Graph classes with given 3-connected components: asymptotic counting and critical phenomena

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This talk

• Families of graphs with given 3-connected components
  ◦ Relation with minor-free graphs

• Asymptotic enumeration dichotomy;
  ◦ SP-like families of graphs
  ◦ Planar-like families of graphs

• Critical phenomena on graphs with fixed density of edges
• Scope: size of the 2, 3-connected core, ...
Objects: simple labelled graphs

- **simple**: no loops, no multiple edges.
- **labelled**: vertices are labelled.

- **graphs**: no embedding.
Tools (I): the symbolic method

Translate set operations in combinatorial structures into formal operations between GF.

In our problem:

• $a_{n,m}$: # of elements with $n$ vertices and $m$ edges.

• Generating function $A(x, y)$:
  ○ The variable $x$ counts the vertices, $y$ counts the edges.
  ○ The GFs are exponential on $x$ and ordinary on $y$.

$$A(x, y) = \sum_{n,m \geq 0} a_{n,m} \frac{x^n}{n!} y^m$$

• Univariate generating function $A(x) = A(x, 1) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$
Tools (II): Singularity Analysis

The smallest singularity $\rho$ of $A(x)$ determines the asymptotics of $a_n$.

- **Location** of $\rho \rightarrow$ exponential growth constant $\gamma$.
- **Behaviour** of $A(x)$ at $\rho \rightarrow$ subexponential terms in asymptotics of $a_n$.

**Transfer Theorems** (Flajolet, Odlyzko)

Let $\alpha \notin \{0, -1, -2, \ldots\}$. If

$$A(x) \sim_{x \rightarrow \rho} a \cdot (1 - x/\rho)^{-\alpha} \iff A(x) = a \cdot (1 - x/\rho)^{-\alpha} + o((1 - x/\rho)^{-\alpha})$$

then

$$g_n \sim \frac{g}{\Gamma(\alpha)} \cdot n^{\alpha-1} \cdot \rho^{-n} \cdot n!$$
General graphs from connected graphs

Let $\mathcal{C}$ be a family of connected graphs.

We define $\mathcal{G}$ as those graphs such that their connected components are in $\mathcal{C}$.

$$\mathcal{G} = \text{SET}(\mathcal{C}) \implies G(x) = \exp(C(x))$$
General graphs from connected graphs

Let $\mathcal{C}$ be a family of *connected* graphs.

We define $\mathcal{G}$ as those graphs such that their *connected components* are in $\mathcal{C}$.

$$\mathcal{G} = \text{SET}(\mathcal{C}) \implies G(x) = \exp(C(x))$$
Connected graphs from 2-connected graphs

Let $\mathcal{B}$ be a family of 2-connected graphs.

We define $\mathcal{C}$ as those connected graphs such that their 2-connected blocks are in $\mathcal{B}$.
Connected graphs from $2$-connected graphs

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Let $\mathcal{B}$ be a family of 2-connected graphs.

We define $\mathcal{C}$ as those connected graphs such that their 2-connected blocks are in $\mathcal{B}$.

In other words, a vertex-rooted connected graph is a tree of 2-connected blocks.

$$C^o = \text{SET}(B^o(v \leftarrow C^o)) \implies xC''(x) = x \exp B'(xC'(x))$$
2-connected graphs from 3-connected graphs

Decomposition in 3-connected components is slightly harder.

Let \( T \) be a family of 3-connected graphs: \( T(x, z) \).

