yC_{k+1}(y^{k-1}))^{k-1}}\cdot \\ & M_1^{\{k+1\}}\left(u, xC_{k+1}(x^{k-1}), yC_{k+1}(y^{k-1})\right). \end{aligned} \tag{4.12}$$

where x marks internal vertices, y marks external vertices and u marks transversal $(k+1)$-agons, $M_1^{\{k+1\}}(u,x,y)$ is defined in Corollary 4.8 and $C_{k+1}(z)$ is the GF associated to plane $(k+1)$-agon dissections (defined in Section 1.2.1).

Proof. Denote by $H_0^{\{k+1\}}(u,v,z_0,z_1,\ldots,z_{k-3},x,y)$ the GF of $(k+1)-$agon dissections of the cylinder, where x marks internal vertices, y marks external vertices, and the rest of the variables take into account the type of cycle polygon in the dissection. The argument is the same as in triangulations, and we omit some details. For a $(k+1)-$agon dissection of the cylinder, we consider a pair of pointed vertices (one on each boundary) on the induced $(k+1)-$agon cyclic dissection. These family of dissections can be counted in two different ways.

Let r,s be variables marking external and internal vertices respectively, that are induced from the underlying cyclic $(k+1)$-agon decomposition. The same argument used in triangulations gives

$$\begin{aligned} & rs\frac{\partial}{\partial r}\frac{\partial}{\partial s}H_0^{\{k+1\}}(ur^{k-1}, vs^{k-1}, z_0rs^{k-2}, z_1r^2s^{k-3}, \ldots, z_{k-3}r^{k-2}s, x, y) \\ = \quad & xy\frac{\partial}{\partial x}\frac{\partial}{\partial y}M_0^{\{k+1\}}\left(u, v, z_0, z_1, \ldots, z_{k-3}, rxC_{k+1}(x^{k-1}), syC_{k+1}(y^{k-1})\right), \end{aligned}$$

where $M_0^{\{k+1\}}$ is considered in Proposition 4.7. Applying Lemma 4.2 we reduce the previous equation to

$$\begin{aligned}
& H_0^{\{k+1\}}(ur^{k-1}, vs^{k-1}, z_0 rs^{k-2}, z_1 r^2 s^{k-3}, \dots, z_{k-3} r^{k-2} s, x, y) \\
= \; & \frac{1}{C_{k+1}(x^{k-1})} \frac{1}{C_{k+1}(y^{k-1})} \left(\frac{\partial}{\partial x} x C_{k+1}(x^{k-1})\right) \left(\frac{\partial}{\partial y} y C_{k+1}(y^{k-1})\right) \cdot \\
& M_0^{\{k+1\}}\left(u, v, z_0, z_1, \dots, z_{k-3}, rxC_{k+1}(x^{k-1}), syC_{k+1}(y^{k-1})\right).
\end{aligned}$$

Setting $z_0 = z_1 = \cdots = z_{k-3} = v = u$, and $r = s = 1$, we get

$$\begin{aligned}
& H^{\{k+1\}}(u, x, y) \\
= \; & \frac{1}{C_{k+1}(x^{k-1})} \frac{1}{C_{k+1}(y^{k-1})} \left(\frac{\partial}{\partial x} x C_{k+1}(x^{k-1})\right) \left(\frac{\partial}{\partial y} y C_{k+1}(y^{k-1})\right) \cdot \\
& M_1^{\{k+1\}}\left(u, xC_{k+1}(x^{k-1}), yC_{k+1}(y^{k-1})\right).
\end{aligned}$$

Using $C_{k+1}(x^{k-1}) = 1 + x^{k-1} C_{k+1}(x^{k-1})^k$ from Section 1.2.1, we obtain (4.12). □

To conclude, if we set $y = x$ and $u = 1$ in Equation (4.12) (using the expression for $M^{\{k+1\}}$ as stated in Corollary 4.9) we obtain the generating function in terms of the total number of vertices:

$$H^{\{k+1\}}(x) = \left(\frac{1 - x^{k-1} C_{k+1}(x^{k-1})^{k-1}}{1 - kx^{k-1} C_{k+1}(x^{k-1})^{k-1}}\right)^2 M^{\{k+1\}}\left(xC_{k+1}(x^{k-1})\right), \tag{4.13}$$

where $M^{\{k+1\}}(x) = M^{\{k+1\}}(1, x)$ and $H^{\{k+1\}}(x) = H^{\{k+1\}}(1, x, x)$. In particular, this expression can be written in terms of x^{k-1}, which gives the enumeration of dissections in terms of the number of cells in the decomposition. $M^{\{k+1\}}(u, x)$ can be written in terms of x^{k-1} (see the corresponding expression in Corollary 4.9), hence the GF $M^{\{k+1\}}(u, xC_{k+1}(x^{k-1}))$ can be written in terms of

$$x^{k-1} C_{k+1}(x^{k-1})^{k-1} = zC_{k+1}(z)^{k-1} = (C_{k+1}(z) - 1)/C_{k+1}(z).$$

This GF enumerates decompositions into $(k+1)$-agons in terms of the number of cells, and is the one used in Section 4.7 in order to obtain asymptotic results. To show some examples, for $k = 3$ we obtain dissections into quadrangles, which has a development of the form

$$H^{\{4\}}(x) = 3x^6 + 112x^8 + 1902x^{10} + 23396x^{12} + 243698x^{14} + 2299064x^{16} + \dots,$$

and for $k = 4$, we obtain

$$H^{\{5\}}(x) = 52x^9 + 1874x^{12} + 37448x^{15} + 586001x^{18} + 8048356x^{21} + \dots$$

In both cases, recall that x marks vertices.

4.6 Unrestricted dissections

A similar approach as in the case of r-agon dissections can be used to obtain the GF for general dissections of the cylinder. By an *unrestricted dissection* we mean a dissection of the cylinder where all vertices lie on its boundary, and the degree of each cell is unrestricted. As in the previous section, we first obtain the GF for cyclic unrestricted dissections, and then we attach planar unrestricted dissection to boundary edges.

In what follows variable $z_{i,j}$ marks cyclic polygons with i external vertices and j internal vertices. As before, x marks internal vertices and y marks external vertices. We write $\mathbf{Z}$ for the infinite vector $(z_{1,2}, z_{2,1}, z_{2,2}, z_{2,3}, \dots)$. We omit the proofs of the forthcoming results, because they are a straightforward modification of the results in the previous section. We start with cyclic unrestricted dissections.

Proposition 4.11 *Let K_1^D and K_2^D be*

$$K_1^D = xyJ_1\left(\sum_{i=2}^{\infty} z_{i,1}y^{i-1}, \sum_{j=2}^{\infty} z_{1,j}x^{j-1}, \sum_{i=2}^{\infty}\sum_{j=2}^{\infty} z_{i,j}y^{i-1}x^{j-1}\right),$$
$$K_2^D = yJ_2\left(\sum_{i=2}^{\infty} z_{i,1}y^{i-1}, \sum_{j=2}^{\infty} z_{1,j}x^{j-1}, \sum_{i=2}^{\infty}\sum_{j=2}^{\infty} z_{i,j}y^{i-1}x^{j-1}\right),$$

where J_1 and J_2 are defined in Proposition 4.6. The GF of cyclic unrestricted dissections of a cylinder is

$$M_0^D(\boldsymbol{Z}, x, y) = x\frac{\partial}{\partial x}\left(\left(\sum_{j=2}^{\infty}\sum_{i=2}^{\infty}(i-1)y^{i-2}x^{j-2}z_{i,j}\right)K_1^D + \sum_{i=2}^{\infty}(i-1)y^{i-2}z_{i,1}K_2^D\right).$$

The previous result is not useful in practical terms, because there are infinitely many variables. The next corollary is a simple consequence of the previous proposition, and the expressions obtained are simpler.

Corollary 4.12 *Let L_1^D and L_2^D be*

$$\begin{aligned} L_1^D &= K_1^D(u,u,u,\ldots,x,y) = xyJ_1\left(u\frac{y}{1-y}, u\frac{x}{1-x}, u\frac{xy}{(1-x)(1-y)}\right),\\ L_2^D &= K_2^D(u,u,u,\ldots,x,y) = yJ_2\left(u\frac{y}{1-y}, u\frac{x}{1-x}, u\frac{xy}{(1-x)(1-y)}\right). \end{aligned}$$

The GF of cyclic dissections of a cylinder is

$$M_1^D(u,x,y) = \frac{ux}{(1-y)^2}\frac{\partial}{\partial x}\left(\frac{1}{1-x}L_1^D + L_2^D\right),$$

where x marks internal vertices, y marks external vertices and u marks cyclic polygons.

The final step consists in setting $y = x$ in the previous corollary.

Corollary 4.13 *The GF of cyclic unrestricted dissections is*

$$M^D(u,x) = \frac{g(u,x)}{(1-2x+x^2-2ux+ux^2)^2(1-x)^6(-1+x+ux)^4},$$

where

$$\begin{aligned} g(u,x) \;=\;& 3x^6(z-1)^6u^3 + x^6(25x^2-54x+27)(x-1)^4u^4 + \\ & 2x^6(43x^3-131x^2+117x-27)(x-1)^3u^5 + \\ & x^6(151x^4-568x^3+684x^2-276x+21)(x-1)^2u^6 + \\ & x^7(x-1)(145x^4-633x^3+924x^2-500x+72)u^7 + \\ & 2x^8(38x^4-186x^3+312x^2-204x+41)u^8 + \\ & 4x^9(x-2)(5x^2-12x+5)u^9 + 2x^{10}(x-2)^2u^{10}, \end{aligned}$$

and x marks vertices and u cyclic polygons.

We are now ready to obtain the GF for unrestricted dissections of the cylinder. The main ideas are as in Theorems 4.4 and 4.10. The difference is the introduction of a variable w which take into account the number of regions. In the next statement, $D(w,x)$ denotes the GF of dissections of the disk, as in Section 1.2.1.

Theorem 4.14 *The GF of dissection of the cylinder is*

$$H^D(u,w,x,y) \;=\; \frac{x^2y^2}{D(w,x)D(w,y)}\frac{\partial}{\partial x}\left(\frac{D(w,x)}{x}\right)\frac{\partial}{\partial y}\left(\frac{D(w,y)}{y}\right)\cdot M_1^D\left(uw,\frac{D(w,x)}{x},\frac{D(w,y)}{y}\right), \tag{4.14}$$

where $M_1^D(u,x,y)$ is as in Corollary 4.12, x,y mark internal and external vertices, u marks cyclic polygons, and w marks regions.

If we set $y=x$ and $u=1$, then $H^D(w,x)=H^D(1,w,x,x)$ is the GF of unrestricted dissections of the cylinder in terms of vertices and regions. The explicit expression for this GF can be obtained in terms of the formula for M^D from Corollary 4.13:

$$\begin{aligned} H^D(w,x) \;=\; & (3w^3+27w^4+54w^5+21w^6)x^6+ \\ & (24w^3+264w^4+792w^5+840w^6+264w^7)x^7+ \\ & (108w^3+1438w^4+5932w^5+10422w^6+8000w^7+2134w^8)x^8+\ldots \end{aligned}$$

In particular, the sequence of coefficients of the form w^rx^r are the same as those obtained in Corollary 4.5 (recall that triangulations maximizes the number of faces for a fixed number of vertices in an unrestricted dissection).

4.7 Asymptotic enumeration

In this section we obtain asymptotic estimates for the number of dissections studied in this chapter. It turns out that they satisfy a general formula of the form $c\cdot n\cdot\rho_\Delta^{-n}(1+O(n^{-1}))$, where c is a constant and ρ_Δ is the radius of convergence of the corresponding GF of planar configurations. The development is similar to that in Section 3.5. The analysis is made over the expressions obtained in Corollary 4.5 and Equations (4.13), (4.14). The asymptotic enumeration is referred to the number of vertices in triangulations and unrestricted dissections. In the case of dissections into $(k+1)$-agons, something more must be said: recall that Equation (4.13) can be written in terms of $z=x^{k-1}$, where z counts the number of faces in the dissection. For our purpose is more natural this parameter (number of faces) than the number of vertices. Consequently, all the asymptotic estimates are given in terms of the number of cells instead of the number of vertices.

In all cases GFs can be expressed in the form $F\left(C_\Delta(z)\right)G_\Delta(z,C_\Delta(z))$, where G_Δ is a bivariate rational function and $C_\Delta(z)$ is the corresponding GF for planar dissections. We write ρ_{k+1} for the radius of convergence of the GF of planar $(k+1)$-agon dissections, and ρ_D for planar unrestricted dissections. In all cases the radius of convergence in planar dissections is the same as in the corresponding dissection of the cylinder. More concretely, we have obtained the following singular developments:

1. Decomposition into triangles: $H\left(Z\right)\sim_{z\to\rho}\frac{1}{16}Z^{-4}$.
2. Dissections into $(k+1)$-agons: $H^{\{k+1\}}\left(Z\right)\sim_{z\to\rho_{k+1}}\frac{(k-1)^2}{16}Z^{-4}$.
3. Unrestricted dissections: $H^D\left(X\right)\sim_{z\to\rho_D}\frac{1}{16}X^{-4}$.

In the former estimates, $Z=\sqrt{1-z/\rho_\Delta}$ and $X=\sqrt{1-x/\rho_\Delta}$. Using Theorem 1.2, we get the estimates

$$[z^n]H^{\{k+1\}}(z)\sim\frac{(k-1)^2}{16}n\cdot\rho_{k+1}^{-1},\qquad [x^n]H^D(1,x)\sim\frac{1}{16}n\cdot\rho_D^{-1}.$$

In table 4.3 this asymptotic is compared with the corresponding families on the projective plane (computed in Section 3.5).

Family	Möbius band	Cylinder
$(k+1)-$agons dissections	$\frac{k-1}{4}\rho_{k+1}^{-n}$	$\frac{(k-1)^2}{16}n\rho_{k+1}^{-n}$
Unrestricted dissections	$\frac{1}{4}\rho_D^{-n}$	$\frac{1}{16}n\rho_D^{-n}$

Table 4.3 Asymptotic enumeration for the planar case and the case we are studying. For the value of $k=2$ in $(k+1)$-agon decompositions we obtain the asymptotic for triangulations.

4.8 Limit laws

In this section we study two statistical parameters which arise from the previous GFs: the size of the core (defined below), and the number of vertices on the external circle in a triangulation of the cylinder. The main differences between this section and Section 3.6 are the tools we use. We apply the Method of Moments (Lemma 1.4 in Section 1.4) together with properties of the *Laplace transform*. For completeness, we include a brief summary of the basic properties of this transform.

Let $\mathbf{Y}$ be a random variable with density function $f_{\mathbf{Y}}(t)$, such that $f_{\mathbf{Y}}(t)=0$ if $t<0$. Recall that the Laplace transform (see [31] for practical applications of this transform) of $f_{\mathbf{Y}}(t)$ is the function $\mathcal{L}\left(f_{\mathbf{Y}}(t)\right)=F_{\mathbf{Y}}(s)$ defined by

$$F_{\mathbf{Y}}(s)=\int_0^{\infty} f_{\mathbf{Y}}(t)e^{-st}dt=\sum_{r=0}^{\infty}\mathbb{E}[\mathbf{Y}^r]\frac{1}{r!}(-s)^r.$$

If there exists $\rho>0$ such that $\mathbb{E}[\mathbf{Y}^r]<\rho^r\cdot r!$, then the Laplace transform is analytic in a neighborhood of the origin, and the density probability function $f_{\mathbf{Y}}(t)$ is uniquely determined by its moments. In fact, the use of Laplace transform is the key tool in the Method of Moments.

To deduce limit laws we use the following strategy: let $(\mathbf{Y}_n)_{n>0}$ be a sequence of random variables. From the GFs obtained in the previous sections, we compute the r-th factorial moments of $(\mathbf{Y}_n)_{n>0}$, from which we deduce the r-th factorial moment of the limit random variable $\mathbf{Y}$. By the Method of Moments we prove that $\mathbf{Y}$ is characterized by its moments. Then we compute the moment generating function of $\mathbf{Y}$, which corresponds to $F_{\mathbf{Y}}(-s)=\mathcal{L}(f_{\mathbf{Y}}(t))$, and finally we apply the inverse Laplace transform to recover $f_{\mathbf{Y}}(t)$. Computing the inverse Laplace transform is not always simple, and we apply some tricks to get the desired density functions. The properties we use in this section about Laplace transform are shown in Table 4.4. The indicator function of a set of real numbers A is denoted by $\mathbb{I}_A(t)$.

Function	Laplace transform
$a_1\cdot f_1(t)+a_2\cdot f_2(t)$	$a_1\cdot F_1(s)+a_2\cdot F_2(s)$
$t^n\cdot f(t)$	$(-1)^n\cdot F^{(n)}(s)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0^-)-\cdots-f^{(n-1)}(0^-)$
$f(at)$	$1/\|a\|\cdot F(s/a)$
$f(t-a)\mathbb{I}_{[a,\infty[}(t)$	$e^{-as}\cdot F(s)$
$\int_0^t f(\tau)d\tau$	$1/s\cdot F(s)$

Table 4.4 Properties of the Laplace transform.

4.8.1 The size of the core in a dissection

We say that the *core of a dissection* of the cylinder is the set of cyclic cells, and the *size* of the core is the number of cells in the core. This is analogous with cyclic triangles in the projective plane, whose distribution was studied in Section 3.6.1.

Recall that in Section 3.6.1 of Chapter 3 we have defined a random variable $\mathbf{Y}$ with density probability function

$$f_{\mathbf{Y}}(t) = \frac{1}{\sqrt{\pi}} e^{-t^2/4} \mathbb{I}_{[0,\infty[}(t). \tag{4.15}$$

We computed the factorial moments, and we deduced that $\mathbf{Y}$ was characterized by its moments. We used an alternative expression for moments, using the Gauss duplication formula:

$$\mathbb{E}[\mathbf{Y}^r] = \frac{2^r}{\sqrt{\pi}} \Gamma\left(\frac{1+r}{2}\right) = \frac{\Gamma(1+r)}{\Gamma(1+r/2)} = \frac{r!}{\Gamma(1+r/2)}.$$

We also need the *complementary error function*, defined by

$$\operatorname{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-r^2} dr.$$

Properties and applications of this function in many different areas can be found in [31]. The important fact here is that $\operatorname{erfc}'(t) = -\frac{2}{\sqrt{\pi}} e^{-t^2}$, which relates the function $\operatorname{erfc}(t)$ with the density probability function of $\mathbf{Y}$ (up to a linear transform of the variable).

4.8.2 The size of the core in triangulations.

The generating function we want to study is

$$H(u,z) = H_0(u,u,z,z) = \left(\frac{1-zC(z)}{1-2zC(z)}\right)^2 M(uzC(z), uzC(z)), \tag{4.16}$$

where $M(x,y)$ is the rational function defined in Lemma 4.3, and $C(z)$ is the Catalan function. Write $\mathbf{Z}_n$ for the discrete random variable equals to the size of the core. The following theorem characterizes the limit law of the sequence $(\mathbf{Z}_n)_{n>0}$:

Theorem 4.15 *Let $\boldsymbol{Z}$ be a random variable with density function*

$$f_{\boldsymbol{Z}}(t) = t \cdot \operatorname{erfc}\left(\frac{t}{2}\right) \cdot \mathbb{I}_{[0,\infty[}(t).$$

Then, $\boldsymbol{Z}_n/\sqrt{n} \xrightarrow{d} \boldsymbol{Z}$.

