Some Properties on the Generalized Hierarchical Product of Graphs

Lali Barrière
Cristina Dalfó
Miquel Àngel Fiol
Margarida Mitjana

Universitat Politècnica de Catalunya

July 23, 2008
Some Properties on the Generalized Hierarchical Product of Graphs

Outline

1. The hierarchical product
2. The generalized hierarchical product
3. Metric parameters
4. Hamiltonian cycles
5. Vertex- and edge-coloring
6. Connectivity
1. The hierarchical product

2. The generalized hierarchical product

3. Metric parameters

4. Hamiltonian cycles

5. Vertex- and edge-coloring

6. Connectivity
Some Properties on the Generalized Hierarchical Product of Graphs

The hierarchical product

Definition [B, Comellas, Dalfó, Fiol, 2008]
For $i = 1, \ldots, N$, $G_i$ graph rooted at 0, $H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$

- vertices $x_N \cdots x_3 x_2 x_1$, $x_i \in V_i$
- if $x_j \sim y_j$ in $G_j$ then
  $x_N \cdots x_{j+1} x_j 0 \cdots 0 \sim x_N \cdots x_{j+1} y_j 0 \cdots 0$

Example

The hierarchical products $K_2 \sqcap K_3$ and $K_3 \sqcap K_2$
Some Properties on the Generalized Hierarchical Product of Graphs

The hierarchical product

\[ G_N \sqcap \cdots \sqcap G_2 \sqcap G_1 \] is a spanning subgraph of \[ G_N \square \cdots \square G_2 \square G_1 \]

Example

The hierarchical product \( P_4 \sqcap P_3 \sqcap P_2 \)
Example

*The hierarchical power* $C_4^3$
Some Properties on the Generalized Hierarchical Product of Graphs

1. The hierarchical product
2. The generalized hierarchical product
3. Metric parameters
4. Hamiltonian cycles
5. Vertex- and edge-coloring
6. Connectivity
Some Properties on the Generalized Hierarchical Product of Graphs

The generalized hierarchical product

For $i = 1, \ldots, N$, $G_i$ graph, and for $i \leq N - 1$, $\emptyset \neq U_i \subseteq V_i$, $H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1)$

- vertices $x_N \ldots x_3 x_2 x_1$, $x_i \in V_i$
- if $x_j \sim y_j$ in $G_j$ and $u_i \in U_i$, $i = 1, \ldots, j - 1$ then $x_N \ldots x_{j+1} x_j u_j - 1 \ldots u_1 \sim x_N \ldots x_{j+1} y_j u_j - 1 \ldots u_1$

Example

$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1)$, with $U_1 = U_2 = \{0, 1\}$
Paticular extreme cases

- (Standard) hierarchical product
  For all $1 \leq i \leq N - 1$, $U_i = \{v_i\}$ \Rightarrow
  $H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1) = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$
  (assuming that $v_i$ is labeled 0)

- Cartesian product
  For all $1 \leq i \leq N - 1$, $U_i = V_i$ \Rightarrow
  $H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1) = G_N \boxtimes \cdots \boxtimes G_2 \boxtimes G_1$
Some Properties on the Generalized Hierarchical Product of Graphs

The generalized hierarchical product

\[ H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1) \]

- \( z \in V_N \times \cdots \times V_{k+1} \Rightarrow H \langle zx_k \ldots x_1 \rangle \) subgraph of \( H \) induced by the set \( \{ zx_k \ldots x_1 | x_i \in V_i, 1 \leq i \leq k \} \)
- \( z \in V_{k-1} \times \cdots \times V_1 \Rightarrow H \langle x_N \ldots x_k z \rangle \) subgraph of \( H \) induced by the set \( \{ x_N \ldots x_k z | x_i \in V_i, k \leq i \leq N \} \)