We define \( B \) as those 2-connected graphs such that can be obtained from \textit{series}, \textit{parallel}, and \( T \)-compositions.

\[
D(x, y) = (1 + y) \exp \left( \frac{xD^2}{1 + xD} + \frac{1}{2x^2} \frac{\partial T}{\partial z}(x, D) \right) - 1
\]

\[
\frac{\partial B}{\partial y}(x, y) = \frac{x^2}{2} \left( \frac{1 + D(x, y)}{1 + y} \right)
\]

where \( D \) is the GF for networks (essentially edge-rooted 2-connected graphs without the edge root).
Summing up: equations

Input: $T(x, z)$

\[
\begin{align*}
\frac{1}{2x^2D} \frac{\partial T}{\partial z}(x, D) - \log \left( \frac{1 + D}{1 + y} \right) + \frac{xD^2}{1 + xD} &= 0 \\
\frac{\partial B}{\partial y}(x, y) &= \frac{x^2}{2} \left( \frac{1 + D(x, y)}{1 + y} \right) \\
C'(x, y) &= \exp \left( B'(xC''(x, y), y) \right) \\
G(x, y) &= \exp(C(x, y))
\end{align*}
\]
Examples of families & excluded minors (I)

Examples.

- Series-parallel graphs
  - Excluded minors: 
  - Allowed 3-connected components: None.
  - $T(x, z) = 0$.

- Planar graphs
  - Excluded minors: 
  - Allowed 3-connected components: 3-connected planar graphs.
  - $T(x, z)$: The number of labelled 2-connected planar graphs [Bender, Gao, Wormald; 02]
Examples of families & excluded minors (II)

• $W_4$-free

  ◦ Excluded minors:

  ◦ Allowed 3-connected components:

  ◦ $T(x, z) = \frac{1}{4!} x^4 z^6$.

• $K_5^-$-free

  ◦ Excluded minors:

  ◦ Allowed 3-connected components:

  ◦ $T(x, z) = \frac{70}{6!} x^6 z^9 - \frac{1}{2} x \left( \log(1 - x z^2) + 2 x z^2 + x^2 z^4 \right)$. 
Examples of families & excluded minors (III)

- $K_{3,3}$-free [Gerke, G, Noy, Weibl; 06]
  - Excluded minors:
    
  - Allowed 3-connected components: 3-connected planar maps.
  - $T(x, z) = \ldots$

- If $\mathcal{G} = \text{Ex}(\mathcal{M})$ and all the excluded minors $\mathcal{M}$ are 3-connected, then $\mathcal{G}$ can be expressed in terms of its 3-connected graphs.

- Problem: finding the set of allowed 3-connected components.
Example: $\mathcal{G} = \text{Planar graphs}$

- Embeddable in the sphere.
- Do not contain $K_5, K_{3,3}$ as a minor.

*Asymptotic enumeration and limit laws of planar graphs* [G., Noy; 05]
Example: $G =$ Planar graphs

- Embeddable in the sphere.
- Do not contain $K_5, K_{3,3}$ as a minor.

Asymptotic enumeration:

$$b_n \sim b \cdot n^{-7/2} \cdot \gamma_B^n \cdot n!$$
$$c_n \sim c \cdot n^{-7/2} \cdot \gamma^n \cdot n!$$
$$g_n \sim g \cdot n^{-7/2} \cdot \gamma^n \cdot n!$$

$\gamma_B \approx 26.1841$
$\gamma \approx 27.2268$

Uniformly distributed random graph $G_n$:

Edges($G_n$) $\sim N(\lambda n, \sigma^2 n)$

Componentes($G_n$) $\sim 1 + P(\nu)$

$\lambda \approx 2.2132$
$\nu \approx 0.0374$
Example: $G = $Series-parallel graphs

- Tree-width smaller or equal than 2.
- Obtained from acyclic graph by series and parallel operations.

On the number of series-parallel and outerplanar graphs [Bodirsky, G, Kang, Noy]
Example: $\mathcal{G} =$ Series-parallel graphs

- Tree-width smaller or equal than 2.
- Obtained from acyclic graph by series and parallel operations.