Proof. We compute the moments of $\mathbf{Z}_n$, as in Theorem 3.9. Consider the expression for $H(u,z)$, which is deduced from Equation (4.16). Near the point $(u,z) = (1,1/4)$, $H(u,z)$ has a singular expansion of the form

$$H(u,z) \sim_{(u,z)\to(1,1/4)} \frac{1}{16} \frac{1}{1-4z} \frac{1}{\left(1-u\left(1-\sqrt{1-4z}\right)\right)^2}.$$

Factorial moments can be computed from the previous expression

$$\sum_{n\geq 0} H(n)\mathbb{E}[(\mathbf{Z}_n)_r] z^n = \frac{\partial^r H(1,z)}{\partial u^r} \sim_{z\to 1/4} \frac{1}{16} \frac{(1+r)!}{(1-4z)^{r/2+2}}.$$

Extracting coefficients using Theorem 1.2, and using the asymptotic estimate $H(n) \sim n \cdot 4^{n-2}$ obtained in Section 4.7, we obtain the following estimate for the r-th factorial moment of $\mathbf{Z}_n$:

$$\mathbb{E}[(\mathbf{Z}_n)_r] \sim \frac{(1+r)! \cdot n^{r/2}}{\Gamma\left(2+r/2\right)} = \frac{2+2r}{2+r} \frac{n^{r/2}}{\Gamma(1+r/2)} r!.$$

As we noticed in Lemma 1.5, the same estimate holds for the ordinary moment $\mathbb{E}[\mathbf{Z}_n^r]$. This can be written as

$$\mathbb{E}\left[\left(\frac{\mathbf{Z}_n}{\sqrt{n}}\right)^r\right] \sim \frac{2+2r}{2+r}\frac{r!}{\Gamma(1+r/2)}.$$

These values are bounded by $r!$, and by the Method of Moments they determine uniquely a limit law $\mathbf{Z}$, such that $\mathbf{Z}_n/\sqrt{n} \xrightarrow{d} \mathbf{Z}$. The next step consists of characterizing $\mathbf{Z}$. At this point, general properties of the Laplace transform are used. Consider the Laplace transform of the density probability function of $\mathbf{Z}$:

$$F_{\mathbf{Z}}(s) = \sum_{r=0}^{\infty} \frac{2+2r}{2+r}\frac{(-s)^r}{\Gamma(1+r/2)} = 2\sum_{r=0}^{\infty}\frac{(-s)^r}{\Gamma(1+r/2)} - 2\sum_{r=0}^{\infty}\frac{1}{r+2}\frac{(-s)^r}{\Gamma(1+r/2)},$$

which can be written in terms of the Laplace transform of $f_{\mathbf{Y}}(t)$ (Equation (4.15)) in the form:

$$F_{\mathbf{Z}}(s) = 2F_{\mathbf{Y}}(s) - 2G(s),$$

where $G(s) = \sum_{r=0}^{\infty} \frac{1}{2+r}\frac{1}{\Gamma(1+r/2)}(-s)^r$ is the Laplace transform of $g(t)$. By linearity (see Table 4.4), $f_{\mathbf{Z}}(t) = 2f_{\mathbf{Y}}(t) - 2g(t)$. We only need to find $g(t)$ to conclude this discussion. The function $G(s)$ can be written as:

$$F_{\mathbf{Y}}(s) = \frac{1}{s}\frac{\partial}{\partial s}\left(s^2 G(s)\right) = 2G(s) + sG'(s).$$

Applying the inverse Laplace transform, we obtain the differential equation

$$\frac{1}{\sqrt{\pi}} e^{-u^2/4}\mathbb{I}_{[0,\infty[}(t) = tg'(t) - g(t),$$

with the initial condition $\int_{-\infty}^{\infty} g(t)dt = 1/2$ (since $2g(t) = 2f_{\mathbf{Y}}(t) - f_{\mathbf{Z}}(t)$). The unique solution is

$$g(t) = \frac{1}{\sqrt{\pi}} e^{-t^2/4}\mathbb{I}_{[0,\infty[}(t) - \frac{t}{2}\mathrm{erfc}\left(\frac{t}{2}\right)\cdot \mathbb{I}_{[0,\infty[}(t),$$

and $f_{\mathbf{Z}}(t) = 2f_{\mathbf{Y}}(t) - 2g(t) = t\cdot \mathrm{erfc}\left(\frac{t}{2}\right)\cdot\mathbb{I}_{[0,\infty[}(t)$, as claimed. □

4.8.3 Size of the core for $(k+1)$ and unrestricted dissections

The previous analysis can be extended to all classes of dissections we have studied in the previous sections. Let $\mathbf{Z}_{k+1,n}$ the random variable equals to the number of cyclic polygons in a random dissection of the cylinder into n $(k+1)-$agons.

Theorem 4.16 *Let $\mathbf{Z}_{k+1}$ be a random variable with density function*

$$f_{\mathbf{Z}_{k+1}}(t) = \frac{t}{a_{k+1}^2}\,\mathrm{erfc}\left(\frac{t}{2a_{k+1}}\right)\cdot\mathbb{I}_{[0,\infty[}(t),$$

where $a_{k+1} = k(k-1)^{-2}\sqrt{(k-1)^3/(2k)}$. Then $\mathbf{Z}_{k+1,n}/\sqrt{n} \xrightarrow{d} \mathbf{Z}_{k+1}$.

Proof. By Theorem 4.10, $H^{\{k+1\}}(u,x) = H^{\{k+1\}}(u,u,x,x)$. Recall that $H^{\{k+1\}}(u,x)$ can be written as $\sum_{r=0}^{\infty} a_r(u)x^{(3+r)(k-1)}$, where r is the number of $(k+1)-$agons in the dissection. We study the new function $H^{\{k+1\}}(u,x^{k-1})$ which is the GF of dissections in terms of faces instead of vertices. Using $C_{k+1}(z) = 1 + zC_{k+1}(z)^k$ we write $zC_{k+1}(z)^{k-1} = (C_{k+1}(z)-1)/C_{k+1}(z)$. Near the singular point $(u,z) = (1,\rho_{k+1})$, $H^{\{k+1\}}(u,z)$ admits a singular expression of the form:

$$\frac{k(k-1)}{12(C_{k+1}(z) - k(C_{k+1}(z)-1))^2}\frac{k(k+1)u((C_{k+1}(z)-1)^2) + 2C_{k+1}(z)(C_{k+1}(z)-1)}{(C_{k+1}(z) - uk(C_{k+1}(z)-1))^2}.$$

From the previous expression, we compute the r-factorial moments of $\mathbf{Z}_{k+1,n}$,

$$\frac{\partial^r H^{\{k+1\}}(1,z)}{\partial u^r} \sim \frac{1}{4}\left(\frac{k}{k-1}\right)^{r+2} \frac{(k-1)^2(1+r)!}{(C_{k+1}(z)-k(C_{k+1}(z)-1))^{r+4}}.$$

Letting $l_{k+1} = \sqrt{2k/(k-1)^3}$ and extracting the nth coefficient we obtain that:

$$\begin{aligned} H^{\{k+1\}}(n)\mathbb{E}[(\mathbf{Z}_{k+1,n})_r] &\sim \frac{1}{4}\left(\frac{k}{k-1}\right)^{r+2} \frac{1}{l_{k+1}^{r+4}(k-1)^{r+2}}[z^n]\frac{(1+r)!}{(1-z/\rho_{k+1})^{r/2+2}} \\ &\sim \frac{1}{4}\frac{k^{r+2}}{(k-1)^{2r+4}}\frac{(1+r)!}{l_k^{r+4}\Gamma(2+r/2)}n^{1+r/2}\rho_k^n. \end{aligned}$$

From Section 4.7 we have the estimate $H^{\{k+1\}}(n) \sim (k-1)^2/16 \cdot n \cdot \rho_{k+1}^n$. Using this estimate and the asymptotic equality between factorial moments and ordinary moments, we get

$$\mathbb{E}\left[\left(\frac{\mathbf{Z}_{k+1,n}}{\sqrt{n}}\right)^r\right] \sim \left(\frac{k}{(k-1)^2 l_k}\right)^r \frac{(1+r)!}{\Gamma(2+r/2)}.$$

From now on, write $a_{k+1} = \frac{k}{(k-1)^2 l_k}$. The procedure now is the same as in Theorem 4.15. The moment generating function of $(\mathbf{Z}_{k+1,n}/\sqrt{n})^r$ is

$$F_{\mathbf{Z}_{k+1}}(s) = \sum_{r=0}^{\infty} a_{k+1}^r \frac{(r+1)!}{\Gamma(2+r/2)\, r!}(-s)^r = F_{\mathbf{Z}}(a_{k+1}s),$$

where $F_{\mathbf{Z}}(s)$ is the Laplace transform from Theorem 4.15. Applying the forth property in Table 4.4 we concludes the proof. $\square$

Observe also that if $k=2$, then $a_{k+1}=1$ and we recover the result for triangulations in Theorem 4.15. The analysis of the problem for the case of unrestricted dissections is similar as the one made in the case of $(k+1)$-agons. We state the result we obtain without proof. As before, let $\mathbf{Z}_{D,n}$ let the random variable equal to the number of cyclic polygon in a random dissection with n vertices.

Theorem 4.17 *Let $\mathbf{Z}_D$ be a random variable with density function*

$$f_{\mathbf{Z}_D}(t) = \frac{t}{a_D^2}\operatorname{erfc}\left(\frac{t}{2a_D}\right)\cdot \mathbb{I}_{[0,\infty[}(t),$$

where $a_D = (2^{7/4}(\sqrt{2}-1))^{-1}$. Then $\mathbf{Z}_{D,n}/\sqrt{n} \xrightarrow{d} \mathbf{Z}_D$.

Notice that the limit laws obtained so far have all the same shape, and are variants (up to a scale factor) of the distribution defined by Equation (4.15).

4.8.4 Distribution of vertices in a triangulation

Consider the following problem: given a random triangulation of the cylinder with n vertices, how many of them lie on the external circle? In this section we show that this parameter gives rise to a very simple and expected limit law.

Let $\mathbf{W}_n$ be the random variable, defined on triangulations of the cylinder with n vertices, equals to the number of vertices that lie in the external circle. We show that the $\mathbf{W}_n$ tend to a limit law $\mathbf{W}$ with density $8/\pi\sqrt{t-t^2}$ for $t \geq 0$. For this, we need to introduce some additional functions.

The *modified Bessel functions of order* α are defined defined as the pair of linearly independent solutions of the differential equation $x^2y'' + xy' + (x^2+\alpha^2)y = 0$. The modified Bessel function *of* *the first kind* and order α, $I_\alpha(x)$ is the solution for which $|y(x)| \to \infty$, as $x \to \infty$. More details and applications can be found in [31]. We also need the *Beta function* $\beta(x,y)$ (see [43]), defined as

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

It is well known that

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \tag{4.17}$$

from which we deduce the following equality if we restrict ourselves to natural integers:

$$\binom{n}{k}(n+1)\beta(n-k+1,k+1) = 1. \tag{4.18}$$

The GF we need in order to compute moments is

$$h(u,z) = H_0(1,1,uz,z),$$

where $H_0(u,v,x,y)$ is as in Theorem 4.4: we set $u = v = 1$ since we are not interested in the associated parameters, and setting $x = uz$, $y = z$, now z marks the total number of vertices, and u marks external vertices.

Theorem 4.18 *Let $\boldsymbol{W}$ be a random variable with density function*

$$f_{\boldsymbol{W}}(t) = \frac{8}{\pi}\sqrt{t-t^2}\mathbb{I}_{[0,1]}(t).$$

Then $\boldsymbol{W}_n/n \xrightarrow{d} \boldsymbol{W}$.

Proof. One easily shows that near the point $(u,z) = (1,1/4)$, $h(u,z)$ can be written in the form:

$$h(u,z) \sim_{(u,z)\to(1,1/4)} \frac{1}{4\sqrt{1-4z}\cdot\sqrt{1-4uz}\cdot\left(\sqrt{1-4z}+\sqrt{1-4uz}\right)^2}.$$

It is more convenient to write it as

$$\frac{1}{4\sqrt{1-4z}^3}\left(\frac{1}{\sqrt{1-4uz}} - \frac{1}{\sqrt{1-4z}+\sqrt{1-4uz}} - \frac{\sqrt{1-4uz}}{\left(\sqrt{1-4z}+\sqrt{1-4uz}\right)^2}\right).$$

The dominant term is the third one. We compute derivatives, and obtain a closed form for $H(n)\mathbb{E}\left[(\mathbf{W}_n)_r\right]$. We apply the Transfer Theorem, and obtain the following estimate:

$$H(n)\mathbb{E}\left[(\mathbf{W}_n)_r\right] \sim \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(3/2+r\right)}{\Gamma(3+r)}n^{r+1}\cdot 4^{n-1}.$$

Using $H(n) \sim n\cdot 4^{n-2}$, and the asymptotic equality between factorial moments and ordinary moments, we obtain the estimate:

$$\mathbb{E}\left[\left(\frac{\mathbf{W}_n}{n}\right)^r\right] \sim \frac{4}{\sqrt{\pi}}\frac{\Gamma\left(3/2+r\right)}{\Gamma(3+r)}.$$

These values are bounded by $4/\sqrt{\pi}$, and the Method of Moments guarantees that $\mathbf{W}_n/n$ tends in distribution to a random variable $\mathbf{W}$, which is characterized by its moments. The moment GF is equal to

$$F_{\mathbf{W}}(s) = \frac{4}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{\Gamma\left(3/2+r\right)}{\Gamma(3+r)\Gamma(1+r)}(-s)^r.$$

Coefficients of this function can be expressed in terms of the Beta function, using the equality $\Gamma(3+r)\cdot\Gamma(1+r) = \beta(3+r,1+r)\cdot\Gamma(4+2r)$ (recall Equation (4.17)). Using the Gauss duplication formula $\Gamma(r)\cdot\Gamma(1/2+r) = 2^{1-2r}\sqrt{\pi}\cdot\Gamma(2r)$, we get

$$F_{\mathbf{W}}(s) = \frac{1}{2}\sum_{r=0}^{\infty}\frac{1}{\beta(3+r,1+r)\Gamma(2+r)\Gamma\left(5/2+r\right)}\left(\frac{-s}{4}\right)^{r}.$$

Using (4.18) we obtain a simplified form for $F_{\mathbf{W}}(s)$:

$$F_{\mathbf{W}}(s) = \sum_{r=0}^{\infty}\binom{2+2r}{r}\frac{1}{(1+r)!}\left(\frac{-s}{4}\right)^{r}.$$

This function can be written in terms of Bessel functions as

$$F_{\mathbf{W}}(s) = \frac{-4}{s}\cdot I_1(-s/2)\cdot e^{-s/2}.$$

Computing the inverse Laplace transform of $F_{\mathbf{W}}(s)$ gives directly the density probability function claimed in the statement of the theorem. □

4.9 A related problem

Instead of taking two independent labellings on $\mathbb{S}_1^1$ and $\mathbb{S}_2^1$, we can also count dissections where only vertices on $\mathbb{S}_1^1$ (the external circle) are labeled. This approach is more natural in the context of map enumeration. In this section we introduce results for these families without proofs. Additionally, this section can be viewed as a brief introduction to the next chapter, where we study this type of dissections.

The main idea in the enumeration of these families is the same as in previous sections: we obtain the GF for cyclic decompositions, and then we make a convenient substitution to obtain the GF for the general family. Denote by $\overrightarrow{M}^{\Delta}(u,x,y)$ the GF of cyclic dissections of the cylinder, where vertices on the internal circle are not labelled. Δ denotes the set of possible degrees of the faces of the decomposition, u marks the number of cyclic polygons, and x, y mark the number of internal and external vertices, respectively. The main relation between $\overrightarrow{M}^{\Delta}(u,x,y)$ and $M^{\Delta}(u,x,y)$ (recall that M^{Δ} is the GF for cyclic dissections) is

$$x\frac{\partial}{\partial x}\overrightarrow{M}^{\Delta}(u,x,y) = M^{\Delta}(u,x,y),$$

which means that a cyclic dissection is obtained from $\overrightarrow{M}^{\Delta}(u,x,y)$ by pointing one internal vertex (the one whose label is $1'$). This equation is translated into

$$\overrightarrow{M}^{\Delta}(u,x,y) = \int_0^x \frac{M^{\Delta}(u,r,y)}{r}dr. \tag{4.19}$$

These equalities are valid for all type of dissections (i.e., for every choice of Δ). The same argument used in Proposition 4.4 (with a modification of the integration lemma used in it) shows that the GF for dissections (triangulations, (k+1)-agons dissections and general dissections) can be written using the expression derived from Equation (4.19). The argument is the same as in Theorems 4.4 and 4.10. The proof for other families is straighforward. The only difference is that we do not consider labels on the internal circle. For the case of $(k+1)$-agon dissections the GF considered is $\overrightarrow{H}^{\{k+1\}}(u,x)$, which is equal to

$$\overrightarrow{H}^{\{k+1\}}(u,x) = \frac{1}{C_{k+1}(x^{k-1})}\left(\frac{\partial}{\partial x}xC_{k+1}(x^{k-1})\right)\overrightarrow{M}^{\{k+1\}}(u,xC_{k+1}(x^{k-1}),xC_{k+1}(x^{k-1})).$$

For dissections the expression obtained is

$$\overrightarrow{H}^{D}(u,x) = \frac{1}{D(u,x)}\left(\frac{\partial}{\partial x}xD(u,x)\right)\overrightarrow{M}^{D}\left(u,\frac{1}{x}D(u,x),\frac{1}{x}D(u,x)\right),$$

in all cases, x marks vertices and u marks cyclic polygons. For the case of triangulations, we obtain

$$\begin{aligned}\overrightarrow{H}(z) &= \frac{-2z^4 - z^3 - 2z^2 + 7z - 2}{z(1-4z)} + \frac{3z^4 + 2z^3 + 7z^2 - 9z + 2}{z(1-4z)} C(z) \\ &= 7z^6 + 77z^7 + 555z^8 + 3318z^9 + \ldots,\end{aligned}$$

which was obtained first in [23]. In our approach, this result is a simple consequence of all the previous computations. For $(k+1)$-agon dissections, the GF for cyclic decompositions is

$$\overrightarrow{M}^{\{k+1\}}(u,x) = \frac{p_k(u,x)}{2(-1+ux^{k-1})^4(-1+kux^{k-1})},$$

where $p_k(u,x)$ is

$$\begin{aligned}p_k(u,x) = {} & -(k-1)(k-2)^3 x^{3k-3} u^3 - \\ & 2(k-1)(4+k)(k-2)^2 x^{4k-4} u^4 + \\ & 6(k-1)(k-2)(-2k-5+k^2) x^{5k-5} u^5 - \\ & 2(k-1)(27+18k-18k^2+2k^3) x^{6k-6} u^6 + \\ & (k-1)(20+48k-24k^2+k^3) x^{7k-7} u^7 + \\ & 2(k-1)(-1-10k+3k^2) x^{8k-8} u^8 + 2(k-1)k x^{9k-9} u^9.\end{aligned}$$

For unrestricted dissections the expression we obtain is

$$\overrightarrow{M}^{D}(u,x) = \frac{-d(u,x)}{(1-2x+x^2+u(x^2-2x))(x-1)^5(-1+x+xu)^3},$$

where $d(u,x)$

$$\begin{aligned}d(u,x) = {} & (x^6(x-1)^3)u^3 + (x^6(-9+7x)(x-1)^2)u^4 + \\ & (3x^6(x-1)(-3+2x)(-2+3x))u^5 + \\ & (x^6(-7+41x-54x^2+19x^3))u^6 + \\ & (x^7(7-19x+8x^2))u^7 + (x^8(x-2))u^8.\end{aligned}$$

As in Section 4.7, we can also obtain asymptotic estimates. This results are summarized in Table 4.5. For triangulations and $(k+1)$-agon dissections, n is referred to the number of polygons, and for unrestricted dissections to the number of vertices.