Lemma

\[(a)\] \( H \langle zx_k \ldots x_1 \rangle = G_k \sqcap G_{k-1}(U_{k-1}) \sqcap \cdots \sqcap G_1(U_1) \)
\[(b)\] \( z \in U_{k-1} \times \cdots \times U_1 \Rightarrow \)
\[H \langle x_N \ldots x_k z \rangle = G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \cdots \sqcap G_k(U_k) \]
\[(c)\] \( z \notin U_{k-1} \times \cdots \times U_1 \Rightarrow H \langle x_N \ldots x_k z \rangle = m K_1, \text{ where } m = n_N \cdots n_k \)
Some Properties on the Generalized Hierarchical Product of Graphs

The generalized hierarchical product

Example

$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1)$, with $U_1 = U_2 = \{0, 1\}$
Some Properties on the Generalized Hierarchical Product of Graphs
The generalized hierarchical product

Example

$$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1), \text{ with } U_1 = U_2 = \{0, 1\}$$
The generalized hierarchical product

\[ K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1), \text{ with } U_1 = U_2 = \{0, 1\} \]
Some Properties on the Generalized Hierarchical Product of Graphs

The generalized hierarchical product

Example

\[ K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1), \text{ with } U_1 = U_2 = \{0, 1\} \]
Some Properties on the Generalized Hierarchical Product of Graphs

The generalized hierarchical product

Example

$K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1)$, with $U_1 = U_2 = \{0, 1\}$
Some Properties on the Generalized Hierarchical Product of Graphs

The generalized hierarchical product

Example

\[ K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1), \text{ with } U_1 = U_2 = \{0, 1\} \]
Some Properties on the Generalized Hierarchical Product of Graphs

The generalized hierarchical product

Example

\[ K_3^3 = K_3 \sqcap K_3(U_2) \sqcap K_3(U_1), \text{ with } U_1 = U_2 = \{0, 1\} \]
Some Properties on the Generalized Hierarchical Product of Graphs

The generalized hierarchical product

**Degree**

- \( v = x_N x_{N-1} \cdots x_2 x_1 \in H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1) \)

\[
\deg_H(v) = \deg_{G_1}(x_1) + \chi_{U_1}(x_1) \deg_{G_2}(x_2) + \cdots + \chi_{U_1}(x_1) \cdots \chi_{U_{N-1}}(x_{N-1}) \deg_{G_N}(x_N)
\]

- The minimum and maximum degree of \( H \) are

\[
\delta_H = \min\{ \delta_{G_1(U_1)}, \delta_{G_1(U_1)} + \delta_{G_2(U_2)}, \ldots, \delta_{G_1(U_1)} + \cdots + \delta_{G_{N-1}(U_{N-1})} + \delta_{G_N} \}
\]

\[
\Delta_H = \max\{ \Delta_{G_1(U_1)}, \Delta_{G_1(U_1)} + \Delta_{G_2(U_2)}, \ldots, \Delta_{G_1(U_1)} + \cdots + \Delta_{G_{N-1}(U_{N-1})} + \Delta_{G_N} \}
\]
The generalized hierarchical product

For all $i = 1, 2, \ldots, N$, $n_i = |V(G_i)|$

- For all $i = 1, 2, \ldots, N$, $G_i$ is $\delta_i$-regular $\Rightarrow$
- $H = G_N \sqcap \cdots \sqcap G_2(U_2) \sqcap G_1(U_1)$ contains exactly
  - $n_N(n_{N-1} - |U_{N-1}|)$ vertices of degree $\delta_N$
  - $n_N|U_{N-1}|(n_{N-2} - |U_{N-2}|)$ vertices of degree $\delta_N + \delta_{N-1}$
  - $\vdots$
  - $n_N|U_{N-1}||U_{N-2}| \cdots |U_2|(n_1 - |U_1|)$ vertices of degree $\delta_N + \cdots + \delta_2$
  - $n_N|U_{N-1}||U_{N-2}| \cdots |U_1|$ vertices of degree $\delta_N + \cdots + \delta_1$
Some Properties on the Generalized Hierarchical Product of Graphs