Asymptotic enumeration:

\[ b_n \sim b \cdot n^{-5/2} \cdot \gamma_B^n \cdot n! \]
\[ c_n \sim c \cdot n^{-5/2} \cdot \gamma^n \cdot n! \] \[ g_n \sim g \cdot n^{-5/2} \cdot \gamma^n \cdot n! \]

\[ \gamma_B \approx 7.8125 \]
\[ \gamma \approx 9.0735 \]

Uniformly distributed random graph $G_n$:

\[ \text{Edges}(G_n) \sim N(\lambda n, \sigma^2 n) \]
\[ \text{Components}(G_n) \sim 1 + P(\nu) \]

\[ \lambda \approx 1.6167 \]
\[ \nu \approx 0.1176 \]
Equations

Planar graphs

\[
\begin{align*}
  u(x, z) &= xz(1 + v(x, z))^2 \\
  v(x, z) &= z(1 + u(x, z))^2 \\
  \frac{\partial T}{\partial z}(x, z) &= x^2 z^2 \left( \frac{1}{1 + xz} + \frac{1}{1 + z} - 1 - \frac{(1 + u)^2(1 + v)^2}{(1 + u + v)^3} \right) \\
  \frac{1}{2x^2 D} \frac{\partial T}{\partial z}(x, D) - \log \left( \frac{1 + D}{1 + y} \right) + \frac{x D^2}{1 + xD} &= 0 \\
  \frac{\partial B}{\partial y}(x, y) &= \frac{x^2}{2} \left( \frac{1 + D(x, y)}{1 + y} \right) \\
  C'(x, y) &= \exp \left( B'(xC''(x, y), y) \right) \\
  G(x, y) &= \exp(C(x, y))
\end{align*}
\]
Equations

Series-parallel graphs

\[
\begin{align*}
\frac{\partial T}{\partial z}(x, z) &= 0 \\
\frac{1}{2x^2D} \frac{\partial T}{\partial z}(x, D) - \log \left( \frac{1 + D}{1 + y} \right) + \frac{xD^2}{1 + xD} &= 0 \\
\frac{\partial B}{\partial y}(x, y) &= \frac{x^2}{2} \left( \frac{1 + D(x, y)}{1 + y} \right) \\
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Equations

Series-parallel graphs

\[ \frac{\partial T}{\partial z}(x, z) = 0 \]

Planar graphs

\[
\begin{align*}
    u(x, z) &= xz(1 + v(x, z))^2 \\
    v(x, z) &= z(1 + u(x, z))^2 \\
    \frac{\partial T}{\partial z}(x, z) &= x^2 z^2 \left( \frac{1}{1 + xz} + \frac{1}{1 + z} + \ldots \right)
\end{align*}
\]

\[
\begin{align*}
    \frac{1}{2x^2 D} \frac{\partial T}{\partial z}(x, D) &- \log \left( \frac{1 + D}{1 + y} \right) + \frac{xD^2}{1 + xD} = 0 \\
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\[
\begin{align*}
    C'(x, y) &= \exp \left( B'(xC'(x, y), y) \right) \\
    G(x, y) &= \exp(C(x, y))
\end{align*}
\]
Equations

Series-parallel graphs

\[ \frac{\partial T}{\partial z}(x, z) = \frac{\partial T}{\partial z} \sim 0 \]

Planar graphs

\[ u(x, z) = xz(1 + v(x, z))^2 \]
\[ v(x \frac{\partial T}{\partial z}) \sim t \cdot (1 - \frac{z}{z_0})^{3/2} \]
\[ \frac{\partial T}{\partial z}(x, z) = x^2 z^2 \left( \frac{1}{1 + xz} + \frac{1}{1 + z} + \ldots \right) \]

\[ \frac{1}{2x^2D} \frac{\partial T}{\partial z}(x, D) - \log \left( \frac{1 + D}{1 + y} \right) + \frac{xD^2}{1 + xD} = 0 \]
\[ \frac{\partial B}{\partial y}(x, y) = \frac{x^2}{2} \left( \frac{1 + D(x, y)}{1 + y} \right) \]

\[ C'(x, y) = \exp \left( B'(xC'(x, y), y) \right) \]
\[ G(x, y) = \exp(C(x, y)) \]
Equations