Family	Singular expansion	Asymptotic growth
Triangulations	$\frac{1}{4}Z^{-2}$	4^{n-1}
(k+1)-agon dissection	$\frac{k-1}{4}Z^{-2}$	$\frac{k-1}{4}\rho_{k+1}^n$
Unrestricted dissection	$\frac{1}{4}X^{-2}$	$\frac{1}{4}\rho_D^n$

Table 4.5 Asymptotic behaviour and estimates for those families: triangulations, $(k+1)$-agon dissections and unrestricted dissections.

The only difference with respect to the results in Section 4.7 is the subexponential term.

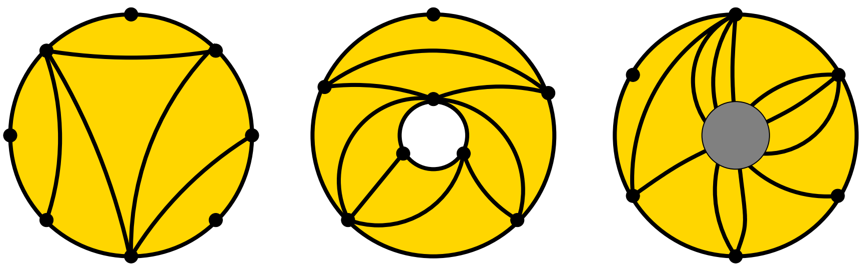
5

Dissections of surfaces with boundaries

The results of this chapter are from [7]. We deal with the generalisation of the problem treated in Chapter 3 and 4 to any compact surface $\mathbb{S}$ with boundaries. We obtain the asymptotic number of simplicial decompositions of a surface $\mathbb{S}$ with n vertices on its boundary. More generally, we determine the asymptotic number of dissections of $\mathbb{S}$ when the faces are δ-gons, such that δ belongs to a set of admissible degrees $\Delta \subseteq \{3, 4, 5, \ldots\}$. We also give limit laws for certain parameters of such dissections.

5.1 Introduction: exact and asymptotic counting

In chapters 3 and 4 of this thesis we have obtained several expressions for the exact enumeration of dissections of the Möbius band and the cylinder. Something that must be noticed is that these GFs become more involved as soon as either the number of connected components of the boundary or the genus of the surface grows. This complexity appears because of the inclusion-exclusion argument which is needed to deal with the "dissection" condition. However, asymptotic expressions are extremely simple (see Sections 3.5 and 4.8). In this chapter we get and explanation for these formulas, despite we do not obtain exact GFs. In other words, we deal with the generalisation of the problem of counting (asymptotically speaking) the number of dissections of any compact surface $\mathbb{S}$ with boundary. We are interested in the asymptotic number of simplicial decompositions of a surface $\mathbb{S}$ having n vertices, all of them lying on its boundary. Some simplicial decompositions of the disc, cylinder and Möbius band are represented in Figure 5.1.



(a) (b) (c)

Figure 5.1 Simplicial decomposition of (a) the disc, (b) the cylinder, (c) the Möbius band.

A map is *triangular* if every face has degree 3. More generally, given a set $\Delta \subseteq \{1, 2, 3, \ldots\}$, a map is Δ-angular if the degree of a face belongs to Δ. A map of this type is called Δ-*angular* map, or shortly, Δ-map. A map is a *dissection* if each face of degree k is incident with k distinct vertices, and the intersection of any pair of faces is either empty, a vertex or an edge. It is easy to see that triangular maps are dissections if and only if they have neither loops nor multiple edges. These maps are called also *simplicial decompositions* of $\mathbb{S}$. In this chapter we enumerate asymptotically the simplicial decompositions of an arbitrary surface $\mathbb{S}$ with boundaries. More precisely, we shall consider the set $\mathcal{D}_{\mathbb{S}}(n)$ of rooted simplicial decomposition of $\mathbb{S}$ having n vertices, all of them lying on the boundary and prove the asymptotic estimate

$$|\mathcal{D}_{\mathbb{S}}(n)| \sim c(\mathbb{S})\, n^{-3\chi(\mathbb{S})/2}\, 4^n, \tag{5.1}$$

where $c(\mathbb{S})$ is a constant which can be determined explicitly. For instance, a disk $\mathbb{D}$ has Euler characteristic $\chi(\mathbb{D}) = 1$ and the number of simplicial decompositions is $|\mathcal{D}_{\mathbb{D}}(n)| = C(n-2) \sim \frac{1}{\sqrt{\pi}} n^{-3/2} 4^n$.

We say that an edge is *non-structuring* if it belongs to the boundary of $\mathbb{S}$ or separates the surface into two parts, one of which is isomorphic to a disk (the other being isomorphic to $\mathbb{S}$); the other edges (in particular those that join distinct boundaries) are called *structuring*. We determine the limit law for the number of structuring edges in simplicial decompositions. In particular, we show that the (random) number $\mathbf{U}_n(\mathcal{D}_{\mathbb{S}})$ of structuring edges in a uniformly random simplicial decomposition of a surface $\mathbb{S}$ with n vertices, rescaled by a factor $n^{-1/2}$, converges in distribution toward a continuous random variable (related to Gamma distributions) which depends only on the Euler characteristic of $\mathbb{S}$.

We also generalise the enumeration and limit law results to Δ-dissections for any set of degrees $\Delta \subseteq \{3, 4, 5, \ldots\}$. Our results are obtained by exploiting a decomposition of the maps in $\mathcal{D}_{\mathbb{S}}(n)$ which is reminiscent of Wright's work on graphs with fixed excess [52, 53], or of work by Chapuy, Marcus and Schaeffer on the enumeration of unicellular maps [12]. This decomposition easily translates into an equation satisfied by the corresponding GF. We then apply classical enumeration techniques based on singularity analysis [19].

This chapter recovers and extends the asymptotic enumeration and limit law results obtained via a recursive approach for the cylinder and Möbius band in [23], [41] and [45] (the last ones correspond with Chapters 3 and 4 of this thesis). As in these papers, we are dealing with maps having all their vertices on the boundary of the surface. This is a sharp restriction which contrasts with most papers in map enumeration. In contrast, most of the literature on maps enumeration deals with maps having vertices outside of the boundary of the underlying surface. We do not deal with this more general problem here. However, a remarkable feature of the asymptotic result (5.1) (and the generalisation we obtain for arbitrary set of degrees $\Delta \subseteq \{3, 4, 5, \ldots\}$) is the linear dependency of the polynomial growth exponent in the Euler characteristic of the underlying surface. Similar results were obtained by a recursive method for general maps by Bender and Canfield in [4] and for maps with certain degree constraints by Gao in [21]. This feature as also been re-derived for general maps using a bijective approach in [12].

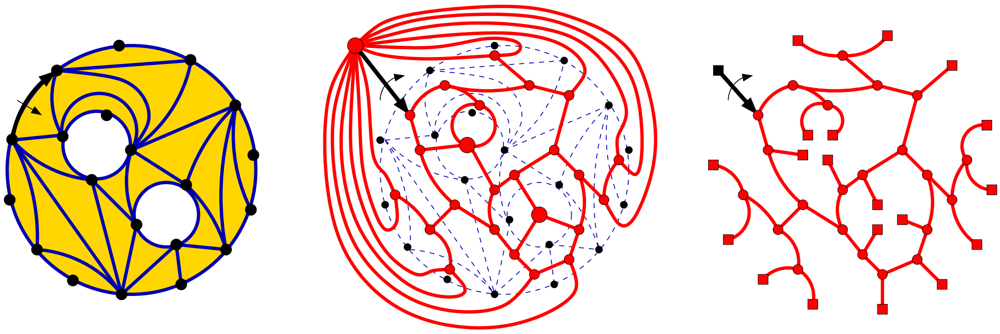
The outline of this chapter is the following. In Section 5.2 we introduce notation and some initial definitions. We enumerate rooted triangular maps in Section 5.3 and then we extend the results to Δ-angular maps for a general set $\Delta \subseteq \{3, 4, 5, \ldots\}$. In Section 5.4 it is proved that the number of rooted Δ-angular maps with n vertices behaves asymptotically as $c(\mathbb{S}, \Delta) n^{-3\chi(\mathbb{S})/2} {\rho_\Delta}^n$, where $c(\mathbb{S}, \Delta)$ and ρ_Δ are constants. In Section 5.5, we prove that the number of Δ-angular maps and of Δ-angular dissection with n vertices on $\mathbb{S}$ are asymptotically equivalent as n goes to infinity. Lastly, in Section 5.6, we study the limit laws of the (reescaled) number of structuring edges in uniformly random Δ-angular dissections of size n. In section 5.7 we give a method for determining the constants $c(\mathbb{S}, \Delta)$ explicitly. Finally, in Section 5.8 we conclude with some remarks and applications of the techniques used.

5.2 Definitions and notation

In this section we introduce some particular terminology used in this chapter. We define also the dual map of a map defined on a surface with boundary, and finally we set our notation for the class of maps we want to study.

We denote $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^{\geq k} = \{k, k+1, k+2, \ldots\}$. For any set $\Delta \subseteq \mathbb{N}$, we denote by $\gcd(\Delta)$ the greatest common divisor of the elements of Δ.

The dual map of a map on a surface with boundary.
Consider a (rooted) map M on a surface $\mathbb{S}$ with boundary. Observe that the (rooted) map M gives raise to a (rooted) map $\overline{M}$ on $\overline{\mathbb{S}}$ by gluing a disk (which become a face of $\overline{M}$) along each component of the boundary of $\mathbb{S}$. We call *external* these faces of $\overline{M}$ and the corresponding vertices of the dual map $\overline{M}^*$. The dual of a map M on a surface $\mathbb{S}$ with boundary is the map on $\overline{\mathbb{S}}$ denoted M^* which is obtained from $\overline{M}^*$ by splitting each external vertex of $\overline{M}^*$ of degree k, by k special vertices called *dangling leaves*. An example is given in Figure 5.2. Observe that for any set $\Delta \subseteq \mathbb{N}^{\geq 2}$, duality establishes a bijection between boundary-rooted Δ-angular maps on $\mathbb{S}$ surface $\mathbb{S}$ and leaf-rooted $(\Delta \cup \{1\})$-valent maps on $\overline{\mathbb{S}}$ having $\beta(\mathbb{S})$ faces, each of them being incident to at least one leaf.



(a) (b) (c)

Figure 5.2 (a) A rooted map M on the surface obtained by removing 3 disjoint discs to the sphere. (b) The map $\overline{M}^*$ on the sphere. (c) The dual map M^* on the sphere.

Sets of maps and generating functions.
A *plane tree* is a map on the sphere having a single face. For any set $\Delta \subseteq \mathbb{N}^{\geq 3}$, we denote by $\mathcal{T}_\Delta(n)$ the (finite) set of $(\Delta \cup \{1\})$-valent leaf-rooted plane trees with n non-root leaves. We denote by $T_\Delta(n)$ the cardinality of $\mathcal{T}_\Delta(n)$ and by $T_\Delta(z) = \sum_{n\geq 0} T_\Delta(n) z^n$ the corresponding generating function. For the special case of $\Delta = \{3\}$, the subscript Δ is omitted so that $\mathcal{T}(n)$ is the set of leaf-rooted *binary trees* with n non-root leaves. Hence, $T(n+1) = C(n)$ is the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$ and $T(z) = \sum_{n\geq 0} T(n) z^n = \frac{1-\sqrt{1-4z}}{2} = zC(z)$. A *doubly-rooted* tree is a leaf-rooted tree having a marked leaf distinct from the root-vertex. Observe that the GF of $(\Delta\cup\{1\})$-valent doubly-rooted trees counted by number of non-root non-marked leaves is $T'_\Delta(z) = \sum_{n\geq 0}(n+1) T_\Delta(n+1) z^n$.

Let $\mathbb{S}$ be a surface with boundary. For any set $\Delta \subseteq \mathbb{N}^{\geq 3}$, we denote by $\mathcal{M}_{\mathbb{S}}^{\Delta}(n)$ the set of boundary-rooted Δ-angular maps on the surface $\mathbb{S}$ having n vertices, all of them lying on the boundary. It is easy to see that the number of edges of maps in $\mathcal{M}_{\mathbb{S}}^{\Delta}(n)$ is bounded, hence the set $\mathcal{M}_{\mathbb{S}}^{\Delta}(n)$ is finite. Indeed, for any map M in $\mathcal{M}_{\mathbb{S}}^{\Delta}(n)$ the Euler relation gives $n - e(M) + f(M) = \chi(\mathbb{S})$, the relation of incidence between faces and edges gives $3f(M) + n \leq 2e(M)$, and solving for $e(M)$ gives $e(M) \leq 2n - 3\chi(\mathbb{S})$. We define $\mathcal{D}_{\mathbb{S}}^{\Delta}(n)$ as the subset of (boundary-rooted) maps in $\mathcal{M}_{\mathbb{S}}^{\Delta}(n)$ which are dissections. We write $\mathcal{M}_{\mathbb{S}}^{\Delta} = \cup_{n\geq 1}\mathcal{M}_{\mathbb{S}}^{\Delta}(n)$ for the set of all Δ-angular maps, $\mathrm{M}_{\mathbb{S}}^{\Delta}(n) = |\mathcal{M}_{\mathbb{S}}^{\Delta}(n)|$ for the number of them having n vertices and $M_{\mathbb{S}}^{\Delta}(z) = \sum_{n\geq 1} \mathrm{M}_{\mathbb{S}}^{\Delta}(n) z^n$ for the corresponding generating function. We adopt similar conventions for the set $\mathcal{D}_{\mathbb{S}}^{\Delta}(n)$. Lastly, the subscript Δ is omitted in all these notations whenever $\Delta = \{3\}$. For instance, $D_{\mathbb{S}}(z)$ is the generating function of boundary-rooted simplicial decompositions of the surface $\mathbb{S}$.

5.3 Enumeration of triangular maps

In this section, we consider triangular maps on an arbitrary surface $\mathbb{S}$ with boundary. We shall enumerate the maps in $\mathcal{M}_{\mathbb{S}} = \mathcal{M}_{\mathbb{S}}^{\{3\}}$ by exploiting a decomposition of the dual of $\{1,3\}$-valent maps on $\overline{\mathbb{S}}$. More precisely, we define a decomposition for maps in the set $\mathcal{A}_{\mathbb{S}}$ of leaf-rooted $\{1,3\}$-valent maps on $\overline{\mathbb{S}}$ having $\beta(\mathbb{S})$ faces. Recall that, by duality, triangular maps in $\mathcal{M}_{\mathbb{S}}$ are in bijection with the maps of $\mathcal{A}_{\mathbb{S}}$ such that each face is incident to at least one leaf (see Figure 5.2).

We denote by $\mathcal{A}_{\mathbb{S}}(n)$ the set of maps in $\mathcal{A}_{\mathbb{S}}$ having n leaves (including the root-vertex). If the surface $\mathbb{S}$ is a disk $\mathbb{D}$, then $\mathcal{A}_{\mathbb{D}}(n)$ is the set of leaf-rooted binary trees having n leaves and $|\mathcal{M}_{\mathbb{D}}(n)| = |\mathcal{A}_{\mathbb{D}}(n)| = C(n-2)$, where $C(n) = \frac{1}{n+1}\binom{2n}{n}$. We now suppose that $\mathbb{S}$ is not a disk (in particular, the Euler characteristic $\chi(\mathbb{S})$ is non-positive). We call *cubic scheme* of the surface $\mathbb{S}$, a leaf-rooted map on $\overline{\mathbb{S}}$ with $\beta(\mathbb{S})$ faces, such that that every non-root vertex has degree 3. Observe that one obtains a cubic scheme of the surface $\mathbb{S}$ by starting from a map in $\mathcal{A}_{\mathbb{S}}$, deleting recursively the non-root vertices of degree 1, and then contracting vertices of degree 2 (replacing the two incident edges by a single edge). This process is represented in Figure 5.3.

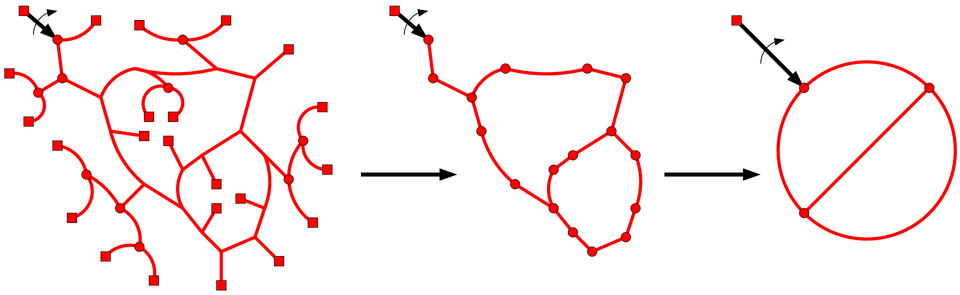


Figure 5.3 The cubic scheme of a map in $\mathcal{A}_{\mathbb{S}}$.

The cubic scheme obtained from a map $A \in \mathcal{A}_{\mathbb{S}}$ (which is clearly independent of the order of deletions of leaves and of contractions of vertices of degree 2) is called the *scheme* of A. The vertices of the scheme S can be identified with some vertices of A. Splitting these vertices gives a set of doubly-rooted trees, each of them associated to an edge of the scheme S (see Figure 5.4). To be more precise, let us choose arbitrarily a canonical end and side for each edge of S. Now, the map A is obtained by replacing each edge e of S by a doubly-rooted binary tree $\tau_e^\bullet$ in such a way that the canonical end and side of the edge e coincide with the root-end and root-side of the tree $\tau_e^\bullet$. It is easy to see that any cubic scheme of the surface $\mathbb{S}$ has $2 - 3\chi(\mathbb{S})$ edges (by using Euler relation together with the relation of incidence between edges and vertices). Therefore, upon choosing an arbitrary labelling and canonical end and side for the edges of every scheme of $\mathbb{S}$, one can define a mapping Φ on $\mathcal{A}_{\mathbb{S}}$ by setting $\Phi(A) = (S, (\tau_1^\bullet, \ldots, \tau_x^\bullet))$, where S is the scheme of the map A and $\tau_i^\bullet$ is the doubly-rooted tree associated with the i-th edge of S. See Figure 5.4.

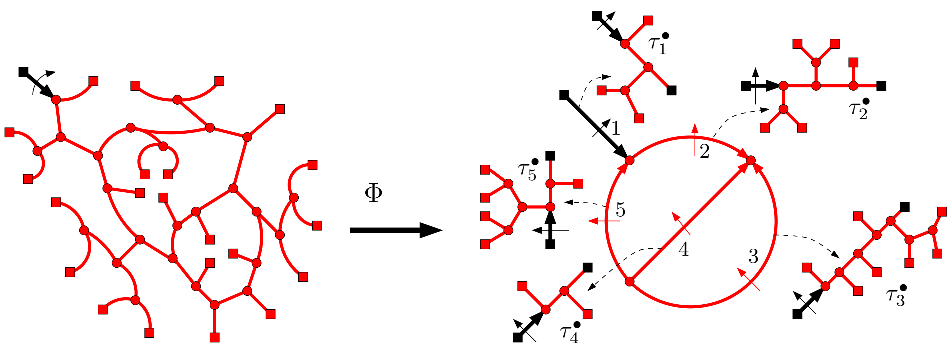


Figure 5.4 The bijection Φ. Root-leaves and marked leaves are indicated by black squares.

Reciprocally, any pair $(S, (\tau_1^\bullet, \ldots, \tau_x^\bullet))$ defines a map in $\mathcal{A}_{\mathbb{S}}$, hence the following lemma.

