Enumeration and limit laws of simplicial decompositions of surfaces with boundaries

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The material of this talk


Background
Objects: surfaces

**Surface** = topological space locally homeomorphic to the plane.

Classification: 2 parameters:

- The Orientability: $\mathbb{T}, \mathbb{P}$.
- The Genus: $\mathbb{T}_g, \mathbb{P}_g$ (and $S^2 (g = 0)$)

**Surface with boundary** = surface where we delete a disjoint union of open disks
Objects: maps

Map = ”Good” embedding of a graph in a surface.

A map has vertices, edges and faces (Contractible!)
Objects: maps

MAPS considered are **ROOTED**.

Crutial concept: **dual map**.

We will deal with maps with topological restrictions.
The Symbolic Method on Generating Functions

COMBINATORIAL RELATIONS between CLASSES

EQUATIONS between GENERATING FUNCTIONS

<table>
<thead>
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<th>Construction</th>
<th>OGF</th>
<th>EGF</th>
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<td>Union</td>
<td>$A \cup B$</td>
<td>$A(x) + B(x)$</td>
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<td>Product</td>
<td>$A \times B$</td>
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<td>Labelled Product</td>
<td>$A \ast B$</td>
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<td>Seq $(A)$</td>
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<td>Pointing</td>
<td>$A^\bullet$</td>
<td>$x \frac{\partial}{\partial x} A(x)$</td>
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<td>Substitution</td>
<td>$A \circ B$</td>
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Singularity analysis on generating functions

GFs: analytic functions in a neighbourhood of the origin.

The smallest singularity of $A(z)$ determines the asymptotics of the coefficients of $A(z)$.

- **POSITION**: exponential growth $\rho$.
- **NATURE**: subexponential growth
- **Transfer Theorems**: Let $\alpha \neq \{0, -1, -2, \ldots\}$. If
  \[ A(z) = a \cdot (1 - z/\rho)^{-\alpha} + o((1 - z/\rho)^{-\alpha}) \]
  then
  \[ a_n = [z^n] A(z) \sim \frac{a}{\Gamma(\alpha)} \cdot n^{\alpha - 1} \cdot \rho^{-n} (1 + o(1)) \]
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Limit laws

Study of parameters $\rightarrow A(u, z) = \sum_{n,m=0}^{\infty} a_{n,m} z^n u^m$.

For a fixed $n$, the numbers $a_{n,m}$ describe a discrete probability law $X_n$

$$p(X_n = m) = \frac{a_{n,m}}{\sum_{m=0}^{\infty} a_{n,m}} = \frac{[u^m z^n] A(u, z)}{[z^n] A(1, z)}$$

Does $X_n$ converge in distribution to a random variable $X$?

We expect normal limit distributions: general theorems

In the problems under study we deal with parameters where these results applies, and others where not

Method of moments & composition schemes
Limit laws

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Method of moments & composition schemes
The method of moments

Let \((X_n)_{n>0}\) and \(X\) be real random variables satisfying:

1. \((A)\) there exists \(R > 0\) such that \(\frac{R^r}{r!} \mathbb{E}[X^r] \to 0\), as \(r \to \infty\),
2. \((B)\) for all \(r \in \mathbb{N}\), \(\mathbb{E}[X_n^r] \to \mathbb{E}[X^r]\), as \(n \to \infty\).

Then \(X_n \xrightarrow{d} X\).

The technical way to apply this result is the use of the Laplace Transform

\[
F_X(s) = \int_0^\infty f_X(t) e^{-st} dt = \sum_{r=0}^{\infty} \mathbb{E}[X^r] \frac{1}{r!} (-s)^r.
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Introduction, the disk, the projective plane and the cylinder
Triangulations of a polygon

**Question:** number of triangulations of a polygon

**Solution:** for $n + 2$ vertices (or $n$ triangles): $C(n) = \frac{1}{n+1} \binom{2n}{n}$.

$$C(z) = 1 + z + 2z^2 + \cdots = \sum_{n=0}^{\infty} C(n) z^n = \frac{1 - \sqrt{1 - 4z}}{2z}.$$  

**Asymptotics:** $C(n) \sim \frac{1}{\sqrt{\pi}} n^{-3/2} 4^n$. 
A general problem
A general problem

We can take a surface without boundary, instead of $S^2$.