Series-parallel graphs

\[
\begin{align*}
\frac{\partial T}{\partial z}(x, z) &= \frac{\partial T}{\partial z} \sim 0 \\
D &\sim d \cdot (1 - \frac{x}{x_0})^{1/2} \\
B &\sim b \cdot \left(1 - \frac{x}{x_0}\right)^{3/2} \\
C &\sim c \cdot \left(1 - \frac{x}{\rho}\right)^{3/2} \\
G &\sim g \cdot \left(1 - \frac{x}{\rho}\right)^{3/2}
\end{align*}
\]

Planar graphs

\[
\begin{align*}
u(x, z) &= xz(1 + v(x, z))^2 \\
v(x \frac{\partial T}{\partial z}) &\sim t \cdot (1 - \frac{z}{z_0})^{3/2} \\
\frac{\partial T}{\partial z}(x, z) &= x^2 z^2 \left(\frac{1}{1 + xz} + \frac{1}{1 + z} + \ldots\right) \\
D &\sim d \cdot (1 - \frac{x}{x_0})^{3/2} \\
B &\sim b \cdot \left(1 - \frac{x}{x_0}\right)^{5/2} \\
C &\sim c \cdot \left(1 - \frac{x}{\rho}\right)^{5/2} \\
G &\sim g \cdot \left(1 - \frac{x}{\rho}\right)^{5/2}
\end{align*}
\]
Results

If either \( \frac{\partial T}{\partial z}(x, z) \)

- has no singularity, or
- the singularity type is \((1 - z/z_0)^\alpha\) with \(\alpha < 1\),

then the situation is alike to the series-parallel case:

\[
D(x) \sim d \cdot (1 - x/x_0)^{1/2} \quad d_n \sim d \cdot n^{-3/2} \cdot x_0^{-n} \cdot n!
\]
\[
B(x) \sim b \cdot (1 - x/x_0)^{3/2} \quad b_n \sim b \cdot n^{-5/2} \cdot x_0^{-n} \cdot n!
\]
\[
C(x) \sim c \cdot (1 - x/\rho)^{3/2} \quad c_n \sim c \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!
\]
\[
G(x) \sim g \cdot (1 - x/\rho)^{3/2} \quad g_n \sim g \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!
\]
Results

If $\frac{\partial T}{\partial z}(x, z)$ has singularity type $(1 - z/z_0)^{3/2}$, then 3 different situations may happen.

Case 1 (Planar case)

\[
D(x) \sim d \cdot (1 - x/x_0)^{3/2} \\
B(x) \sim b \cdot (1 - x/x_0)^{5/2} \\
C(x) \sim c \cdot (1 - x/\rho)^{5/2} \\
G(x) \sim g \cdot (1 - x/\rho)^{5/2}
\]

\[
d_n \sim d \cdot n^{-5/2} \cdot x_0^{-n} \cdot n! \\
b_n \sim b \cdot n^{-7/2} \cdot x_0^{-n} \cdot n! \\
c_n \sim c \cdot n^{-7/2} \cdot \rho^{-n} \cdot n! \\
g_n \sim g \cdot n^{-7/2} \cdot \rho^{-n} \cdot n!
\]
Results

If $\frac{\partial T}{\partial z}(x, z)$ has singularity type $(1 - z/z_0)^{3/2}$, then 3 different situations may happen.

Case 2 (Series-parallel case)

\[
D(x) \sim d \cdot (1 - x/x_0)^{1/2}
\]
\[
B(x) \sim b \cdot (1 - x/x_0)^{3/2}
\]
\[
C(x) \sim c \cdot (1 - x/\rho)^{3/2}
\]
\[
G(x) \sim g \cdot (1 - x/\rho)^{3/2}
\]

\[
d_n \sim d \cdot n^{-3/2} \cdot x_0^{-n} \cdot n!
\]
\[
b_n \sim b \cdot n^{-5/2} \cdot x_0^{-n} \cdot n!
\]
\[
c_n \sim c \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!
\]
\[
g_n \sim g \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!
\]
Results

If \( \frac{\partial T}{\partial z}(x, z) \) has singularity type \((1 - z/z_0)^{3/2}\), then 3 different situations may happen.