Lemma 5.1 *The mapping* Φ *is a bijection between*

- *the set* $\mathcal{A}_{\mathbb{S}}$ *of leaf-rooted* $\{1,3\}$*-valent maps on* $\overline{\mathbb{S}}$ *having* $\beta(\mathbb{S})$ *faces,*
- *the pairs made of a cubic scheme of* $\mathbb{S}$ *and a sequence of* $2-3\chi(\mathbb{S})$ *doubly-rooted binary trees.*

Moreover, the number of non-root leaves of a map $A \in \mathcal{A}_{\mathbb{S}}$ *is equal to the total number of leaves which are neither marked- nor root-leaves in the associated sequence of doubly-rooted trees.*

Lemma 5.1 immediately translates into an equation relating $A_{\mathbb{S}}(z) = \sum_{n\geq 0} |\mathcal{A}_{\mathbb{S}}(n)| z^n$ and the generating function $T'(z)$ of doubly-rooted binary trees:

$$A_{\mathbb{S}}(z) \;=\; a(\mathbb{S}) z \left(T'(z)\right)^{2-3\chi(\mathbb{S})},$$

where $a(\mathbb{S})$ is the number of cubic schemes of the surface $\mathbb{S}$. A way of determining the constant $a(\mathbb{S})$ is given in Section 5.7. Since the cubic schemes of $\mathbb{S}$ are trivially in bijection with rooted $\{3\}$-valent maps on $\overline{\mathbb{S}}$ having $\beta(\mathbb{S})$ faces, the constant $a(\mathbb{S})$ counts these maps. Moreover, since $T(z) = \sum_{n\geq 0} C(n) z^{n+1} = (1-\sqrt{1-4z})/2$ one gets $T'(z) = (1-4z)^{-1/2}$ and

$$A_{\mathbb{S}}(z) = a(\mathbb{S}) z (1-4z)^{3\chi(\mathbb{S})/2-1}.$$

From this expression, one can obtain an exact formula (depending on the parity of $\chi(\mathbb{S})$) for the cardinal $|\mathcal{A}_{\mathbb{S}}(n)| = [z^n]A_{\mathbb{S}}(z)$. However, we shall content ourselves with the following asymptotic result:

$$|\mathcal{A}_{\mathbb{S}}(n)| = [z^n]A_{\mathbb{S}}(z) \;=_{n\to\infty}\; \frac{a(\mathbb{S})}{4\Gamma\left(1-3\chi(\mathbb{S})/2\right)}\; n^{-3\chi(\mathbb{S})/2}\, 4^n \;\left(1+O\left(n^{-1}\right)\right). \tag{5.2}$$

It only remains to bound the number of maps in $\mathcal{A}_{\mathbb{S}}(n)$ which are not the dual of maps in $\mathcal{M}_{\mathbb{S}}(n)$ to prove the following lemma.

Lemma 5.2 *The number* $M_{\mathbb{S}}(n)$ *of boundary-rooted triangular maps on* $\mathbb{S}$ *having* n *vertices satisfies:*

$$M_{\mathbb{S}}(n) = A_{\mathbb{S}}(n)\left(1+O(n^{-1/2})\right).$$

A doubly-rooted tree is said *one-sided* if there are no-leaf on one of the sides of the path going from the root-vertex to the marked leaf; it is said *two-sided* otherwise.

Proof. Let $\widetilde{\mathcal{A}}_{\mathbb{S}}$ be the class of maps in $\mathcal{A}_{\mathbb{S}}$ which are not the dual of maps in $\mathcal{M}_{\mathbb{S}}$, and let $\widetilde{A}_{\mathbb{S}}(z) = A_{\mathbb{S}}(z) - M_{\mathbb{S}}(z)$ be the corresponding GF. Let A be a map in $\mathcal{A}_{\mathbb{S}}$ which is not the dual of a triangular map in $\mathcal{M}_{\mathbb{S}}$. Then A has a face incident to no leaf and its image $\left(S, (\tau_1^\bullet, \ldots, \tau_{2-3\chi(\mathbb{S})}^\bullet)\right)$ by the bijection Φ is such that one of the doubly-rooted trees $\tau_1^\bullet, \ldots, \tau_{2-3\chi(\mathbb{S})}^\bullet$ is one-sided. Thus,

$$\widetilde{A}_{\mathbb{S}}(z) \leq \left(2-3\chi(\mathbb{S})\right) a(\mathbb{S}) z \widetilde{T}(z) (T'(z))^{1-3\chi(\mathbb{S})},$$

where $\widetilde{T}(z)$ is the generating function of one-sided doubly-rooted binary trees (counted by number of non-root, non-marked leaves). The number of one-sided doubly-rooted binary trees having n leaves which are neither marked nor the root-vertex is $2T(n+1)$ if $n > 0$ and 1 for $n = 0$, hence $\widetilde{T}(z) = 2T(z)/z - 1$. The coefficients of the series $z\widetilde{T}(z)(T'(z))^{1-3\chi(\mathbb{S})}$ can be determined explicitly and gives

$$[z^n]\widetilde{A}_{\mathbb{S}}(z) = O\left([z^n] z\widetilde{T}(z)(T'(z))^{1-3\chi(\mathbb{S})}\right) = O\left(n^{-3\chi(\mathbb{S})/2-1/2} 4^n\right) = O\left(\frac{[z^n]A_{\mathbb{S}}(z)}{\sqrt{n}}\right).$$

This completes the proof of Lemma 5.2. □

To conclude this section, the following theorem is a simple consequence of the estimate stated in Expression (5.2) and the estimate obtained in Lemma 5.2.

Theorem 5.3 *Let $\mathbb{S}$ be any surface with boundary distinct from the disc. The asymptotic number of boundary-rooted triangulations on $\mathbb{S}$ with n vertices, all of them lying on the boundary is*

$$M_{\mathbb{S}}(n) = \frac{a(\mathbb{S})}{4\Gamma(1-3\chi(\mathbb{S})/2)}\ n^{-3\chi(\mathbb{S})/2}\ 4^n \left(1 + O\left(n^{-1/2}\right)\right),$$

where $a(\mathbb{S})$ is the number of cubic schemes of $\mathbb{S}$ and Γ is the usual Gamma function.

5.4 Enumeration of Δ-angular maps

We now extend the results of the previous section to Δ-angular maps, where Δ is any subset of $\mathbb{N}^{\geq 3}$. We first deal with the case of a disk $\mathbb{D}$. By duality, the problem corresponds to count leaf-rooted $(\Delta \cup \{1\})$-valent trees by number of leaves. This is partially done in Example VII.13 in [19] and we follow the method developed there. Then, we count Δ-angular maps on arbitrary surfaces by exploiting an extension of the bijection Φ.

5.4.1 Counting trees by number of leaves

In this subsection, we enumerate the $(\Delta \cup \{1\})$-valent plane trees by number of leaves. Recall that leaf-rooted $(\Delta \cup \{1\})$-valent trees are the dual of boundary rooted Δ-angular maps on a disk $\mathbb{D}$. The main result of this subsection is the following proposition, which gives the asymptotic number of such maps.

Proposition 5.4 *Let Δ be any subset of $\mathbb{N}^{\geq 3}$ and let p be the greatest common divisor of the set $\{\delta - 2,\ \delta \in \Delta\}$. Then, the number of non-root leaves of $(\Delta \cup \{1\})$-valent trees are congruent to 1 modulo p. Moreover, the asymptotic number of leaf-rooted $(\Delta \cup \{1\})$-valent trees having $1 + np$ non-root leaves is*

$$T_\Delta(1+np) =_{n\to\infty} \frac{p\gamma_\Delta}{2\sqrt{\pi}}\ (1+np)^{-3/2}\, {\rho_\Delta}^{-(np+1)}\ \left(1 + O\left(n^{-1}\right)\right),$$

where τ_Δ, ρ_Δ and γ_Δ are the unique positive constants satisfying:

$$\sum_{\delta\in\Delta}(\delta-1){\tau_\Delta}^{\delta-2} = 1,\quad \rho_\Delta = \tau_\Delta - \sum_{\delta\in\Delta}{\tau_\Delta}^{\delta-1},\ \gamma_\Delta = \sqrt{\frac{2\rho_\Delta}{\sum_{\delta\in\Delta}(\delta-1)(\delta-2){\tau_\Delta}^{\delta-3}}}. \tag{5.3}$$

Some comments must be said in reference to the constants τ_Δ, ρ_Δ and γ_Δ. The positive constant τ_Δ satisfying Expression (5.3) clearly exists, is unique, and is less than 1. Indeed, the function $f : \tau \mapsto \sum_{\delta\in\Delta}(\delta-1)\tau^{\delta-2}$ is well-defined and strictly increasing on $[0,1[$ and $f(0) = 1$ while $\lim_{\tau\to 1} f(\tau) > 1$. Moreover, $\rho_\Delta = \tau_\Delta\left(1 - \sum_{\delta\in\Delta}{\tau_\Delta}^{\delta-2}\right)$ is clearly positive. Hence, τ_Δ, ρ_Δ and γ_Δ are well defined. Notice that in the case when Δ is made of a single element $\delta = k+1$, and one gets $\tau_\Delta = k^{-1/(k-1)}$, $\rho_\Delta = \left(\frac{(k-1)^{k-1}}{k^k}\right)^{1/(k-1)}$, and $\gamma_\Delta = \sqrt{2k^{-(k+1)/(k-1)}}$.

In the rest of this subsection, we prove Proposition 5.4 and, on our way, establish several lemmas which are needed for the next subsection. We first introduce a change of variable in order to deal with the periodicity of the number of leaves.

Lemma 5.5 *Let $\Delta \subseteq \mathbb{N}^{\geq 3}$, $p = \gcd(\delta - 2,\ \delta \in \Delta)$ and $T_\Delta(z)$ the generating function of $(\Delta \cup \{1\})$-valent leaf-rooted trees. There exists a unique power series $Y_\Delta(t)$ in t such that*

$$T_\Delta(z) = zY_\Delta(z^p).$$

Moreover, the series Y_Δ satisfies

$$Y_\Delta(t) = 1 + \sum_{k\in K} t^k Y_\Delta(t)^{kp+1} \tag{5.4}$$

where K is the subset of $\mathbb{N}^{\geq 1}$ defined by $\Delta = \{2 + kp,\ k \in K\}$.

Proof. The fact that the number of non-rooted leaves is congruent to 1 modulo p is easily shown by induction on the number of leaves. Hence it exists a power series $Y_\Delta(t)$ such that $T_\Delta(z) = zY_\Delta(z^p)$, and it is unique with this property. We now use the classical *decomposition of trees at the root* (which corresponds to splitting the vertex adjacent to the root-leaf) represented in Figure 5.5 (recall also Equation (1.1)).

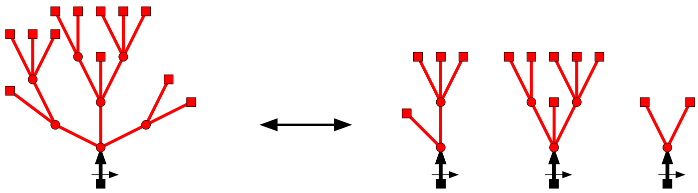


Figure 5.5 Decomposition of trees at the root

This decomposition gives

$$T_\Delta(z) = z + \sum_{\delta \in \Delta} T_\Delta(z)^{\delta - 1}, \tag{5.5}$$

and one obtains Equation (5.4) by substituting $T_\Delta(z)$ by $zY_\Delta(z^p)$ in Equation (5.5) and then substituting z^p by t. □

We now analyze the singularity of the generating function $Y_\Delta(t)$ and deduce from it the asymptotic behaviour of its coefficients. The main machinery is provided by a theorem of Meir and Moon [37] (which appears as Theorem VII.3 of [19]), on the singular behaviour of generating functions defined by a *smooth implicit-function schema*. More concretely, let $W(t)$ be a function analytic at 0, with $W(0) = 0$ and $[t^n]W(t) \geq 0$ for all $n \geq 0$. The function is said to satisfy a *smooth implicit-function schema* if there exists a bivariate power series $G(t, w) = \sum_{m,n \geq 0} g_{m,n} t^m w^n$ satisfying $W(t) = G(t, W(t))$ and the following conditions:

(a) There exist positive numbers $R, S > 0$ such that $G(t, w)$ is analytic in the domain $|t| < R$ and $|w| < S$.

(b) The coefficients of G satisfy $g_{m,n} \geq 0$, $g_{0,0} = 0$, $g_{0,1} \neq 1$, $g_{m,n} > 0$ for some $m \geq 0$ and some $n \geq 2$.

(c) There exist two numbers r, s, such that $0 < r < R$ and $0 < s < S$, satisfying the system of equations

$$G(r, s) = s \quad \text{and} \quad G_w(r, s) = 1,$$

which is called the *characteristic system* (where G_w denotes the derivative of G with respect to its second variable).

A series $W(t) = \sum_{n \geq 0} w_n t^n$ is said *aperiodic* if there exists integers $i < j < k$ such that the coefficients of w_i, w_j, w_k are non-zero and $\gcd(j - i, k - i) = 1$.

Lemma 5.6 ([37]) *Let $W(t)$ be an aperiodic function satisfying the smooth implicit-function schema defined by $G(t, w)$, and let (r, s) be the positive solution of the characteristic system. Then, the series $W(t)$ is analytic in a domain dented at $t = r$. Moreover, at any order $n \geq 0$, an expansion of the form*

$$W(t) \; =_{t \to r} \; \sum_{k=0}^{n} \alpha_k (1 - t/r)^{k/2} + o\left((1 - t/r)^{n/2}\right),$$

is valid in this domain, with $\alpha_0 = s$ and $\alpha_1 = -\sqrt{\frac{2rG_t(r,s)}{G_{w,w}(r,s)}}$.

This lemma is the key point to prove the following one:

Lemma 5.7 *Let $\Delta \subseteq \mathbb{N}^{\geq 3}$ and let p, τ_Δ, ρ_Δ and γ_Δ be as defined in Proposition 5.4. The generating function $Y_\Delta(t)$ of leaf-rooted $(\Delta \cup \{1\})$-valent trees (defined in Lemma 5.5) is analytic in a domain dented at $t = {\rho_\Delta}^p$. Moreover, at any order $n \geq 0$, an expansion of the form*

$$Y_\Delta(t) \;\; =_{t \to {\rho_\Delta}^p} \;\; \sum_{k=0}^{n} \alpha_k \left(1 - \frac{t}{{\rho_\Delta}^p}\right)^{k/2} + o\left(\left(1 - \frac{t}{{\rho_\Delta}^p}\right)^{n/2}\right)$$

is valid in this domain, with $\alpha_0 = \dfrac{\tau_\Delta}{\rho_\Delta}$ and $\alpha_1 = -\dfrac{\gamma_\Delta}{\rho_\Delta\sqrt{p}}$.

Proof. We first check that Lemma 5.6 applies to the series $W(t) = Y_\Delta(t) - 1$. Clearly, the GF $W(t)$ is analytic at 0, has non-negative coefficients and $W(0) = 0$. Moreover, by Lemma 5.5, $W(t) = G(t, W(t))$ for $G(t, w) = \sum_{k \in K} t^k (w + 1)^{kp+1}$.

We now check that $W(t)$ is aperiodic. Since $p = \gcd(\delta - 2,\ \delta \in \Delta)$, there exist $k, l \in K$ such that $\gcd(k, l) = 1$ (this includes the case $k = l = 1$). It is easy to see that there exists $(\Delta \cup \{1\})$-valent trees such with $(\alpha k + \beta l)p + 1$ non-rooted leaves for all $\alpha, \beta \leq 0$. Hence, $[t^{\alpha k + \beta l}]W(t) \neq 0$ for all $\alpha, \beta > 0$. This shows that the generating function $W(t)$ is aperiodic.

Finally, the conditions (a) and (b) clearly hold for $R = S = 1$. The condition (c) holds for $r = {\rho_\Delta}^p$ and $s = \tau_\Delta/\rho_\Delta - 1$. Indeed with these values

$$G(r, s) = \sum_{k \in K} r^k (s + 1)^{kp+1} = \sum_{k \in K} {\rho_\Delta}^{kp} \left(\frac{\tau_\Delta}{\rho_\Delta}\right)^{kp+1} = \frac{1}{\rho_\Delta} \sum_{\delta \in \Delta} {\tau_\Delta}^{\delta - 1} = \frac{\tau_\Delta - \rho_\Delta}{\rho_\Delta} = s$$

and

$$G_w(r, s) = \sum_{k \in K} (kp + 1) r^k (s + 1)^{kp} = \sum_{k \in K} (kp + 1)\tau_\Delta^{kp} = \sum_{\delta \in \Delta} (\delta - 1)\tau_\Delta^{\delta - 2} = 1.$$

Thus, the series $W(t) = Y_\Delta(t) - 1$ satisfies the conditions of Lemma 5.6. It only remains to show that $\alpha_1 = -\sqrt{\dfrac{2rG_t(r, s)}{G_{w,w}(r, s)}} = -\dfrac{\gamma_\Delta}{\rho_\Delta\sqrt{p}}$. One has

$$\begin{aligned} G_t(r, s) &= \sum_{k \in K} k r^{k-1} (s + 1)^{kp+1} \\ &= \frac{s + 1}{rp} \left(\sum_{k \in K} (kp + 1)(r(s + 1)^p)^k - \sum_{k \in K} (r(s + 1)^p)^k \right) \\ &= \frac{s + 1}{rp} \left(1 - \frac{s}{s + 1}\right) = \frac{1}{rp}, \end{aligned}$$

and

$$G_{w,w}(r, s) = \sum_{k \in K} kp(kp + 1) r^k (s + 1)^{kp-1} = \rho_\Delta \sum_{\delta \in \Delta} (\delta - 1)(\delta - 2){\tau_\Delta}^{\delta - 3} = \frac{2{\rho_\Delta}^2}{{\gamma_\Delta}^2}.$$

Hence, $\alpha_1 = -\dfrac{\gamma_\Delta}{\rho_\Delta\sqrt{p}}$. This completes the proof of Lemma 5.7. □

Lemma 5.7 ensures that the generating function $Y_\Delta(t)$ satisfies the required condition in order to apply the Transfer Theorem between singularity types and coefficient asymptotic. Application of the Transfer Theorem gives the result in Proposition 5.4.

5.4.2 Counting Δ-angular maps on general surfaces

We consider a surface $\mathbb{S}$ with boundary distinct from a disk and denote by $\mathcal{A}_{\mathbb{S}}^{\Delta}$ the set of leaf-rooted $(\Delta \cup \{1\})$-valent maps on $\overline{\mathbb{S}}$. Recall that, by duality, the maps in $\mathcal{M}_{\mathbb{S}}^{\Delta}$ (that is, boundary-rooted Δ-angular on the surface $\overline{\mathbb{S}}$) are in bijection with the maps in $\mathcal{A}_{\mathbb{S}}^{\Delta}$ such that every face is incident with at least one leaf. We first extend the bijection Φ defined in Section 5.3 to maps in $\mathcal{A}_{\mathbb{S}}^{\Delta}$. This decomposition leads to consider leaf-rooted trees with *legs*, that is, marked vertices at distance 2 from the root-leaf. We call scheme of the surface $\mathbb{S}$ a leaf-rooted map on $\overline{\mathbb{S}}$ having $\beta(\mathbb{S})$ faces and such that the degree of any non-root vertex is at least 3. Recall that a scheme is cubic if the degree of any non-root vertex is 3. By combining Euler relation with the relation of incidence between vertices and edges, it is easy to see that the number of edges of a scheme of $\mathbb{S}$ is at most $e = 2 - 3\chi(\mathbb{S})$, with equality if and only if the scheme is cubic. In particular, this implies that the number of scheme is finite.

Let A be a map in $\mathcal{A}_{\mathbb{S}}^{\Delta}$. One obtains a scheme S of $\mathbb{S}$ by recursively deleting the non-root vertices of degree 1 and then contracting the vertices of degree 2; see Figure 5.3 for an illustrative example. The vertices of the scheme S can be identified with some vertices of A. The split of these vertices gives a sequence of doubly-rooted Δ-valent trees (each of them associated with an edge of S), and a sequence of leaf-rooted Δ-valent trees with legs (each of them associated with a non-root vertex of S); see Figure 5.6.

