The Hierarchical Product of Graphs

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The Hierarchical Product of Graphs

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   - Generalization of the hierarchical product
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Introduction

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Motivation

Complex networks: randomness, heterogeneity, modularity


Hierarchical networks: degree distribution, modularity

Our work

- Deterministic graphs
- Algebraic methods
Our work

- Deterministic graphs
- Algebraic methods
- Far from "real networks"

but a beautiful mathematical object !!!
Previous work

The Hierarchical Product of Graphs
Graphs and matrices

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2 Graphs and matrices

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   Spectral properties of $G \sqcap K^m_2$
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   Hypertrees and generalized hypertrees
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6 Conclusions
Spectrum of a matrix $M$

$M$ $n \times n$ matrix on $\mathbb{R}$

- Characteristic polynomial of $M$
  \[ \Phi_M(x) := \det(xI - M) \]

- Spectrum of $M$
  \[ \text{sp}M := \text{set of roots of } \Phi_M(x), \text{ called eigenvalues of } M \]
  \[ \lambda \in \text{sp}M \Rightarrow \dim \ker(\lambda I - M) \geq 1 \]

- Eigenvectors, eigenspaces
  \[ \mathbf{v} \text{ is a } \lambda\text{-eigenvector if } M\mathbf{v} = \lambda \mathbf{v} \]
  \[ \lambda \in \text{sp}M, \ E_\lambda := \text{ set of } \lambda\text{-eigenvectors of } M \]
  \[ E_\lambda \text{ is a subspace of } \mathbb{R}^n \]
Adjacency matrix and Laplacian matrix

$G = (V, E)$, $V = \{1, 2, \ldots n\}$ ⇒

- Adjacency matrix of $G$:

$$A(G) = (a_{i,j})_{1 \leq i, j \leq n} \quad a_{i,j} = \begin{cases} 1, & \text{if } i \sim j \\ 0, & \text{if } i \not\sim j \end{cases}$$

$$\text{tr}(A) = 0, \quad \sum_j a_{i,j} = \delta_i$$

(Ordinary) spectrum of $G := \text{spectrum of } A(G)$.

- Laplacian matrix of $G$:

$$L(G) = (\ell_{i,j})_{1 \leq i, j \leq n} \quad \ell_{i,j} = \begin{cases} \delta_i, & \text{if } i = j \\ -1, & \text{if } i \sim j \\ 0, & \text{if } i \not\sim j, i \neq j \end{cases}$$

$$L(G) = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n) - A(G)$$

Laplacian spectrum of $G := \text{spectrum of } L(G)$. 
Example: \( G = P_3 \)

\[
A(G) = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} \quad L(G) = \begin{bmatrix}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{bmatrix}
\]
Example: $G = P_3$

$$A(G) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow \Phi_A(x) = \det \begin{bmatrix} x & -1 & 0 \\ -1 & x & -1 \\ 0 & -1 & x \end{bmatrix} = x^3 - 2x$$

Eigenvalues and eigenvectors

$$\Phi_A(x) = (x - \sqrt{2}) \cdot x \cdot (x + \sqrt{2}) \Rightarrow \lambda_1 = \sqrt{2}, \lambda_2 = 0, \lambda_3 = -\sqrt{2}$$

$w_1 = (\sqrt{2}, 2, \sqrt{2})$

$w_2 = (1, 0, -1)$

$w_3 = (\sqrt{2}, -2, \sqrt{2})$
Example: $G = P_3$

$$L(G) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow$$

$$QL(x) = \det \begin{bmatrix} x - 1 & 1 & 0 \\ 1 & x - 2 & 1 \\ 0 & 1 & x - 1 \end{bmatrix} = x^3 - 4x^2 + 3x$$

Laplacian eigenvalues and eigenvectors

$$QL(x) = x \cdot (x - 1) \cdot (x - 3) \Rightarrow \mu_1 = 3, \mu_2 = 1, \mu_3 = 0$$

$$\mathbf{w}_1 = (1, -2, 1)$$
$$\mathbf{w}_2 = (1, 0, -1)$$
$$\mathbf{w}_3 = (1, 1, 1)$$
Properties

\( G = (V, E) \) graph ⇒

- \( A \) adjacency matrix
- \( \Phi_A(x) = \Phi_G(x) = \det(xI - A) \) characteristic polynomial
- \( \text{sp}(A) = \text{sp}(G) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\} \)
- \( \text{ev}(A) = \text{ev}(G) = \{\lambda_0 > \lambda_1 > \cdots > \lambda_d\} \)