The generalized hierarchical product

**Associativity**

**Proposition**

For $i = 1, 2, 3$, let $G_i$ be a graph and, for $i = 1, 2$, $U_i \subseteq V_i$. The generalized hierarchical product satisfies

$$G_3 \cap G_2(U_2) \cap G_1(U_1) = G_3 \cap (G_2 \cap G_1(U_1))(U_2 \times U_1)$$

$$= (G_3 \cap G_2(U_2)) \cap G_1(U_1)$$

Can be easily generalized to the case of $N$ factors.
Some Properties on the Generalized Hierarchical Product of Graphs

The generalized hierarchical product
Some Properties on the Generalized Hierarchical Product of Graphs

The generalized hierarchical product
Some Properties on the Generalized Hierarchical Product of Graphs

1. The hierarchical product

2. The generalized hierarchical product

3. Metric parameters

4. Hamiltonian cycles

5. Vertex- and edge-coloring

6. Connectivity
Metric parameters

Definition (Distance through $U$)

$G = (V, E), \emptyset \neq U \subset V$

- $p_G(U)(x, y)$ path from $x$ to $y$ through $U$, if intersects $U$
- $\text{dist}_G(U)(x, y) = \min \text{length}(p_G(U)(x, y))$

Remark $\text{dist}_G(U)(x, x) = 0 \iff x \in U$

- Mean distance through $U$: $d_G(U) = \frac{1}{n^2} \sum_{x, y \in V} \text{dist}_G(U)(x, y)$
- Eccentricity through $U$: $\text{ecc}_G(U)(x) = \max_{y \in V} \text{dist}_G(U)(x, y)$
- Radius through $U$: $r_G(U) = \min_{x \in V} \text{ecc}_G(U)(x)$
- Diameter through $U$: $D_G(U) = \max_{x \in V} \text{ecc}_G(U)(x)$
Some Properties on the Generalized Hierarchical Product of Graphs

Metric parameters

\[ x = (x_2, x_1), \ y = (y_2, y_1) \in H = G_2 \sqcap G_1(U_1) \Rightarrow \]
\[ \text{dist}_H(x, y) = \text{dist}_{G_2}(x_2, y_2) + \text{dist}_{G_1(U_1)}(x_1, y_1) \]

**Theorem**

\[ H = G_2 \sqcap G_1(U_1), \ U_1 \subset V_1, \ n_2 = |V_2| \]

(a) **Mean distance:**
\[ d_H = d_{G_2} + \frac{1}{n_2} \left( d_{G_1} + (n_2 - 1)d_{G_1(U_1)} \right) \]

(b) **Eccentricity of** \( x = (x_2, x_1) \in V: \)
\[ \text{ecc}_H(x) = \text{ecc}_{G_2}(x_2) + \text{ecc}_{G_1(U_1)}(x_1) \]

(c) **Radius:**
\[ r_H = r_{G_2} + r_{G_1(U_1)} \]

(d) **Diameter:**
\[ D_H = D_{G_2} + D_{G_1(U_1)} \]
Some Properties on the Generalized Hierarchical Product of Graphs

Hamiltonian cycles

1. The hierarchical product
2. The generalized hierarchical product
3. Metric parameters
4. Hamiltonian cycles
5. Vertex- and edge-coloring
6. Connectivity
Hamiltonian cycles: sufficient conditions

Proposition

If $G_1$, $G_2$ Hamiltonian and $G_1[U_1]$ contains two consecutive edges of a Hamiltonian cycle of $G_1$, then $H = G_2 \cap G_1(U_1)$ is Hamiltonian.

Sketch A Hamiltonian cycle in $G_2 \cap G_1(U_1)$ going through three copies of $G_2$ and $n_2$ copies of $G_1$. 
Proposition

If $G_1$, $G_2$ Hamiltonian, $n_2 = |V_2|$ is even, and $G_1[U_1]$ contains one edge of a Hamiltonian cycle of $G_1$, then $H = G_2 \sqcap G_1(U_1)$ is Hamiltonian.