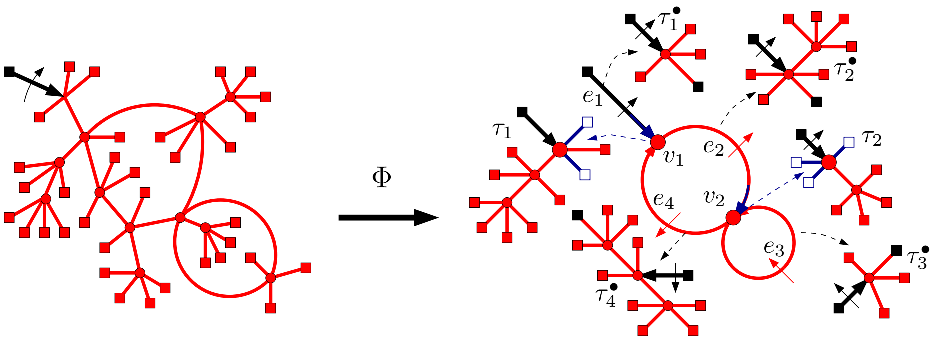


Figure 5.6 The bijection Φ: decomposition of a map whose scheme has $e = 4$ edges and $v = 2$ non-root vertices of respective degrees 3 and 4. Legs are indicated by white squares.

More precisely, if the scheme S has k edges and l non-root vertices having degree $d_1, \ldots, d_l$, then the map A is obtained by substituting each edge of S by a doubly-rooted $(\Delta \cup \{1\})$-valent tree, and each vertex of S of degree d by a leaf-rooted $(\Delta \cup \{1\})$-valent tree with $d-1$ legs (these substitution can be made unambiguously provided that one chooses a canonical end and side for each each edge of S and a canonical incident end for each vertex of S). We define the mapping Φ on $\mathcal{A}_{\mathbb{S}}^{\Delta}$ by setting $\Phi(A) = (S, (\tau_1^\bullet, \ldots, \tau_e^\bullet), (\tau_1, \ldots, \tau_v))$, where S is the scheme of A having e edges and v non-root vertices, $\tau_i^\bullet$ is the doubly-rooted tree associated with the ith edge of S and τ_j is the tree with legs associated with the jth vertex of S. Reciprocally any triple $(S, (\tau_1^\bullet, \ldots, \tau_e^\bullet), (\tau_1, \ldots, \tau_v))$ defines a map in $\mathcal{A}_{\mathbb{S}}^{\Delta}$, hence the following lemma.

Lemma 5.8 *Let S be a scheme of $\mathbb{S}$ with e edges and v non-root vertices having degree $d_1, \ldots, d_v$. Then the mapping Φ gives a bijection between*

- *the set map in $\mathcal{A}_{\mathbb{S}}^{\Delta}$ having scheme S,*
- *pairs made of a sequence of e doubly-rooted $(\Delta \cup \{1\})$-valent trees and a sequence of v leaf-rooted $(\Delta \cup \{1\})$-valent trees with $d_1 - 1, \ldots, d_v - 1$ legs respectively.*

Moreover, the number of non-root leaves of a map $A \in \mathcal{A}_{\mathbb{S}}^{\Delta}$ is equal to the total number of leaves which are neither legs nor marked-leaves nor root-leaves in the associated sequences of trees.

We denote by $A_{\mathbb{S}}^{\Delta}(z)$ the generating function of the set $\mathcal{A}_{\mathbb{S}}^{\Delta}$ of leaf-rooted $(\Delta \cup \{1\})$-valent maps counted by number of leaves. By partitioning the set of maps in $\mathcal{A}_{\mathbb{S}}^{\Delta}$ according to their scheme, one gets

$$A_{\mathbb{S}}^{\Delta}(z) = \sum_{S \text{ scheme}} F_S^{\Delta}(z), \tag{5.6}$$

where the sum is over the schemes of $\mathbb{S}$ and $F_S^{\Delta}(z)$ is the generating function of the subset of maps in $\mathcal{A}_{\mathbb{S}}^{\Delta}$ having scheme S counted by the number of leaves. Moreover, for a scheme S with e edges and v non-root vertices of respective degrees $d_1, \ldots, d_v$, Lemma 5.8 gives

$$F_S^{\Delta}(z) \ = \ z \ (T'_{\Delta}(z))^e \prod_{i=1}^{v} T_{\Delta, d_i - 1}(z), \tag{5.7}$$

where $T_{\Delta,\ell}(z)$ is the generating function of $(\Delta \cup \{1\})$-valent leaf-rooted trees with ℓ legs counted by number of leaves which are neither legs nor root-leaves. It is now convenient to introduce a change of variable to deal with the periodicity of the number of leaves of maps in $\mathcal{A}_{\mathbb{S}}^{\Delta}$.

Lemma 5.9 *Let $\Delta \subseteq \mathbb{N}^{\geq 3}$ and let $p = \gcd(\delta - 2,\ \delta \in \Delta)$. For any positive integer ℓ, there exists a power series $Y_{\Delta,\ell}(t)$ in t such that $T_{\Delta,\ell}(z) = z^{1-\ell} Y_{\Delta,\ell}(z^p)$. Moreover, the generating function of leaf-rooted $(\Delta \cup \{1\})$-valent maps on $\overline{\mathbb{S}}$ satisfies*

$$\begin{aligned} A_{\mathbb{S}}^{\Delta}(z) &= z^{2\chi(\mathbb{S})} B_{\mathbb{S}}^{\Delta}(z^p), \\ B_{\mathbb{S}}^{\Delta}(t) &= \sum_{S \text{ scheme}} (ptY'_{\Delta}(t) + Y_{\Delta}(t))^{|E(S)|} \prod_{u \in V(S)} Y_{\Delta, \deg(u)-1}(t), \end{aligned} \tag{5.8}$$

where the sum is over all the schemes S of $\mathbb{S}$ and $E(S)$, $V(S)$ are respectively the set of edges and non-root vertices of the scheme S.

Observe that Equation (5.8) shows that the coefficient $[z^n]A_{\mathbb{S}}^{\Delta}(z)$ is 0 unless n is congruent to $2\chi(\mathbb{S})$ modulo p. In other words, the number of leaves of $(\Delta \cup \{1\})$-valent maps on $\overline{\mathbb{S}}$ is congruent to $2\chi(\mathbb{S})$ modulo p.

Proof. By Lemma 5.5, the number of non-root leaves of $(\Delta \cup \{1\})$-valent trees is congruent to 1 modulo p. Hence, the number of non-root, non-marked leaves of a $(\Delta \cup \{1\})$-valent trees with ℓ legs is congruent to $1 - \ell$ modulo p. This ensures the existence of a power series $Y_{\Delta,\ell}(t)$ such that $T_{\Delta,\ell}(z) = z^{1-\ell} Y_{\Delta,\ell}(z^p)$.

Let S be a scheme with e edges and v non-root vertices of respective degrees $d_1, \ldots, d_v$. Plugging the identities $T_{\Delta,\ell}(z) = z^{1-\ell} Y_{\Delta,\ell}(z^p)$ into Equation (5.7) gives

$$F_S^{\Delta}(z) \ = \ z^{1+2v-\sum_{i=1}^{v} d_i} \left(\frac{d}{dz} \left[z Y_{\Delta}(z^p) \right] \right)^e \prod_{i=1}^{v} Y_{\Delta, d_i - 1}(z^p). \tag{5.9}$$

The sum $\sum_{i=1}^{v} d_i$ is the number of edge-ends which are not the root-ends. Hence, $\sum_{i=1}^{v} d_i = 2e - 1$ and by Euler relation, $1 + 2v - \sum_{i=1}^{v} d_i \ = \ 2 + 2v - 2e \ = \ 2\left(\chi(\overline{\mathbb{S}}) - \beta(\mathbb{S})\right) \ = \ 2\chi(\mathbb{S})$ (since the scheme S is a map on $\overline{\mathbb{S}}$ having $v+1$ vertices, e edges and $\beta(\mathbb{S})$ faces). Moreover, $\frac{d}{dz}\left[zY_{\Delta}(z^p)\right] = pz^p Y'_{\Delta}(z^p) + Y_{\Delta}(z^p)$. Thus continuing developing Equation (5.9), gives

$$F_S^{\Delta}(z) \ = \ z^{2\chi(\mathbb{S})} \ (pz^p Y'_{\Delta}(z^p) + Y_{\Delta}(z^p))^e \prod_{i=1}^{v} Y_{\Delta, d_i - 1}(z^p), \tag{5.10}$$

Replacing z^p by t in the right-hand-side of (5.10) and summing over all the schemes of $\mathbb{S}$ gives Equation (5.8) from Equation (5.6). □

We now study the singularities of the generating functions of trees with legs.

Lemma 5.10 *Let $\Delta \subseteq \mathbb{N}^{\geq 3}$, let $p = \gcd(\delta - 2,\ \delta \in \Delta)$ and let ρ_Δ, γ_Δ be the constants defined by Expression (5.3). For any positive integer ℓ, the generating function $Y_{\Delta,\ell}(t)$ is analytic in a domain dented at $t = {\rho_\Delta}^p$. Moreover, there exists a constant κ_ℓ such that the expansion*

$$Y_{\Delta,\ell}(t) =_{t\to{\rho_\Delta}^p} \kappa_\ell + O\left(\sqrt{1 - \frac{t}{{\rho_\Delta}^p}}\right),$$

is valid in this domain. In particular, $\kappa_2 = \left(\frac{\rho_\Delta}{\gamma_\Delta}\right)^2$.

Proof. By considering the decomposition at the root of $(\Delta \cup \{1\})$-valent leaf-rooted trees with ℓ legs, one gets

$$T_{\Delta,\ell}(z) = \sum_{\delta\in\Delta,\ \delta>\ell} \binom{\delta-1}{l} T_\Delta(z)^{\delta-\ell-1}. \tag{5.11}$$

In particular, for $\ell = 1$ and using Equation (5.5) this gives

$$\begin{aligned} T_{\Delta,1}(z) &= \sum_{\delta\in\Delta} (\delta-1) T_\Delta(z)^{\delta-2} = \frac{1}{T'_\Delta(z)} \frac{d}{dz} \left(\sum_{\delta\in\Delta} T_\Delta(z)^{\delta-1} \right) \\ &= \frac{1}{T'_\Delta(z)} \frac{d}{dz} (T_\Delta(z) - z) = 1 - \frac{1}{T'_\Delta(z)}. \end{aligned} \tag{5.12}$$

Moreover for $\ell > 1$, Equation (5.11) gives

$$T_{\Delta,\ell}(z) = \frac{1}{\ell T'_\Delta(z)} \frac{d}{dz} T_{\Delta,\ell-1}(z).$$

Making the change of variable $t = z^p$ gives

$$Y_{\Delta,1}(t) = 1 - \frac{1}{Z_\Delta(t)} \quad \text{and} \quad Y_{\Delta,\ell}(t) = \frac{ptY'_{\Delta,\ell-1}(t) + (2-\ell)Y_{\Delta,\ell-1}(t)}{\ell Z_\Delta(t)}, \tag{5.13}$$

where $Z_\Delta(t) = ptY'_\Delta(t) + Y_\Delta(t)$ is such that $T'_\Delta(z) = Z_\Delta(z^p)$. In particular,

$$T_{\Delta,2}(t) = \frac{ptZ'_\Delta(t)}{2Z_\Delta(t)^3}. \tag{5.14}$$

We continue proving that for all $\ell > 0$, the generating function $Y_{\Delta,\ell}(t)$ is analytic in a domain dented at $t = {\rho_\Delta}^p$. By Lemma 5.7, the generating function $Z_\Delta(t) = ptY'_\Delta(t) + Y_\Delta(t)$ is analytic in a domain dented at $t = {\rho_\Delta}^p$. Given Equation (5.13), the same property holds for the series $Y_{\Delta,\ell}(t)$ for all $\ell > 0$ provided that $Z_\Delta(t)$ does not cancel in a domain dented at $t = {\rho_\Delta}^p$. It is therefore sufficient to prove that $|Z_\Delta(t)| > 1/2$ for all $|t| < {\rho_\Delta}^p$ or equivalently, $|T'_\Delta(z)| > 1/2$ for $|z| < \rho_\Delta$.

First observe that $\lim_{z\to\rho_\Delta} T_\Delta(z) = \tau_\Delta$ by Lemma 5.7. Hence, for all $|z| < \rho_\Delta$, $|T_\Delta(z)| \leq T_\Delta(|z|) < \tau_\Delta$ and by (5.3)

$$\left| \sum_{\delta\in\Delta} (\delta-1) T_\Delta(z)^{\delta-2} \right| \leq \sum_{\delta\in\Delta} (\delta-1) \left|T_\Delta(z)\right|^{\delta-2} < \sum_{\delta\in\Delta} (\delta-1) \tau_\Delta^{\delta-2} = 1.$$

Moreover, by differentiating Expression (5.5) with respect to z, one gets

$$T'_\Delta(z) = 1 + T'_\Delta(z) \sum_{\delta\in\Delta} (\delta-1) T_\Delta(z)^{\delta-2}.$$

Hence, the inequality

$$|T'_\Delta(z)| \geq \frac{1}{1 + \left|\sum_{\delta\in\Delta} (\delta-1) T_\Delta(z)^{\delta-2}\right|} > \frac{1}{2}$$

holds. We now prove that for all $\ell > 0$, the series $Y_{\Delta,\ell}(t)$ has an expansion of the form $\kappa_\ell + O\left(\sqrt{1-t/\rho_\Delta{}^p}\right)$ valid in a domain dented at $t = \rho_\Delta{}^p$. By Lemma 5.7, the series $Y_\Delta(t)$ is analytic in a domain at $t = \rho_\Delta{}^p$. Thus, its expansion at $t = \rho_\Delta{}^p$ can be differentiated term by term. For the series $Z_\Delta(t) = ptY'_\Delta(t) + Y_\Delta(t)$ this gives an expansion of the form

$$Z_\Delta(t) =_{t\to\rho_\Delta{}^p} \left(1-\frac{t}{\rho_\Delta{}^p}\right)^{-1/2}\left[\sum_{k=0}^{n}\beta_k\left(1-\frac{t}{\rho_\Delta{}^p}\right)^{k/2} + o\left(\left(1-\frac{t}{\rho_\Delta{}^p}\right)^{n/2}\right)\right], \tag{5.15}$$

where $\beta_0 = -\frac{p}{2}\alpha_1 = \frac{\gamma_\Delta\sqrt{p}}{2\rho_\Delta}$. Thus, by induction on ℓ, the series $Y_{\Delta,\ell}(z)$ has an expansion of the form

$$Y_{\Delta,\ell}(z) =_{t\to\rho_\Delta{}^p} \sum_{k=0}^{n}\kappa_{\ell,k}\left(1-\frac{t}{\rho_\Delta{}^p}\right)^{k/2} + o\left((1-\frac{t}{\rho_\Delta{}^p})^{n/2}\right).$$

In particular, Equation (5.14) gives $\kappa_2 = \kappa_{2,0} = \frac{p}{4\beta_0{}^4} = (\rho_\Delta/\gamma_\Delta)^2$. □

Using the previous lemmas we are in condition to prove the main result of this section:

Theorem 5.11 *Let $\mathbb{S}$ be any surface with boundary distinct from the disc, let $\Delta \subseteq \mathbb{N}^{\geq 3}$ and let p be the greatest common divisor of $\{\delta-2,\ \delta\in\Delta\}$. Then, the number of vertices of Δ-angular maps (having all their vertices on the boundary) is congruent to $2\chi(\mathbb{S})$ modulo p. Moreover, the asymptotic number of boundary-rooted Δ-angular maps of $\mathbb{S}$ having $np+2\chi(\mathbb{S})$ vertices is $M_{\mathbb{S}}^{\Delta}(np+2\chi(\mathbb{S}))$, with asymptotic estimate*

$$\frac{a(\mathbb{S})p\,(8\gamma_\Delta\rho_\Delta)^{\chi(\mathbb{S})}}{4\Gamma(1-3\chi(\mathbb{S})/2)}\ (np+2\chi(\mathbb{S}))^{-3\chi(\mathbb{S})/2}\,\rho_\Delta{}^{-(np+2\chi(\mathbb{S}))}\ \left(1+O\left(n^{-1/2}\right)\right),$$

where ρ_Δ and γ_Δ are the constants determined by Equations (5.3), and $a(\mathbb{S})$ is the number of cubic schemes of $\mathbb{S}$.

Theorem 5.11 generalizes Theorem 5.3 since for $\Delta = \{3\}$, one has $p = 1$, $\rho_\Delta = 1/4$ and $\gamma_\Delta = 1/2$.

Proof. By Equation (5.15), the following equality

$$\begin{aligned} ptY'_\Delta(t) + Y_\Delta(t) &= Z_\Delta(t) \\ &=_{t\to\rho_\Delta{}^p} \frac{\gamma_\Delta\sqrt{p}}{2\rho_\Delta}\left(1-\frac{t}{\rho_\Delta{}^p}\right)^{-1/2}\left(1+O\left(1-\frac{t}{\rho_\Delta{}^p}\right)\right) \end{aligned} \tag{5.16}$$

holds in a domain dented at $t = \rho_\Delta{}^p$. Thus, by Lemma 5.10, the generating function

$$G_S(t) = (ptY'_\Delta(t) + Y_\Delta(t))^e\prod_{i=1}^{v}Y_{\Delta,d_i-1}(t)$$

associated to a scheme S with e edges and v vertices of respective degrees $d_1,\ldots,d_v$ has an expansion of the form

$$G_S(t) =_{t\to\rho_\Delta{}^p} \left(\prod_{i=1}^{v}\kappa_{d_i-1}\right)\left(\frac{\gamma_\Delta\sqrt{p}}{2\rho_\Delta}\right)^e\left(1-\frac{t}{\rho_\Delta{}^p}\right)^{-e/2}\left(1+O\left(\sqrt{1-\frac{t}{\rho_\Delta{}^p}}\right)\right),$$

valid in a domain dented at $t = \rho_\Delta{}^p$. Moreover, as mentioned above, the number of edges of a scheme of $\mathbb{S}$ is at most $e = 2-3\chi(\mathbb{S})$, with equality if and only if the scheme is cubic. There are $a(\mathbb{S}) > 0$ cubic schemes and all of them have $v = 1-2\chi(\mathbb{S})$ non-root vertices. Thus, Lemma 5.9 gives $A_{\mathbb{S}}^{\Delta}(z) = z^{2\chi(\mathbb{S})}B_{\mathbb{S}}^{\Delta}(z^p)$ with

$$\begin{aligned} B_{\mathbb{S}}^{\Delta}(t) &= \sum_{S\text{ scheme}} G_S(t) \\ &=_{t\to\rho_\Delta{}^p} \frac{p\,a(\mathbb{S})}{4}\left(\frac{8\gamma_\Delta}{p^{3/2}\rho_\Delta}\right)^{\chi(\mathbb{S})}\left(1-\frac{t}{\rho_\Delta{}^p}\right)^{3\chi(\mathbb{S})/2-1}\left(1+O\left(\sqrt{1-\frac{t}{\rho_\Delta{}^p}}\right)\right), \end{aligned}$$

where the expansion is valid in a domain dented at $t = \rho_\Delta{}^p$. Applying standard techniques (see [19]) one can obtain the asymptotic behaviour of the coefficient $[z^{np+2\chi(\mathbb{S})}]A_{\mathbb{S}}^{\Delta}(z) = [t^n]B_{\mathbb{S}}^{\Delta}(t)$ from the singular behaviour of the series $B_{\mathbb{S}}^{\Delta}(t)$. More concretely, an estimate for $[t^n]B_{\mathbb{S}}^{\Delta}(t)$ is

$$\frac{p\,a(\mathbb{S})}{4\Gamma(1-3\chi(\mathbb{S})/2)}\left(\frac{8\gamma_\Delta}{p^{3/2}\rho_\Delta}\right)^{\chi(\mathbb{S})} n^{-3\chi(\mathbb{S})/2}\ \rho_\Delta{}^{-np}\left(1+O\left(n^{-1/2}\right)\right),$$

which can be simplified into the expression

$$\frac{p\,a(\mathbb{S})\,(8\gamma_\Delta\rho_\Delta)^{\chi(\mathbb{S})}}{4\Gamma(1-3\chi(\mathbb{S})/2)}\ (np+2\chi(\mathbb{S}))^{-3\chi(\mathbb{S})/2}\ \rho_\Delta{}^{-(np+2\chi(\mathbb{S}))}\left(1+O\left(n^{-1/2}\right)\right). \tag{5.17}$$

In order to conclude the proof of Theorem 5.11, it suffices to compare the number of maps in $\mathcal{M}_{\mathbb{S}}^{\Delta}(n)$ with the number of maps in $\mathcal{A}_{\mathbb{S}}^{\Delta}(n)$ and prove that

$$M_{\mathbb{S}}^{\Delta}(n) =_{n\to\infty} A_{\mathbb{S}}^{\Delta}(n)\left(1+O\left(n^{-1/2}\right)\right). \tag{5.18}$$

One can write a proof of Equation (5.18) along the line of the proof of Lemma 5.2. We omit such a proof since a stronger statement (Theorem 5.12) is proved in the next section. □

5.5 From maps to dissections

In this section we prove that the number of maps and the number of dissections are asymptotically equivalent. More precisely, we prove the following theorem.