We consider the real projective plane $\mathbb{P}$. 
Triangulations vs simplicial decompositions

In the disk: triangulation = simplicial decomposition (Contractible space)

For general surfaces, this equality is not true

We are only concerned with SIMPLICIAL decompositions
**RESULT: the GF for simplicial decompositions**

We rediscover a result of Edelman and Reiner

The GF associated to triangulations of the Möebius band, with all vertices on the boundary, is

\[
P(z) = \frac{(2 - 9z + 6z^2 + 7z^3 - 2z^4)C(z) - (2 - 7z + z^2 + 5z^3)}{z(1 - 4z)}
\]

\[
= z^5 + 14z^6 + 113z^7 + 720z^8 + 4033z^9 + \cdots
\]

where \( z \) marks polygons
Sketch of the proof

**First step**: combinatorial decomposition

\[ P = (C \times \triangle \times \mathcal{P}) \cup (\mathcal{P} \times \triangle \times C) \cup (\triangle \times \mathcal{R}) \rightarrow P(z) = z \left( 2C(z)P(z) + R(z) \right) \]

**Second step**: inclusion-exclusion argument over the disk
Just to show
A composition scheme for the projective plane

Composition scheme

\[ M(z) = \frac{z^5 (1 + 3z - 2z^2)}{(1 - 2z)(1 - z)^3}. \]

\[ u \frac{\partial}{\partial u} P(u, z) = z \frac{\partial}{\partial z} M(uzC(z)) \rightarrow P(u, z) = \left(1 + \frac{zC'(z)}{C(z)}\right) M(uzC(z)). \]
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Simplicial decompositions of the cylinder

We are also interested in simplicial decompositions of the cylinder (NOT rooted maps!)

A decomposition of the cylinder by the root is possible, but involved! (Gao, Xiao, Wang)

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A composition scheme for the cylinder
RESULT: the GF for simplicial decompositions

Let \( M(x, y) \) be the GF

\[
2 \frac{x^2 y^3 (y^2 - 3y + 3)}{(y - 1)^3} + \frac{(2y^3 - 4y^2 + 2y - 1)y}{(x - 1)(y - 1)^2} + \frac{y}{x + y - 1} - \frac{y(y - 1)}{(x + y - 1)^2} + \frac{y(-1 + 4y^3 - 8y^2 + 4y)}{(x - 1)^2(y - 1)^2} + 2\frac{y^2}{(x - 1)^3}
\]

where \( x, y \) marks internal and external vertices, respectively.

Then, the GF associated to simplicial decompositions of the cylinder verifies

\[
H(u, x, y) = \frac{1 - xC(x)}{1 - 2xC(x)} \frac{1 - yC(y)}{1 - 2yC(y)} M(uxC(x), uyC(y)),
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Some non-standard limit laws

We define in this family 2 random variables

- The size of the core $Y_n$: $H(u, z, z)$

$$H(u, z, z) = \left( \frac{1 - zC(z)}{1 - 2zC(z)} \right)^2 M(uzC(z), uzC(z)),$$

- Vertices over the external circle $W_n$: $H(1, z, zu)$

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In both cases, we cannot apply “general results”
RESULT: limit laws

We get limit distributions for both random variables:

\[ f_Y(t) = t \cdot \text{erfc} \left( \frac{t}{2} \right) \cdot \mathbb{I}_{[0,\infty]}(t) \]

\[ f_W(t) = \frac{8}{\pi} \sqrt{t - t^2} \cdot \mathbb{I}_{[0,1]}(t). \]

Then,

\[ Y_n/\sqrt{n} \overset{d}{\to} Y, \]

\[ W_n/n \overset{d}{\to} W. \]

Main ideas:

- Get the moments (Method of Moments)
- Encapsulate them in a convenient function
- Apply inverse Laplace transform
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Main ideas:

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Additional results

We have obtained their singularity behaviour

\[ \left[z^n\right] P(z) \sim \frac{1}{4} 4^n; \quad \left[z^n\right] H(1, z, z) \sim \frac{1}{16} n4^n \]

GFs are complex, ASYMPTOTICS are simple

Simpliciality is a strange property in a triangulation?

ARE THERE GENERAL ARGUMENTS?
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ARE THERE GENERAL ARGUMENTS?
Enumeration and limit laws of dissections of general surfaces with boundary
Simplicial decompositions of general surfaces

We can generalize simplicial decompositions to general surfaces \textbf{WITH} boundary

\[ \mathcal{M}_S(n) := \text{set of triangulations of } S \]
\[ \mathcal{D}_S(n) := \text{set of simplicial decompositions of } S \]
\[ \mathcal{D}_S(n) \subset \mathcal{M}_S(n) \]
RESULT: asymptotic enumeration

Let $S$ be a surface. Let $\chi(S)$ be its Euler characteristic.

$$|D_S(n)| \sim |M_S(n)| \sim \frac{a(S)}{\Gamma(1 - 3\chi(S)/2)} n^{-3/2\chi(S)} 4^n$$

where

- $\beta(S) := \# \text{ connected components of } \partial S$
- $a(S) := \# \text{rooted cubic maps in } \overline{S} \text{ with } \beta(S) \text{ faces}$
- $\Gamma(\diamond) := \text{Gamma function}$

Tools:

- “Tree decomposition of triangulations”
- generating function language + asymptotic analysis
RESULT: asymptotic enumeration

Let $\mathcal{S}$ be a surface. Let $\chi(\mathcal{S})$ be its Euler characteristic.