Sketch A Hamiltonian cycle in $G_2 \sqcap G_1(U_1)$ going through two copies of $G_2$ and $n_2$ copies of $G_1$ when $n_2$ is even.
Some Properties on the Generalized Hierarchical Product of Graphs

1 The hierarchical product
2 The generalized hierarchical product
3 Metric parameters
4 Hamiltonian cycles
5 Vertex- and edge-coloring
6 Connectivity
Vertex-coloring

$\chi(G)$ chromatic number of $G$

- $\chi(G_2 \Box G_1) = \max\{\chi(G_2), \chi(G_1)\}$ [Sabidussi, 1957]

Proposition

$\chi(G_2 \cap G_1(U_1)) = \max\{\chi(G_2), \chi(G_1)\}$

Proof.

\[ G_1, \ G_2 < G_2 \cap G_1(U_1) < G_2 \Box G_1 \]
Edge-coloring

\(\chi'(G)\) chromatic index of \(G\)

- \(\Delta_G \leq \chi'(G) \leq \Delta_G + 1\) [Vizing, 1964]
  
  (\(G\) of class 1 if \(\chi'(G) = \Delta_G\), \(G\) of class 2 if \(\chi'(G) = \Delta_G + 1\))

- if one of the graphs \(G_1\) or \(G_2\) is of class 1 then \(G_2 \square G_1\) is of class 1 [Mahmoodian, 1981]

**Proposition**

\(\chi'(G_2 \sqcap G_1) = \max\{\Delta_{G_2} + d_0, \chi'(G_1)\}\)

**Proof.**
Proposition
\[ \max\{\Delta_{G_2} + \Delta_{G_1}(U_1), \chi'(G_1)\} \leq \chi'(H) \leq \max\{\chi'(G_2) + \Delta_{G_1}(U_1), \chi'(G_1)\} \]

Corollary
- if \( G_1 \) is of class 1 and \( \exists u \in U_1, \deg_{G_1}(u) = \Delta_{G_1} \), then \( \chi'(H) = \Delta_{G_2} + \Delta_{G_1} \)
- if \( G_2 \) is of class 1 then \( \chi'(H) = \max\{\Delta_{G_2} + \Delta_{G_1}(U_1), \chi'(G_1)\} \)
Some Properties on the Generalized Hierarchical Product of Graphs

Connectivity

1. The hierarchical product
2. The generalized hierarchical product
3. Metric parameters
4. Hamiltonian cycles
5. Vertex- and edge-coloring
6. Connectivity
Some Properties on the Generalized Hierarchical Product of Graphs

Connectivity

**Vertex connectivity of** $H = G_2 \sqcap G_1(U_1)$

Cartesian product: $H = G_2 \square G_1$, i.e., $U_1 = V_1$

$$\kappa(G_2 \square G_1) = \min\{\kappa_1|V_2|, \kappa_2|V_1|, \delta_1 + \delta_2\},$$

$\kappa_i = \kappa(G_i)$ and $\delta_i = \delta(G_i)$ [Špacapan 2007]

**Connectivity relative to** $U$

$G = (V, E)$ graph, $U \subsetneq V$

$$\kappa(U|\overline{U}) = \min\{|S| : S \subset V, \exists u \in G - S \text{ s.t. } \nexists \text{ path from } u \text{ to } U\}$$

**Proposition**

$U_1 \subsetneq V_1 \Rightarrow \kappa(G_2 \sqcap G_1(U_1)) \leq \min\{|\kappa_1|V_2|, \kappa(U_1|\overline{U}_1), \delta_H\}$,

*where* $\delta_H = \min\{\delta_{G_1(\overline{U}_1)}, \delta_{G_1(U_1)} + \delta_{G_2}\}$. 
See the poster on Algebraic properties of the generalized hierarchical product !!!
Thank you !!!