Theorem 5.12 *Let $\mathbb{S}$ be a surface with boundary and let $\Delta \subseteq \mathbb{N}^{\geq 3}$. The number of maps in $\mathcal{D}_{\mathbb{S}}^{\Delta}(n)$, in $\mathcal{M}_{\mathbb{S}}^{\Delta}(n)$ and in $\mathcal{A}_{\mathbb{S}}^{\Delta}(n)$ satisfy*

$$D_{\mathbb{S}}^{\Delta}(n) \ = \ M_{\mathbb{S}}^{\Delta}(n)\left(1+O\left(n^{-1/2}\right)\right) \ = \ A_{\mathbb{S}}^{\Delta}(n)\left(1+O\left(n^{-1/2}\right)\right).$$

By Theorem 5.12, the asymptotic enumeration of Δ-angular maps given by Theorem 5.11 also applies to Δ-angular dissections. The rest of this section is devoted to the proof of Theorem 5.12.

The inequalities $D_{\mathbb{S}}^{\Delta}(n) \leq M_{\mathbb{S}}^{\Delta}(n) \leq A_{\mathbb{S}}^{\Delta}(n)$ are obvious, hence it suffices to prove

$$A_{\mathbb{S}}^{\Delta}(n) - D_{\mathbb{S}}^{\Delta}(n) = O\left(\frac{A_{\mathbb{S}}^{\Delta}(n)}{\sqrt{n}}\right). \tag{5.19}$$

For this purpose, we give a sufficient condition for a map A in $\mathcal{A}_{\mathbb{S}}^{\Delta}$ to be the dual of a dissection in $D_{\mathbb{S}}^{\Delta}(n)$.

Let $\tau^\bullet$ be a doubly-rooted tree and let $e_1, \ldots, e_k$ be some edges appearing in this order on the path from the root-leaf to the marked leaf. One obtains $k+1$ doubly-rooted trees $\tau_0^\bullet, \ldots, \tau_k^\bullet$ by cutting the edges $e_1, \ldots, e_k$ in their middle (the middle point of e_i, $i = 1 \ldots k$, corresponds to the marked vertex of $\tau_{i-1}^\bullet$ and to the root-vertex of $\tau_i^\bullet$); see Figure 5.7. A doubly-rooted tree $\tau^\bullet$ is *balanced* if there exist three edges e_1, e_2, e_3 on the path from the root-leaf to the marked leaf such that cutting at these edges gives 4 doubly-rooted trees which are all two-sided. For instance, the tree at the left of Figure 5.7 is balanced.

Lemma 5.13 *Let $\mathbb{S}$ be a surface with boundary, let A be a map in $\mathcal{A}_{\mathbb{S}}^{\Delta}$ and let $(S, (\tau_1^\bullet, \ldots, \tau_e^\bullet), (\tau_1, \ldots, \tau_v))$ be its image by the bijection Φ. If all the doubly-rooted trees $\tau_1^\bullet, \ldots, \tau_e^\bullet$ are balanced, then A is the dual of a dissection M in $\mathcal{D}_{\mathbb{S}}^{\Delta}$.*

Proof. We suppose that all the trees $\tau_1^\bullet, \ldots, \tau_e^\bullet$ are balanced. Since the trees $\tau_1^\bullet, \ldots, \tau_e^\bullet$ are two-sided, the map A is the dual of a map M in $\mathcal{M}_{\mathbb{S}}^{\Delta}$. We want to show that the map M is a dissection, and for this purpose we will examine the *runs* of A (defined below).

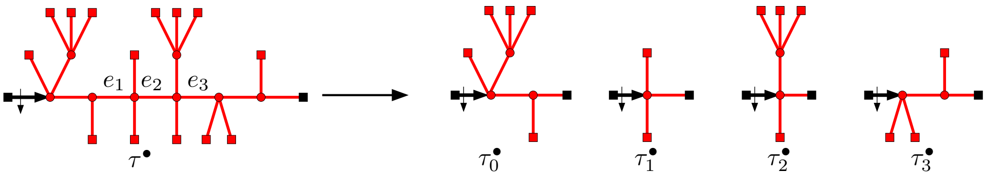


Figure 5.7 Cutting the tree $\tau^\bullet$ at the edges e_1, e_2, e_3.

A *run* for the map A is a sequence of vertices and edges $R = v_0, e_1, v_1, \dots, e_k, v_k$ (where e_i is an edge with endpoints v_{i-1} and v_i for $i = 1 \dots k$) encountered when turning around a face of A from one leaf to the next (that is, v_0 and v_k are leaves while $v_1, \dots, v_{k-1}$ are not). Since the surface $\mathbb{S}$ needs not be orientable, we consider both directions for turning around faces, so that $\overline{R} = v_k, e_k, \dots, v_1, e_1, v_0$ is also a run called the *reverse run*. It is clear from the definition of duality that the vertices of M are in bijection with the runs of A considered up to reversing, while the faces of M are in bijection with the non-leaf vertices of A. From these bijections, it is easy to see that the map M is a dissection if and only if

(i) no vertex appears twice in a run;

(ii) for any pair of distinct runs R, R' such that R' is not the reverse of R, the intersection $R \cap R'$ (that is, the set of vertices and edges which appear in both runs) is either empty, made of one vertex, or made of one edge and its two endpoints.

Indeed, Condition (i) ensures that no vertex of M is incident twice to the same face (equivalently, no face of M is incident twice to the same vertex) and Condition (ii) ensures that any pair of vertices of M which are both incident to two faces f, f' are the endpoints of an edge incident to both f and f' (equivalently, the intersection of two faces of M is either empty, a vertex, or an edge).

Observe that the runs of any tree having no vertex of degree 2 satisfy the Conditions (i) and (ii). We now compare the runs of A to the runs of some trees. For $i = 1 \dots e$, we choose some edges $f_{i,1}$, $f_{i,2}$,$f_{i,3}$ of the doubly-rooted tree $\tau_i^\bullet$ such that cutting at this edges gives 4 doubly-rooted trees $\tau_{i,j}^\bullet$, $j = 1 \dots 4$ which are all two-sided. Observe that by cutting the map A *at all but two* of the edges $f_{i,j}$, for $i = 1 \dots e$ and $j = 1, 2, 3$, one obtains a disjoint union of plane trees (none of which has a vertex of degree 2). Moreover, since all the trees $\tau_{i,j}^\bullet$ are two-sided, no run of the map A contains more than one of the edges $f_{i,j}$. Therefore, any run of A is also the run of a tree (having no vertex of degree 2). Hence, no vertex appears twice in a run of A and Condition (i) holds. Similarly, any pair of intersecting runs of A is a pair of runs of a tree. Thus, Condition (ii) holds for A. □

We now consider the generating function $\widehat{T}_\Delta(z)$ of doubly-rooted Δ-valent trees which are not balanced (counted by number of non-root, non-marked leaves). Since the number of non-root non-marked leaves of Δ-valent trees is congruent to $p = \gcd(\delta - 2,\ \delta \in \Delta)$, there exists a generating function $\widehat{Y}_\Delta(t)$ such that $\widehat{T}_\Delta(z) = \widehat{Y}_\Delta(z^p)$.

Lemma 5.14 *The generating function $\widehat{Y}_\Delta(t)$ of non-balanced trees is analytic in a domain dented at $t = \rho_\Delta{}^p$, and there is a constant $\widehat{\kappa}$ such that, in this domain, $\widehat{Y}_\Delta(t)$ admits the expansion*

$$\widehat{Y}_\Delta(t) =_{t \to \rho_\Delta{}^p} \widehat{\kappa} + O\left(\sqrt{1 - \frac{t}{\rho_\Delta{}^p}}\right).$$

Proof. Let $\tau^\bullet$ be a non-balanced doubly-rooted tree. If the tree $\tau^\bullet$ is two-sided, then there exists an edge e_1 on the path from the root-vertex to the marked vertex such that cutting the tree $\tau^\bullet$ at e_1 and at the edge e_1' following e_1 gives three doubly-rooted trees $\tau_1^\bullet$, $\tau_2^\bullet$, $\tau^{\bullet\prime}$ such that $\tau_1^\bullet$ is one-sided, $\tau_2^\bullet$ is a tree with one leg, and $\tau^{\bullet\prime}$ has no two edges e_2', e_3' such that cutting at these edges gives three two-sided trees. Continuing this decomposition and translating it into generating functions gives:

$$\widehat{T}_\Delta(z) \leq \widetilde{T}_\Delta(z) + \widetilde{T}_\Delta(z)T_{\Delta,1}(z)\left[\widetilde{T}_\Delta(z) + \widetilde{T}_\Delta(z)T_{\Delta,1}(z)\left[\widetilde{T}_\Delta(z) + \widetilde{T}_\Delta(z)^2T_{\Delta,1}(z)\right]\right],$$

where $\widetilde{T}_\Delta(z)$ is the generating function of one-sided trees and $T_{\Delta,1}(z)$ is the generating function of trees with one leg. By performing the change of variable $t = z^p$ one gets

$$\widehat{Y}_\Delta(t) \leq \widetilde{Y}_\Delta(t) + \widetilde{Y}_\Delta(t)Y_{\Delta,1}(t)\left[\widetilde{Y}_\Delta(t) + \widetilde{Y}_\Delta(t)Y_{\Delta,1}(t)\left[\widetilde{Y}_\Delta(t) + \widetilde{Y}_\Delta(t)^2Y_{\Delta,1}(t)\right]\right]. \tag{5.20}$$

The number of one-sided trees having n non-marked non-root leaves is $2T(n+1) = 2C(n)$ for $n > 0$ and 1 for $n = 0$. Hence, $\widetilde{T}_\Delta(z) = 2T_\Delta(z)/z - 1$ and $\widetilde{Y}_\Delta(t) = 2Y_\Delta(t) - 1$. Thus, Lemma 5.7 implies that the series $\widetilde{Y}_\Delta(t)$ is analytic and has an expansion of the form

$$\widetilde{Y}_\Delta(t) =_{t\to\rho_\Delta{}^p} \frac{2\tau_\Delta}{\rho_\Delta} + O\left(\sqrt{1 - \frac{t}{\rho_\Delta{}^p}}\right),$$

valid in a domain dented at $t = \rho_\Delta{}^p$. Similarly, Lemma 5.10 implies that the series $Y_{\Delta,1}(t)$ is analytic and has an expansion of the form

$$Y_{\Delta,1}(t) =_{t\to\rho_\Delta{}^p} \kappa_1 + O\left(\sqrt{1 - \frac{t}{\rho_\Delta{}^p}}\right),$$

valid in a domain dented at $t = \rho_\Delta{}^p$. The lemma follows from these expansions and Equation (5.20). □

We are now ready to bound the number of maps in $\mathcal{A}_{\mathbb{S}}^{\Delta}(n)$ which are not the dual of a dissection and prove Equation (5.19). Let $\widehat{\mathcal{A}}_{\mathbb{S}}^{\Delta}$ be the class of maps in $\mathcal{A}_{\mathbb{S}}^{\Delta}$ which are not the dual of a dissection in $\mathcal{D}_{\mathbb{S}}^{\Delta}$ and let $\widehat{A}_{\mathbb{S}}^{\Delta}(z) = A_{\mathbb{S}}^{\Delta}(z) - D_{\mathbb{S}}^{\Delta}(z)$ be the corresponding generating function. For any scheme S of $\mathbb{S}$, we also define $\widehat{F}_S^{\Delta}(z)$ as the generating function of maps in $\widehat{\mathcal{A}}_{\mathbb{S}}^{\Delta}$ which have scheme S. By Lemma 5.13, for any map A in $\widehat{\mathcal{A}}_{\mathbb{S}}^{\Delta}$, the image $\Phi(A) = (S, (\tau_1^\bullet, \ldots, \tau_e^\bullet), (\tau_1, \ldots, \tau_v))$ is such that one of the e doubly-rooted trees $\tau_1^\bullet, \ldots, \tau_e^\bullet$ is not balanced. Hence,

$$\widehat{F}_S^{\Delta}(z) \;\leq\; e\,\frac{\widehat{T}_\Delta(z)}{T'_\Delta(z)}\,F_S^{\Delta}(z), \tag{5.21}$$

where $T'_\Delta(z)$ is the generating function of doubly-rooted trees and $\widehat{T}_\Delta(z)$ is the generating function of non-balanced doubly-rooted $(\Delta \cup \{1\})$-valent trees. By summing Expression (5.21) over all schemes S of $\mathbb{S}$, one gets

$$\begin{aligned}
\widehat{A}_{\mathbb{S}}^{\Delta}(z) &= \sum_{S \text{ scheme}} \widehat{F}_S^{\Delta}(z) \leq (2 - 3\chi(\mathbb{S}))\,\frac{\widehat{T}_\Delta(z)}{T'_\Delta(z)} \sum_{S \text{ scheme}} F_S^{\Delta}(z) \\
&= (2 - 3\chi(\mathbb{S}))\,\frac{\widehat{T}_\Delta(z)}{T'_\Delta(z)} A_{\mathbb{S}}^{\Delta}(z),
\end{aligned} \tag{5.22}$$

since $(2 - 3\chi(\mathbb{S}))$ is the maximal number of edges of schemes of $\mathbb{S}$. Plugging these identity into Equation (5.22) and replacing z^p by t gives

$$\widehat{B}_{\mathbb{S}}^{\Delta}(t) \leq \frac{(2 - 3\chi(\mathbb{S}))\,\widehat{Y}_\Delta(t)}{Z_\Delta(t)} B_{\mathbb{S}}^{\Delta}(t),$$

where $\widehat{B}_{\mathbb{S}}^{\Delta}(t)$ is defined by $\widehat{A}_{\mathbb{S}}^{\Delta}(z) = z^{2\chi(\mathbb{S})}\widehat{B}_{\mathbb{S}}^{\Delta}(z^p)$ and $Z_\Delta(t)$ is defined by $T'_\Delta(z) = Z_\Delta(z^p)$. As shown before (see the proof of Lemma 5.7), the series $Z_\Delta(t) = ptY'_\Delta(t) + Y_\Delta(t)$ is analytic and non-zero in a domain dented at $t = \rho_\Delta{}^p$. Moreover, the expansion

$$Z_\Delta(t) \sim_{t\to\rho_\Delta{}^p} \frac{\gamma_\Delta\sqrt{p}}{2\rho_\Delta}\left(1 - \frac{t}{\rho_\Delta{}^p}\right)^{-1/2}$$

is valid in this domain. Given the expansion of $Z_\Delta(t)$ given by Equation (5.15) and the expansion of $\widehat{Y}_\Delta(t)$ given by Lemma 5.14, one obtains the expansion

$$\widehat{B}_{\mathbb{S}}^{\Delta}(t) =_{t\to\rho_\Delta{}^p} O\left(\sqrt{1-\frac{t}{\rho_\Delta{}^p}}\, B_{\mathbb{S}}^{\Delta}(t)\right)$$

in a domain dented at $t=\rho_\Delta{}^p$. By the Transfer Theorem between singularity types and coefficients asymptotic one obtains

$$[t^n]\widehat{B}_{\mathbb{S}}^{\Delta}(t) =_{n\to\infty} O\left(\frac{[t^n]B_{\mathbb{S}}^{\Delta}(t)}{\sqrt{n}}\right).$$

This completes the proof of Equation (5.19) and Theorem 5.11. □

5.6 Limit laws

In this section, we study the limit law of the number of structuring edges in Δ-dissections. Our method is based on generating function manipulation with the Method of Moments. We first make some remarks over the Method of Moments in Subsection 5.6.1, and then apply it in Subsection 5.6.2 in order to determine the limit law of the number of structuring edges of Δ-angular dissections.

5.6.1 A modification on the Method of Moments

As we have shown in the previous chapters, the Method of Moments provides a tool to deduce limit laws. In this chapter we need a modification of the method in the following way: we are studying several combinatorial classes, and we want to obtain limit laws for all of them. Suppose that $\mathcal{C}$ is a combinatorial class. We denote by $\mathbf{U}_n(\mathcal{C})$ the integer valued random variable corresponding to the value of a parameter $U:\mathcal{C}\to\mathbb{R}$ for elements $C\in\mathcal{C}$ chosen uniformly at random among those of size n (we suppose here that $\mathcal{C}(n)\neq\emptyset$).