Case 3 (Mixed case)

\[
\begin{align*}
D(x) & \sim d \cdot (1 - x/x_0)^{3/2} & d_n & \sim d \cdot n^{-5/2} \cdot x_0^{-n} \cdot n! \\
B(x) & \sim b \cdot (1 - x/x_0)^{5/2} & b_n & \sim b \cdot n^{-7/2} \cdot x_0^{-n} \cdot n! \\
C(x) & \sim c \cdot (1 - x/\rho)^{3/2} & c_n & \sim c \cdot n^{-5/2} \cdot \rho^{-n} \cdot n! \\
G(x) & \sim g \cdot (1 - x/\rho)^{3/2} & g_n & \sim g \cdot n^{-5/2} \cdot \rho^{-n} \cdot n!
\end{align*}
\]
Asymptotic enumeration

Some parameters can be expressed in terms of $T(x, z)$:

- Growth constants $x_0$ (2-connected) and $\rho$ (connected, all graphs).
- Asymptotic constants $d, b, c, g$.
- $\lambda$ and $\sigma$ of the Normally distributed edges.
- Parameter $\nu$ of the Poisson distributed connected components.
Graphs with fixed density of edges

- Almost all graphs in $G$ have the same edge density $\lambda$.
- What about graphs in $G$ with other edge densities?
  - Singularity analysis of $G(x, y)$ when $y \neq 1$

Asymptotic enumeration of graphs with $(\lambda n - \sqrt{n}, \lambda n + \sqrt{n})$ edges:

$$g(\lambda) \cdot n^{\alpha} \cdot \rho(\lambda)^{-n} \cdot n!$$

where $\alpha = -5/2$ (series-parallel-like case) or $\alpha = -7/2$ (planar-like case).
Critical phenomena

The singularity type of $G(x, y)$ may change with the value of $y$.

Example: $\mathcal{T} = \{3$-connected cubic planar graphs$\}$.

- $c(\lambda)_n$: connected graphs of $G$ with $(\lambda n - \sqrt{n}, \lambda n + \sqrt{n})$ edges.
  
  - If $\lambda < 1.1844\ldots$,
    
    $$c(\lambda)_n \sim c_\lambda \cdot n^{-7/2} \cdot \rho(\lambda)^{-n} \cdot n!$$
  
  - If $\lambda > 1.1844\ldots$,
    
    $$c(\lambda)_n \sim c_\lambda \cdot n^{-5/2} \cdot \rho(\lambda)^{-n} \cdot n!$$
Critical phenomena

The singularity type of $G(x, y)$ may change with the value of $y$.

Example: $\mathcal{T} = \{\text{3-connected cubic planar graphs}\}$.

- (top) Probability of small core
- (bottom) Size of 2-connected large core
Critical phenomena

The singularity type of $G(x, y)$ may change with the value of $y$.

Example: $\mathcal{T} = \{3$-connected cubic planar graphs$\}$.

- $b(\lambda)_n$: 2-connected graphs of $G$ with $(\lambda n - \sqrt{n}, \lambda n + \sqrt{n})$ edges.
  - If $\lambda < 1.31725 \ldots$,
    \[ b(\lambda)_n \sim b_\lambda \cdot n^{-7/2} \cdot x_0(\lambda)^{-n} \cdot n! \]
  - If $\lambda > 1.31725 \ldots$,
    \[ b(\lambda)_n \sim b_\lambda \cdot n^{-5/2} \cdot x_0(\lambda)^{-n} \cdot n! \]
Critical phenomena

The singularity type of $G(x, y)$ may change with the value of $y$.

Example: $\mathcal{T} = \{3$-connected cubic planar graphs$\}$.

- (top) Probability of small core
- (bottom) Number of edges of 3-connected large core