$$ |\mathcal{D}_\mathcal{S}(n)| \sim |\mathcal{M}_\mathcal{S}(n)| \sim \frac{a(\mathcal{S})}{\Gamma(1 - 3\chi(\mathcal{S})/2)} n^{-3/2\chi(\mathcal{S})} 4^n $$

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Tools:

- “Tree decomposition of triangulations”
- generating function language + asymptotic analysis
Sketch of the proof: the tree-decomposition

**Dual graph** of the triangulation

**Prune** the map, **contract** vertices of degree 2 $\rightarrow$ **cubic scheme**

$$A_S := \text{cubic scheme} \times 2 - 3\chi(S) \text{ double rooted binary trees}$$
Sketch of the proof: generating functions

\[ \mathcal{D}_S(n) \subseteq \mathcal{M}_S(n) \subseteq \mathcal{A}_S(n) \]

- Double rooted binary trees
  \[ T^\circ(z) = \frac{\partial}{\partial z} (zC(z)) = \frac{1}{\sqrt{1-4z}} \]

  \[ A_S(z) = \sum_{n>0} |A_S(n)| z^n = \sum_{\text{cubic schemes on } S} (T^\circ(z))^{2-3\chi(S)} \]

  \[ A_S(z) = a(S) \left( \frac{1}{\sqrt{1-4z}} \right)^{2-3\chi(S)} \rightarrow |A_S(n)| \sim \frac{a(S)n^{-3/2\chi(S)}}{\Gamma(1-3\chi(S)/2)} 4^n \]

- Construct \( \mathcal{D}^*_S(n) \)

\[ \mathcal{D}^*_S(n) \subseteq \mathcal{D}_S(n) \subseteq \mathcal{M}_S(n) \subseteq \mathcal{A}_S(n) \]

\[ |\mathcal{D}^*_S(n)| \sim |\mathcal{A}_S(n)| \]
Sketch of the proof: generating functions

\[ D_S(n) \subseteq M_S(n) \subseteq A_S(n) \]

- **Double rooted binary trees**
  \[ T^o(z) = \frac{\partial}{\partial z} (zC(z)) = \frac{1}{\sqrt{1-4z}} \]
  \[ A_S(z) = \sum_{n>0} |A_S(n)| z^n = \sum_{\text{cubic schemes on } S} \left( T^o(z) \right)^{2-3\chi(S)} \]

- **Construct** \( D_S^*(n) \)

\[ D_S^*(n) \subseteq D_S(n) \subseteq M_S(n) \subseteq A_S(n) \]
\[ |D_S^*(n)| \sim |A_S(n)| \]
Generalisation

- **Simplicial decompositions:** $\Delta = \{3\}$.
- **General sets:** $\Delta \subseteq \mathbb{N}^{\geq 3} \rightarrow M_\Delta^S(n), D_\Delta^S(n)$

We use a similar tree-decomposition, but more involved.

Cubic schemes $\Leftrightarrow$ Largest number of edges!
RESULT: asymptotic enumeration

Let $\mathbb{S}$ be a surface. Let $\chi(\mathbb{S})$ be Euler characteristic.

$$\sum_{\delta \in \Delta} (\delta - 1) \tau_{\Delta}^{\delta-2} = 1; \quad \rho_{\Delta} = \tau_{\Delta} - \sum_{\delta \in \Delta} \tau_{\Delta}^{\delta-1};$$

$$\gamma_{\Delta} = \sqrt{\frac{2\rho_{\Delta}}{\sum_{\delta \in \Delta} (\delta - 1)(\delta - 2) \tau_{\Delta}^{\delta-3}}}$$

Let $p$ the gcd of the elements in the set $\{\delta - 2 : \delta \in \Delta\}$. Then

$$|\mathcal{D}_{\mathbb{S}}^{\Delta}(n)| \sim \frac{a(\mathbb{S})p (8\gamma_{\Delta}\rho_{\Delta})^{\chi(\mathbb{S})}}{4 \Gamma(1 - 3\chi(\mathbb{S})/2)} n^{-3\chi(\mathbb{S})/2} \rho_{\Delta}^{-n}.$$
A structural limit law

As in the cylinder and the projective plane, we have a structure where we paste planar dissections

\[
\begin{align*}
\text{Structuring edges := edges which either do not separate the surface or separate it properly.}
\end{align*}
\]

\[
\begin{align*}
X_n: \text{ random variable in } \mathcal{M}_S(n), \mathcal{D}_S(n).
\end{align*}
\]
RESULT: number of structuring edges

$\frac{X_n}{\sqrt{n}}$ converges in distribution to $X_{\chi(S)}$:

$$g_{\chi(S)}(t) = 2 \frac{\left( \frac{\rho \Delta}{\gamma \Delta \sqrt{p}} t \right)^{-3\chi(S)} \exp \left( - \left( \frac{\rho \Delta}{\gamma \Delta \sqrt{p}} t \right)^2 \right)}{\Gamma \left( (1 - 3\chi(S))/2 \right)} I_{[0,\infty]}(t)$$

GAMMA distributions
Thank you
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