Lemma 5.15 *Let $\mathcal{C}'$ be a subclass of the combinatorial class $\mathcal{C}$ such that $|\mathcal{C}(n)|\sim|\mathcal{C}'(n)|$, and let $X:\mathcal{C}\to\mathbb{R}$ be a parameter. Then the sequence of random variables $(\boldsymbol{X}_n(\mathcal{C}'))_{n\in\mathbb{N}}$ converge in distribution if and only if the sequence $(\boldsymbol{X}_n(\mathcal{C}))_{n\in\mathbb{N}}$ does. In this case, they have the same limit.*

Proof. We need to prove that, for all $x\in\mathbb{R}$, $\left|\mathbf{p}(\{\mathbf{X}_n(\mathcal{C}')\le x\})-\mathbf{p}(\{\mathbf{X}_n(\mathcal{C})\le x\})\right| \to_{n\to\infty} 0$. To this end, we consider a coupling of the variables $\mathbf{X}_n(\mathcal{C})$ and $\mathbf{X}_n(\mathcal{C}')$ obtained in the following way. Let $\mathfrak{C}$ and $\mathfrak{C}'$ be independent random variables whose value is an element chosen uniformly at random in $\mathcal{C}(n)$ and $\mathcal{C}'(n)$, respectively. Let $\mathfrak{C}''$ be the random variable whose value is $\mathfrak{C}$ if $\mathfrak{C}\in\mathcal{C}'(n)$ and $\mathfrak{C}'$ otherwise is uniformly random in $\mathcal{C}'(n)$. Hence, the random variable $\mathbf{X}_n(\mathcal{C})$ and $\mathbf{X}_n(\mathcal{C}')$ have the same distribution as $X(\mathfrak{C})$ and $X(\mathfrak{C}'')$ respectively. Thus, for all $x\in\mathbb{R}$,

$$\begin{aligned}|\mathbf{p}(\{\mathbf{X}_n(\mathcal{C}')\le x\})-\mathbf{p}(\{\mathbf{X}_n(\mathcal{C})\le x\})| &\le \mathbf{p}(\{\mathfrak{C}''\neq\mathfrak{C}\})\le\mathbf{p}(\{\mathfrak{C}\notin\mathcal{C}'(n)\})\\ &\le 1-\frac{|\mathcal{C}'(n)|}{|\mathcal{C}(n)|} \to_{n\to\infty} 0.\end{aligned}$$

□

5.6.2 Number of structuring edges in Δ-angular dissections

We are now ready to study the limit law of the number of structuring edges in Δ-angular dissections. Recall that for a map M in $\mathcal{M}_{\mathbb{S}}^{\Delta}$, an edge is said *structuring* if either it does not separate the surface $\mathbb{S}$ or if it separates $\mathbb{S}$ into two parts, none of which is homeomorphic to a disc. For a map $A\in\mathcal{A}_{\mathbb{S}}^{\Delta}$, we call *structuring* the edges of the submap of A obtained by recursively deleting all leaves. These are the edges whose deletion either does not disconnect the map or disconnect it in two parts, none of which is reduced to a tree. With this definition, if the maps $M \in \mathcal{M}_{\mathbb{S}}^{\Delta}$ and $A \in \mathcal{A}_{\mathbb{S}}^{\Delta}$ are dual of each other, then their structuring edges correspond by duality. We denote by U the parameter corresponding to the number of structuring edges so that for $\mathcal{C} \in \{\mathcal{A}_{\mathbb{S}}^{\Delta}, \mathcal{M}_{\mathbb{S}}^{\Delta}, \mathcal{D}_{\mathbb{S}}^{\Delta}\}$, the random variable $\mathbf{U}_n(\mathcal{C})$ gives the number of structuring edges of a map $C \in \mathcal{C}$ chosen uniformly at random among those of size n. Recall that the size of maps in $\mathcal{A}_{\mathbb{S}}^{\Delta}$ is the number of leaves, while the size of maps in $\mathcal{M}_{\mathbb{S}}^{\Delta}$ is the number of vertices. Hence, the bivariate GFs $A_{\mathbb{S}}^{\Delta}(u,z)$, $M_{\mathbb{S}}^{\Delta}(u,z)$ and $D_{\mathbb{S}}^{\Delta}(u,z)$ associated to the parameter $\mathbf{U}$ for the classes $\mathcal{A}_{\mathbb{S}}^{\Delta}$, $\mathcal{M}_{\mathbb{S}}^{\Delta}$ and $\mathcal{D}_{\mathbb{S}}^{\Delta}$ satisfy that $A_{\mathbb{S}}^{\Delta}(z) = A_{\mathbb{S}}^{\Delta}(1,z)$, $M_{\mathbb{S}}^{\Delta}(z) = M_{\mathbb{S}}^{\Delta}(1,z)$ and $D_{\mathbb{S}}^{\Delta}(z) = D_{\mathbb{S}}^{\Delta}(1,z)$.

Recall that the size of maps in $\mathcal{C}_{\mathbb{S}}^{\Delta} \in \{\mathcal{A}_{\mathbb{S}}^{\Delta}, \mathcal{M}_{\mathbb{S}}^{\Delta}, \mathcal{D}_{\mathbb{S}}^{\Delta}\}$ is congruent to $2\chi(\mathbb{S})$ modulo $p = \gcd(\delta - 2,\ \delta \in \Delta)$. Moreover, it will be shown shortly that the average number of structuring edges of maps $\mathcal{A}_{\mathbb{S}}^{\Delta}(n)$ is $O(\sqrt{n})$. This leads us to consider the following rescaled random variables: for any set $\Delta \subseteq \mathbb{N}^{\geq 3}$ we define the rescaled random variables $\mathbf{X}_n(\mathcal{C}_{\mathbb{S}}^{\Delta})$, for the class $\mathcal{C}_{\mathbb{S}}^{\Delta} \in \{\mathcal{A}_{\mathbb{S}}^{\Delta}, \mathcal{M}_{\mathbb{S}}^{\Delta}, \mathcal{D}_{\mathbb{S}}^{\Delta}\}$ by

$$\mathbf{X}_n(\mathcal{C}_{\mathbb{S}}^{\Delta}) \;=\; \frac{\mathbf{U}_{np+2\chi(\mathbb{S})}(\mathcal{C}_{\mathbb{S}}^{\Delta})}{\sqrt{np+2\chi(\mathbb{S})}},$$

where $p = \gcd(\delta - 2,\ \delta \in \Delta)$.

We also define some continuous random variables $\mathbf{X}_k$ as follows. For all non-negative integer k, we denote by $\mathbf{X}_k$ the real random variable with probability density function

$$g_k(t) = \frac{2\,t^{3k}\,e^{-t^2}}{\Gamma\left(\frac{1+3k}{2}\right)}\,\mathbb{I}_{[0,\infty[}(t). \tag{5.23}$$

Equivalently, $\mathbf{X}_k$ corresponds with the square root of a Gamma distribution with parameters 1 and $(3k+1)/2$. The main result for structuring edges is the following theorem:

Theorem 5.16 *Let $\mathbb{S}$ be any surface with boundary distinct from the disc, let $\Delta \subseteq \mathbb{N}^{\geq 3}$ and $p = \gcd(\delta - 2,\ \delta \in \Delta)$. The sequences of random variables $\left(\boldsymbol{X}_n(\mathcal{M}_{\mathbb{S}}^{\Delta})\right)_{n\in\mathbb{N}}$ and $\left(\boldsymbol{X}_n(\mathcal{D}_{\mathbb{S}}^{\Delta})\right)_{n\in\mathbb{N}}$ corresponding respectively to the rescaled number of structuring edges in Δ-angular maps and dissections both converge in distribution to the random variable $\boldsymbol{X}_{\mathbb{S}}^{\Delta} = \left(\frac{\gamma_\Delta}{\rho_\Delta}\right) \boldsymbol{X}_{-\chi(\mathbb{S})}$ where γ_Δ and ρ_Δ are the constants defined by Expression (5.3).*

In the case $\Delta = 3$, one has $p = 1$, $\rho_\Delta = 1/4$, $\gamma_\Delta = 1/2$. Hence, by Theorem 5.16 the rescaled number of structuring edges of uniformly random simplicial decompositions $\mathbf{U}_n(\mathcal{D}_{\mathbb{S}})/\sqrt{n}$ converges to the random variable $2\mathbf{X}_{-\chi(\mathbb{S})}$ whose probability density function is

$$f(t) \;=\; \frac{1}{2} g_{-\chi(\mathbb{S})}\left(\frac{t}{2}\right) \;=\; \frac{1}{\Gamma\!\left(\frac{1-3\chi(\mathbb{S})}{2}\right)} \left(\frac{t}{2}\right)^{-3\chi(\mathbb{S})} \quad e^{-t^2/4}\ \mathbb{I}_{[0,\infty[}(t),$$

The rest of this section is devoted to the proof of Theorem 5.16.

By duality, the classes $\mathcal{M}_{\mathbb{S}}^{\Delta}$ and $\mathcal{D}_{\mathbb{S}}^{\Delta}$ can be considered as subclasses of $\mathcal{A}_{\mathbb{S}}^{\Delta}$. Moreover, by Lemma 5.12, $|\mathcal{M}_{\mathbb{S}}^{\Delta}(n)| \sim |\mathcal{D}_{\mathbb{S}}^{\Delta}(n)| \sim |\mathcal{A}_{\mathbb{S}}^{\Delta}(n)|$. Thus, by Lemma 5.15 it is sufficient to prove that the rescaled random variable $\mathbf{X}_n(\mathcal{A}_{\mathbb{S}}^{\Delta})$ converges to $\mathbf{X}_{\mathbb{S}}^{\Delta}$ in distribution. In order to apply the Method of Moments in the version it appears in Lemma 1.5, we first check that the variable $\mathbf{X}_{\mathbb{S}}^{\Delta}$ satisfies condition (A).

Lemma 5.17 *The random variable $\boldsymbol{X}_{\mathbb{S}}^{\Delta}$ satisfies condition (A) in Lemma 1.4, and its rth moment is*

$$\mathbb{E}\left[\left(\boldsymbol{X}_{\mathbb{S}}^{\Delta}\right)^r\right] \;=\; \left(\frac{\gamma_\Delta}{\rho_\Delta}\right)^r \frac{\Gamma\!\left(\frac{r+1-3\chi(\mathbb{S})}{2}\right)}{\Gamma\!\left(\frac{1-3\chi(\mathbb{S})}{2}\right)}. \tag{5.24}$$

Proof. By definition of $\mathbf{X}_{\mathbb{S}}^{\Delta}$, $\mathbb{E}\left[\left(\mathbf{X}_{\mathbb{S}}^{\Delta}\right)^r\right] = \left(\frac{\gamma_\Delta}{\rho_\Delta}\right)^r \mathbb{E}\left[\left(\mathbf{X}_{-\chi(\mathbb{S})}\right)^r\right]$ where $\mathbf{X}_k$ is defined in Equation (5.23). The moments of $\mathbf{X}_k$ can be obtained by making the change of variable $u = t^2$:

$$\begin{aligned}\mathbb{E}\left[\mathbf{X}_k{}^r\right] &= \frac{2}{\Gamma\left(\frac{1+3k}{2}\right)}\int_0^\infty t^{r+3k}e^{-t^2}dt = \frac{1}{\Gamma\left(\frac{1+3k}{2}\right)}\int_0^\infty u^{(r+3k-1)/2}e^{-u}du \\ &= \frac{\Gamma\left(\frac{r+1+3k}{2}\right)}{\Gamma\left(\frac{1+3k}{2}\right)}.\end{aligned}$$

Thus, Expression (5.24) holds. Moreover, since $\Gamma\left(\frac{r+1+3k}{2}\right) \leq \Gamma(r+1) = r!$ for r large enough, Condition (A) from Lemma 1.4 holds for any R less than $\left(\frac{\rho_\Delta}{\gamma_\Delta\sqrt{p}}\right)$. □

We now study the moments of the random variables $\mathbf{U}_n(\mathcal{A}_{\mathbb{S}}^{\Delta})$ corresponding to the number of structuring edges. For this purpose, we shall exploit once again the decomposition Φ of the maps in $\mathcal{A}_{\mathbb{S}}^{\Delta}$ (Figure 5.6). This decomposition leads us to consider the *spine edges* of doubly-rooted trees, that is, the edges on the path from the root-leaf to the marked leaf. Indeed, if the image of a map $A \in \mathcal{A}_{\mathbb{S}}^{\Delta}$ by the decomposition Φ is $(S, (\tau_1^\bullet, \ldots, \tau_e^\bullet), (\tau_1, \ldots, \tau_v))$, where the doubly-rooted trees $\tau_1^\bullet, \ldots, \tau_{e-1}^\bullet$ correspond to the $e-1$ non-root edges of the scheme S, then the structuring edges of the map A are the spine edges of the doubly-rooted trees $\tau_1^\bullet, \ldots, \tau_{e-1}^\bullet$.

We denote by V the parameter corresponding to the number of spine edges and by

$$T_\Delta^\bullet(u,z) = \sum_{\tau^\bullet} u^{V(\tau^\bullet)} z^{|\tau^\bullet|}$$

the associated bivariate generating function (here the sum is over all doubly-rooted $(\Delta \cup \{1\})$-valent trees and $|\tau^\bullet|$ is the number of leaves which are neither marked nor the root-leaf). The decomposition of doubly-rooted trees into a sequence of trees with one leg by the decomposition represented in Figure 5.8 shows that

$$T_\Delta^\bullet(u,z) = \frac{u}{1 - uT_{\Delta,1}(z)} = \left(\frac{1}{u} - T_{\Delta,1}(z)\right)^{-1},$$

where $T_{\Delta,1}(z)$ is the generating function of $(\Delta \cup \{1\})$-valent trees with one leg. Moreover, plugging the expression of $T_{\Delta,1}(z)$ given by Equation (5.12) gives

$$T_\Delta^\bullet(u,z) = \left(\frac{1}{u} - 1 + \frac{1}{T_\Delta'(z)}\right)^{-1}. \tag{5.25}$$

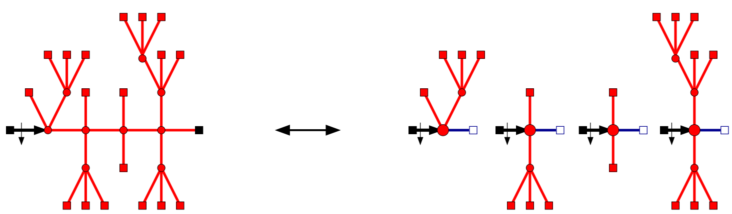


Figure 5.8 Decomposition of doubly-rooted trees as a sequence of trees with one leg.

We now translate the bijection induced by Φ (Lemma 5.8) in terms of generating functions. For a scheme S of $\mathbb{S}$, we denote by $F_S^\Delta(u,z)$ the generating function of maps in $\mathcal{A}_{\mathbb{S}}^{\Delta}$ having scheme S counted by number of leaves and structuring edges. This gives

$$A_{\mathbb{S}}^{\Delta}(u,z) = \sum_{S \text{ scheme}} F_S^\Delta(u,z). \tag{5.26}$$

The bijection induced by Φ (Lemma 5.8) and the correspondence between spine edges and structuring edges gives

$$F_S^{\Delta}(u,z) \;=\; z\,T'_{\Delta}(z)\,(T^{\bullet}_{\Delta}(u,z))^{e_S-1}\prod_{i=1}^{v_S} T_{\Delta,d_i(S)-1}(z), \tag{5.27}$$

for a scheme S with e_S edges and v_S non-root vertices of respective degrees $d_1(S),\ldots,\, d_{v_S}(S)$. Combining Equations (5.25), (5.26) and (5.27) gives

$$A_{\mathbb{S}}^{\Delta}(u,z) \;=\; z\,T'_{\Delta}(z)\sum_{S \text{ scheme}}\left(\frac{1}{u}-1+\frac{1}{T'_{\Delta}(z)}\right)^{1-e_S}\prod_{i=1}^{v_S} T_{\Delta,d_i(S)-1}(z),$$

where the sum is defined over the schemes S of $\mathbb{S}$ having e_S edges and v_S non-root vertices of respective degree $d_1(S),\ldots d_{v_S}(S)$. Making the change of variable $t=z^p$ gives

$$B_{\mathbb{S}}^{\Delta}(u,t) \;=\; Z_{\Delta}(t)\sum_{S \text{ scheme}}\left(\frac{1}{u}-1+\frac{1}{Z_{\Delta}(t)}\right)^{1-e_S}\prod_{i=1}^{v_S} Y_{\Delta,d_i(S)-1}(t), \tag{5.28}$$

where $B_{\mathbb{S}}^{\Delta}(u,t)$ is defined by $A_{\mathbb{S}}^{\Delta}(u,z) = z^{2\chi(\mathbb{S})}B_{\mathbb{S}}^{\Delta}(u,z^p)$ and $Z_{\Delta}(t) = ptY'_{\Delta}(t)+Y_{\Delta}(t)$ satisfies $T'_{\Delta}(z)=Z_{\Delta}(z^p)$.

By differentiating Equation (5.28) with respect to the variable u and keeping only the dominant part in the asymptotic $(u,t)\to(1,\rho_{\Delta}{}^p)$ (recall that $Z_{\Delta}(t)\to_{t\to\rho_{\Delta}{}^p}\infty$ by Equation (5.16)) one gets a singular expression for $\frac{\partial^r}{\partial u^r}B_{\mathbb{S}}^{\Delta}(u,t)$ when $t\to\rho_{\Delta}{}^p$:

$$Z_{\Delta}(t)\sum_{S \text{ scheme}}\frac{(e_S+r-2)!}{u^{2r}(e_S-2)!}\left(\frac{1}{u}-1+\frac{1}{Z_{\Delta}(t)}\right)^{1-r-e_S}\prod_{i=1}^{v_S} Y_{\Delta,d_i(S)-1}(t).$$

Finally, writing $u=1$,

$$\frac{\partial^r}{\partial u^r}B_{\mathbb{S}}^{\Delta}(1,t)\sim_{t\to\rho_{\Delta}{}^p}\sum_{S \text{ scheme}}\frac{(e_S+r-2)!}{(e_S-2)!}\,Z_{\Delta}(t)^{e_S+r}\prod_{i=1}^{v_S} Y_{\Delta,d_i(S)-1}(t).$$

The singular expansion of the series $Y_{\Delta,\ell}(t)$ and $Z_{\Delta}(t)$ given by Lemma 5.10 and Equation (5.16) gives,

$$\frac{\partial^r}{\partial u^r}B_{\mathbb{S}}^{\Delta}(1,t)\sim\sum_{S \text{ scheme}}\frac{(e_S+r-2)!}{(e_S-2)!}\left(\frac{\gamma_{\Delta}\sqrt{p}}{2\rho_{\Delta}}\right)^{e_S+r}\left(1-\frac{t}{\rho_{\Delta}{}^p}\right)^{-\frac{e_S+r}{2}}\prod_{i=1}^{v_S}\kappa_{d_i(S)-1}.$$

Since the maximum number of edges e_S of a scheme S of $\mathbb{S}$ is $2-3\chi(\mathbb{S})$, with equality only for the $a(\mathbb{S})$ cubic schemes, and since that cubic schemes have $v_S=1-2\chi(\mathbb{S})$ non-root vertices, one gets

$$\frac{\partial^r}{\partial u^r}B_{\mathbb{S}}^{\Delta}(1,t)\sim a(\mathbb{S})\,\kappa_2{}^{1-2\chi(\mathbb{S})}\,\frac{(r-3\chi(\mathbb{S}))!}{(-3\chi(\mathbb{S}))!}\left(\frac{\gamma_{\Delta}\sqrt{p}}{2\rho_{\Delta}}\right)^{r+2-3\chi(\mathbb{S})}\left(1-\frac{t}{\rho_{\Delta}{}^p}\right)^{-\frac{r+2-3\chi(\mathbb{S})}{2}}. \tag{5.29}$$

The generating function $\frac{\partial^r}{\partial u^r}B_{\mathbb{S}}^{\Delta}(1,t)$ is analytic in a domain dented at $t=\rho_{\Delta}^p$. Hence, the asymptotic expansion (5.29) implies

$$[t^n]\frac{\partial^r}{\partial u^r}B_{\mathbb{S}}^{\Delta}(1,t)\sim a(\mathbb{S})\kappa_2^{1-2\chi(\mathbb{S})}\,\frac{(r-3\chi(\mathbb{S}))!}{(-3\chi(\mathbb{S}))!}\left(\frac{\gamma_{\Delta}\sqrt{p}}{2\rho_{\Delta}}\right)^{r+2-3\chi(\mathbb{S})}\frac{n^{\frac{r-3\chi(\mathbb{S})}{2}}\rho_{\Delta}{}^{-np}}{\Gamma(\frac{r+2-3\chi(\mathbb{S})}{2})}.$$

Using $\kappa_2=\left(\frac{\rho_{\Delta}}{\gamma_{\Delta}}\right)^2$ and the asymptotic of $[t^n]B_{\mathbb{S}}^{\Delta}(t)$ given by Equation (5.17) one gets

$$\frac{[t^n]\frac{\partial^r}{\partial u^r}B_{\mathbb{S}}^{\Delta}(u,t)}{(np+2\chi(\mathbb{S}))^{r/2}[t^n]B_{\mathbb{S}}^{\Delta}(t)}\to_{n\to\infty}\left(\frac{\gamma_{\Delta}}{2\rho_{\Delta}}\right)^r\frac{(r-3\chi(\mathbb{S}))!}{(-3\chi(\mathbb{S}))!}\,\frac{\Gamma\left(\frac{2-3\chi(\mathbb{S})}{2}\right)}{\Gamma\left(\frac{r+2-3\chi(\mathbb{S})}{2}\right)}.$$

The right-hand-side of this equation can be simplified by writing factorials in terms the Gamma function and using Gauss duplication formula. This gives

$$\frac{[t^n]\frac{\partial^r}{\partial u^r}B_{\mathbb{S}}^{\Delta}(1,t)}{(np+2\chi(\mathbb{S}))^{r/2}[t^n]B_{\mathbb{S}}^{\Delta}(t)} \to_{n\to\infty} \left(\frac{\gamma_\Delta}{\rho_\Delta}\right)^r \frac{\Gamma\left(\frac{r+1-3\chi(\mathbb{S})}{2}\right)}{\Gamma\left(\frac{1-3\chi(\mathbb{S})}{2}\right)}.$$

Comparing this expression with the r-th moment of $\mathbf{X}_{\mathbb{S}}^{\Delta}$ (Lemma 5.17) gives

$$\frac{[z^{np+2\chi(\mathbb{S})}]\frac{\partial^r}{\partial u^r}A_{\mathbb{S}}^{\Delta}(u,z)}{(np+2\chi(\mathbb{S}))^{r/2}\,[z^{np+2\chi(\mathbb{S})}]A_{\mathbb{S}}^{\Delta}(1,z)} = \frac{[t^n]\frac{\partial^r}{\partial u^r}B_{\mathbb{S}}^{\Delta}(u,t)}{(np+2\chi(\mathbb{S}))^{r/2}\,[t^n]B_{\mathbb{S}}^{\Delta}(t)},$$

which tends to $\mathbb{E}\left[\left(\mathbf{X}_{\mathbb{S}}^{\Delta}\right)^r\right]$ when n tends to ∞. This is exactly Condition (B') in Lemma 1.5 for the convergence of $\mathbf{X}_n(\mathcal{C}_{\mathbb{S}}^{\Delta}) = \frac{\mathbf{U}_{np+2\chi(\mathbb{S})}(\mathcal{C}_{\mathbb{S}}^{\Delta})}{\sqrt{np+2\chi(\mathbb{S})}}$ to $\mathbf{X}_{\mathbb{S}}^{\Delta}$. Theorem 5.16 then follows from Lemmas 1.5 and 5.15. □

5.7 Determining the constants: functional equations for cubic maps.

In this section, we give equations determining the constant $a(\mathbb{S})$ appearing in Theorems 5.3 and 5.11. Recall that for any surface $\mathbb{S}$ with boundary, $a(\mathbb{S})$ denotes the number of cubic schemes of $\mathbb{S}$. Our method for determining $a(\mathbb{S})$ is inspired by the works of Bender and Canfield [4] and Gao [21]. Proofs are omitted.

We call *k-marked near-cubic* maps the rooted maps having k marked vertices distinct from the root-vertex and such that every non-root, non-marked vertex has degree 3. For any integer $g \geq 0$ we consider the orientable surface of genus g without boundary $\mathbb{T}_g$. For any integer $k \geq 0$, we denote by $\mathcal{O}_{g,k}$ the set of k-marked near-cubic maps on $\mathbb{T}_g$ and we denote by $O_{g,k}(x, x_1, x_2, \ldots, x_k) = O_{g,k}(z, x, x_1, x_2, \ldots, x_k)$ the corresponding generating function. More precisely,

$$O_{g,k}(x, x_1, x_2, \ldots, x_k) = \sum_{M\in\mathcal{O}_{g,k}} x^{d(M)}x_1^{d_1(M)}\ldots x_k^{d_k(M)}z^{e(M)},$$

where $e(M)$ is the number of edges, $d(M)$ is the degree of the root-vertex and $d_1(M), \ldots, d_k(M)$ are the respective degrees of the marked vertices (for a natural canonical order of the marked vertices that we do not explicit here).Similarly, we consider the non-orientable surface of genus g without boundary $\mathbb{P}_g$. We denote by $\mathcal{P}_{g,k}$ the set of k-marked near-cubic maps on $\mathbb{P}_g$ and we denote by $P_{g,k}(x, x_1, x_2, \ldots, x_k) = P_{g,k}(z, x, x_1, x_2, \ldots, x_k)$ the corresponding generating function. Recall that for any surface $\mathbb{S}$ with boundary, the cubic schemes of $\mathbb{S}$ have $e = 2 - 3\chi(\mathbb{S})$ edges so that

$$a(\mathbb{S}) \;=\; \begin{cases} [x^1z^{2-3\chi(\mathbb{S})}]O_{1-\chi(\overline{\mathbb{S}})/2,0}(x,z) & \text{if the surface } \mathbb{S} \text{ is orientable,} \\ [x^1z^{2-3\chi(\mathbb{S})}]P_{2-\chi(\overline{\mathbb{S}}),0}(x,z) & \text{otherwise.} \end{cases} \tag{5.30}$$

We now give a system of functional equation determining the series $O_{g,k}$ and $P_{g,k}$ uniquely.

Proposition 5.18 *The series $(O_{g,k})_{g,k\in\mathbb{N}}$ are completely determined (as power series in z with* *polynomial coefficient in $x, x_1, \ldots, x_k$) by the following system of equations:*

$$\begin{aligned}
0 \quad = \quad & -O_{g,k}(x, x_1, \ldots, x_k) + \\
& c_0 \; + \; \frac{z}{x}\left(O_{g,k}(x, x_1, \ldots, x_k) - c_0 - x[x^1]O_{g,k}(x, x_1, \ldots, x_k)\right) + \\
& \frac{x x_k z}{x - x_k}\left(x O_{g,k-1}(x, x_1, \ldots, x_{k-1}) - x_k O_{g,k-1}(x_k, x_1, \ldots, x_{k-1})\right) + \\
& x^2 z \sum_{i=0}^{g} \sum_{j=0}^{k} O_{i,j}(x, x_1, \ldots, x_j) O_{g-i,k-j}(x, x_{j+1}, \ldots, x_k) + \\
& x^3 z \sum_{j=1}^{k+1} \left(\frac{\partial}{\partial x_j} O_{g-1,k+1}(x, x_1, \ldots, x_{k+1})\right)\Bigg|_{x_j = x}\Bigg|_{x_{j+1} = x_j} \cdots \Bigg|_{x_{k+1} = x_k}, \qquad (5.31)
\end{aligned}$$

where $c_0 = [x^0]O_{g,k}(x, x_1, \ldots, x_k)$ is equal to 1 if $g = k = 0$ and 0 otherwise.

Similarly, the series $Q_{g,k} = O_{g/2,k} + P_{g,k}$ (where $O_{g/2,k}$ is 0 if g is odd) are determined by the following system of equations:

$$\begin{aligned}
0 \quad = \quad & -Q_{g,k}(x, x_1, \ldots, x_k) + \\
& c_0 \; + \; \frac{z}{x}\left(Q_{g,k}(x, x_1, \ldots, x_k) - c_0 - x[x^1]Q_{g,k}(x, x_1, \ldots, x_k)\right) + \\
& \frac{x x_k z}{x - x_k}\left(x Q_{g,k-1}(x, x_1, \ldots, x_{k-1}) - x_k Q_{g,k-1}(x_k, x_1, \ldots, x_{k-1})\right) + \\
& x^2 z \sum_{i=0}^{g} \sum_{j=0}^{k} Q_{i,j}(x, x_1, \ldots, x_j) Q_{g-i,k-j}(x, x_{j+1}, \ldots, x_k) + \\
& 2x^3 z \sum_{j=1}^{k+1} \left(\frac{\partial}{\partial x_j} Q_{g-2,k+1}(x, x_1, \ldots, x_{k+1})\right)\Bigg|_{x_j = x}\Bigg|_{x_{j+1} = x_j} \cdots \Bigg|_{x_{k+1} = x_k} + \\
& x^3 z \frac{\partial}{\partial x} Q_{g-1,k}(x, x_1, \ldots, x_k). \qquad (5.32)
\end{aligned}$$

The proof of Proposition 5.18 is omitted. We only indicate that the third summand in the right-hand-side of Equation (5.31) (resp. Equation (5.32)) corresponds to maps in $\mathcal{O}_{g,k}$ (resp. $\mathcal{O}_{g/2,k} \cup \mathcal{P}_{g,k}$) such that the root-edge joins the root-vertex to a non-root non-marked vertex; the forth term correspond to maps such that the root-edge joins the root-vertex to a marked vertex; the fifth summand corresponds to maps such that the root-edge is a loop which separates the surface $\mathbb{T}_g$ (resp. $\mathbb{P}_g$) into two connected-components; the sixth summand (resp. sixth and seventh summands) corresponds to maps such that the root-edge is a loop which does not separate the surface.

Proposition (5.18) together with Equation 5.30 give a recursive way for computing the constants $a(\mathbb{S})$ for any surface $\mathbb{S}$. The first values are given for orientable surfaces in Table 5.1 and for non-orientable surfaces and in Table 5.2. The first line in Table 5.1 corresponds to rooted planar cubic maps with β edges. These maps where enumerated in [39] and a explicit formula exists in this case:

$$a(\mathbb{S}) \quad = \quad \frac{2^{\beta(\mathbb{S})}(3\beta(\mathbb{S}) - 6)!!}{8\,\beta(\mathbb{S})!(\beta(\mathbb{S}) - 2)!!}.$$

The first column in Table 5.1 corresponds to rooted cubic maps with a single face on the g-torus. These maps where first enumerated by Lehman and Walsh [33]. Indeed, a special case of [33, Equation (9)] gives the following formula

$$a(\mathbb{S}) \quad = \quad \frac{2(6g - 3)!}{12^g g!(3g - 2)!},$$

which was also proved bijectively in [10]. Related results for the case $\beta(\mathbb{S}) = 1$ can be found in [2].

Orientable	$\beta(\mathbb{S})=1$	$\beta(\mathbb{S})=2$	$\beta(\mathbb{S})=3$	$\beta(\mathbb{S})=4$
Genus 0: $\chi(\overline{\mathbb{S}}) = 2$	0	1	4	32
Genus 1: $\chi(\overline{\mathbb{S}}) = 0$	1	28	664	14912
Genus 2: $\chi(\overline{\mathbb{S}}) = -2$	105	8112	396792	15663360
Genus 3: $\chi(\overline{\mathbb{S}}) = -4$	50050	6718856	51778972	30074896256

Table 5.1 The number $a(\mathbb{S})$ of cubic schemes of orientable surfaces.

Non-orientable	$\beta(\mathbb{S})=1$	$\beta(\mathbb{S})=2$	$\beta(\mathbb{S})=3$	$\beta(\mathbb{S})=4$
Genus 1: $\chi(\overline{\mathbb{S}}) = 1$	1	9	118	1773
Genus 2: $\chi(\overline{\mathbb{S}}) = 0$	6	174	4236	97134
Genus 3: $\chi(\overline{\mathbb{S}}) = -1$	128	6786	249416	7820190
Genus 4: $\chi(\overline{\mathbb{S}}) = -2$	3780	301680	15139800	610410600

Table 5.2 The number $a(\mathbb{S})$ of cubic schemes of non-orientable surfaces.

5.8 Concluding remarks

We have found a solution of the problem of enumerating Δ-angular maps on arbitrary surfaces with boundary, with the restriction that all vertices lie on the boundary. The technique used in this chapter has consisted in exploiting a tree structure of dual maps.

This same philosophy can be used to enumerate related constructions. For instance, the seminal ideas used here can be adapted to obtain upper bounds for the number of non-crossing partitions on surfaces with boundary. In this case, we are not under the map enumeration framework, and we need some structural results to restrict ourselves to the case of maps.

This enumeration of non-crossing partitions has been used in [46] to build a framework for the design of $2^{\mathcal{O}(k)} \cdot n$ step dynamic programming algorithms for surface-embedded graphs on n vertices of branchwidth at most k. The approach is based on a new type of branch decomposition called *surface cut decomposition*, which generalizes sphere cut decompositions for planar graphs, and where dynamic programming should be applied for each particular problem. The key idea is that the size of the tables of a dynamic programming algorithm over a surface cut decomposition can be upper bounded in terms of the non-crossing partitions in surfaces with boundary.

Bibliography

[1] E. Artin. *Gamma Function.* Holt-Rinehart and Winston Inc., 1964. English translation of *Einführung in die Theorie der Gammafunktion*, 1931.

[2] R. Bacher and A. Vdovina. Counting 1-vertex triangulations of oriented surfaces. *Discrete Math.*, 246(1-3):13–27, 2002.

[3] C. Banderier, P. Flajolet, G. Schaeffer, and M. Soria. Random maps, coalescing saddles, singularity analysis, and airy phenomena. *Random Strucures Algorithms*, 19:194–246, 2001.

[4] E. Bender and R. Canfield. The number of rooted maps on an orientable surface. *J. Combin. Theory Ser. B*, 53(2):293–299, 1991.

[5] E. A. Bender, Z. Gao, and N. C. Wormald. The number of 2-connected labelled planar graphs. *Electron. J. Combin.*, 9:194–246, 2002.

[6] O. Bernardi, M. Noy, and D. Welsh. On the growth of minor-closed families of graphs. Available at `arXiv:0710.2995v1`, [math.CO].

[7] O. Bernardi and J. Rué. Counting simplicial decompositions in surfaces with boundaries. Manuscript available at `http://lanl.arxiv.org/abs/0901.1608`.

[8] P. Billingsley. *Probability and measure.* Wiley Series in Probability and Mathematical Statistics, 1995.

[9] M. Bodirsky, O. Giménez, M. Kang, and M. Noy. Enumeration and limit laws for series-parallel graphs. *European J. Combin.*, 28(8):2091–2105, 2007.

[10] G. Chapuy. The structure of dominant unicellular maps, and a connnection between maps of positive genus and planar labelled trees. To appear at *Probability Theory and Related Fields.* Available at `arXiv:0804.0546`.

[11] G. Chapuy, É. Fusy, M. Kang, and B. Shoilekova. A complete grammar for decomposing a family of graphs into 3-connected components. *Electron. J. Combin.*, 15(1):Research Paper 148, 39, 2008.

[12] G. Chapuy, M. Marcus, and G. Schaeffer. A bijection for rooted maps on orientable surfaces. Submitted. See ArXiv:0712.3649.

[13] R. Diestel. *Graph decompositions. A study in infinite graph theory.* Oxford U. Press, New York, 1990.

[14] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, third edition, 2005.

[15] M. Drmota, O. Giménez, and M. Noy. Degree distribution in random planar graphs. Submitted.

[16] P. Edelman and V. Reiner. Catalan triangulations of the möbius band. *Graphs Combin.*, 13(3):231–243, 1997.

[17] P. Flajolet and M. Noy. Analytic combinatorics of non-crossing configurations. *Discrete Math.*, 204(1-3):203–229, 1999.

[18] P. Flajolet and A. Odlyzko. Singularity analysis of generating functions. *SIAM J. Discrete Math.*, 3(2):216–240, 1990.

[19] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge Univ. Press, 2008.

[20] A. Gagarin, G. Labelle, P. Leroux, and T. Walsh. Structure and enumeration of two-connected graphs with prescribed three-connected components. *Advances in Applied Mathematics*, 43(1):46 – 74, 2009.

[21] Z. Gao. The number of degree restricted maps on general surfaces. *Discrete Math.*, 123(1-3):47–63, 1993.

[22] Z. Gao and N. C. Wormald. The size of the largest components in random planar maps. *SIAM J. Discrete Math.*, (12):217–228, 1999.

[23] Z. Gao, Y. Xiao, and J. Wang. The average number of non-separating edges in Catalan triangulations of the Mbius band and the cylinder. Unpublished manuscript.

[24] S. Gerke, O. Giménez, M. Noy, and A. Weißl. The number of graphs not containing $K_{3,3}$ as a minor. *Electron. J. Combin.*, 15(1):Research Paper 114, 20, 2008.

[25] O. Giménez and M. Noy. *Counting planar graphs and related families of graphs, Surveys in Combinatorics* 2009. Lecture Note Series, No. 365. London Mathematical Society, 2009.

[26] O. Giménez and M. Noy. The number of planar graphs and properties of random planar graphs. *J. Amer. Math. Soc.*, (22):309–329, 2009.

[27] O. Giménez, M. Noy, and J. Rué. Graph classes with given 3-connected components: asymptotic enumeration and random graphs. In preparation.

[28] O. Giménez, M. Noy, and J. Rué. Graph classes with given 3-connected components: asymptotic counting and critical phenomena. *Electronic Notes in Discrete Mathematics*, 29:521 – 529, 2007.

[29] G. Grimmet and D. Stirzaker. *Probability and random processes*. Oxford Univ. Press, 2001.

[30] H.-K. Hwang. On convergence rates in the central limit theorems for combinatorial structures. *Eur. J. Comb.*, 19(3):329–343, 1998.

[31] G. Korn and T. Korn. *Mathematical Handbook for Scientists and Engineers*. McGraw-Hill Companies, 1967.

[32] S. K. Lando and A. K. Zvonkin. *Graphs on Surfaces and Their Applications*, volume 141 of *Encyclopaedia of Mathematical Sciences: Lower-Dimensional Topology II*. Springer-Verlag, 2004.

[33] A. Lehman and T. Walsh. Counting rooted maps by genus I. *J. Combin. Theory Ser. B*, 13:192–218, 1972.

[34] W. Massey. *A basic course in Algebraic Topology*. Springer-Verlag, 1991.

[35] C. McDiarmid. Random graphs on surfaces. *J. Combin. Theory Ser. B*, (98):778–797, 2008.

[36] C. McDiarmid, A. Steger, and D. J. A. Welsh. Random planar graphs. *J. Combin. Theory Ser. B*, (93):187–205, 2005.

[37] A. Meier and J. Moon. On an asymptotic method in enumeration. *J. Combin. Theory, Ser. A*, 51(1):77–89, 1989.

[38] B. Mohar and C. Thomassen. *Graphs on surfaces.* J. Hopkins Univ. Press, 2001.

[39] R. Mullin, E. Nemeth, and P. Schellenberg. The enumeration of almost cubic maps. In *Proceedings of the Louisiana Conference on Combinatorics*, pages 281–295, 1970.

[40] J. P. Nolan. *Stable Distributions - Models for Heavy Tailed Data.* Birkhäuser, Boston, 2010. In progress, Chapter 1 online at `academic2.american.edu/∼jpnolan`.

[41] M. Noy and J. Rué. Counting polygon dissections in the projective plane. *Adv. in Appl. Math.*, 41(4):599–619, 2008.

[42] K. Panagiotou and A.Steger. Maximal biconnected subgraphs of random planar graphs. *Proc. ACM-SIAM Symposium on Discrete Algorithms (SODA 09)*, pages 432–440, 2009.

[43] R. Remmert. *Classical Topics in Complex Function Theory.* Springer-Verlag, 1998.

[44] N. Robertson and P. D. Seymour. Graph minors. I-XX. *J. Combin. Theory Ser. B*, 1993-2004.

[45] J. Rué. Enumeration and limit laws of dissections in a cylinder. In preparation.

[46] J. Rué, D. M. Thilikos, and I. Sau-Valls. Dynamic programming for graphs on surfaces. In preparation.

[47] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[48] B. A. Trakhtenbrot. Towards a theory of non-repeating contact schemes (in russian). *Trudi Mat. Inst. Akad. Nauk SSSR*, (51):226–269, 1958.

[49] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, (15):249–271, 1963.

[50] W. T. Tutte. *Connectivity in graphs.* Mathematical Expositions, No. 15. University of Toronto Press, Toronto, Ont., 1966.

[51] T. R. S. Walsh. Counting labelled three-connected and homeomorphically irreducible two-connected graphs. *J. Combin. Theory Ser. B*, (32):1–11, 1982.

[52] E. Wright. The number of connected sparsely edged graphs. *J. Graph Theory*, 1:317–330, 1977.

[53] E. Wright. The number of connected sparsely edged graphs. II. Smooth graphs and blocks. *J. Graph Theory*, 2:299–305, 1978.