Basic properties

1. \( A \) symmetric ⇒ \( \forall \lambda_i \in \mathbb{R}; A \) diagonalizes; \( \lambda_i \in \mathbb{Q} \Rightarrow \lambda_i \in \mathbb{Z} \)
2. \( G = G_1 \cup \cdots \cup G_k \) connected comp. ⇒ \( \Phi_G(x) = \prod_i \Phi_{G_i}(x) \)
3. \( G \) connected ⇒ \( \lambda_0 = \rho(G) \) spectral radius of \( G \)
   \( \forall i, |\lambda_i| \leq \rho(G) \)
   if \( m \geq 1 \) ⇒ \( \rho(G) \geq 1 \) and there is a negative eigenvalue
4. \( \mathbf{w} = (w_1, \ldots, w_n) \) eigenvector of eigenvalue \( \lambda \) ⇒
   \[ A\mathbf{w} = \lambda\mathbf{w} \iff \forall i, \sum_{j \sim i} w_j = \lambda w_i \]
   (assign weight \( w_i \) to vertex \( i \))
An easy case

\[ G = K_n \]

\[ A(K_n) = J - I, \text{ where } J = (1) \]
\[ sp(J) = \{n^1, 0^{n-1}\} \]
\[ E_n = (1, 1, \ldots 1) \]
\[ E_0 \perp E_n \]

\[ sp(K_n) = \{(n - 1)^1, (-1)^{n-1}\} \]
\[ E_{n-1} = (1, 1, \ldots 1) \]
\[ E_{-1} \perp E_n \]
Not so basic properties

1. \[ \frac{\delta_1 + \cdots + \delta_n}{n} \leq \lambda_0 \leq \max_i \delta_i \]
   
   If \( G \) is \( \delta \)-regular, then \( \lambda_0 = \delta \) and \( \mathbf{w}_0 = (1, 1, \ldots, 1) \)

2. \( D = \text{diam} G \Rightarrow D \leq d = |\text{ev}(G)| - 1 \)

3. \( G \) bipartite \( \Leftrightarrow \) \( sp(G) \) symmetric (with respect to 0)

4. \( \omega_G \) clique number of \( G \), \( \chi_G \) chromatic number of \( G \) \( \Rightarrow \)
   \[ \omega_G \leq 1 - \frac{\lambda_0}{\lambda_d} \leq \chi_G \leq 1 + \lambda_0 \]

5. \( G \) regular, \( \alpha_G \) independence number of \( G \) \( \Rightarrow \)
   \[ \alpha_G \leq \frac{n}{1 + \frac{\lambda_0}{-\lambda_d}} \]

The Hierarchical Product of Graphs
Graphs and matrices

Spectrum of some graphs

• $\text{sp}(K_{m,n}) = \{ \pm \sqrt{mn}, 0^{m+n-2} \}$

• $\omega = e^{\frac{2\pi i}{n}} \Rightarrow \text{sp}(C_n) = \{ \omega^r + \omega^{-r} = 2 \cos \frac{2\pi r}{n} : 0 \leq r \leq n - 1 \}$

$A(C_4) = \begin{bmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
\end{bmatrix}$ $\Rightarrow \Phi_{C_4}(x) = (x^2 - 4) \cdot x^2$

$\omega = e^{\frac{2\pi i}{4}} = i \Rightarrow \lambda_0 = \omega^4 + \omega^{-4} = 2, \lambda_1 = \omega^3 + \omega^{-3} = 0,$
$\lambda_2 = \omega + \omega^{-1} = 0, \lambda_3 = \omega^2 + \omega^{-2} = -2$

• $\text{sp}(P_n) = \{ 2 \cos \frac{\pi r}{n+1} : 1 \leq r \leq n \}$

$\begin{cases}
\text{sp}(G) = \{ \lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d} \} \\
\text{sp}(H) = \{ \mu_0^{k_0}, \mu_1^{k_1}, \ldots, \mu_{d'}^{k_{d'}} \} \\
\end{cases} \Rightarrow \text{sp}(G \square H) = \{ (\lambda_i + \mu_j)^{m_i+k_j} : 0 \leq i \leq d, 0 \leq j \leq d' \}$
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\[ \Phi_G(x) \text{ coefficients} \]

\[ \Phi_G(x) = x^n + c_1 x^{n-1} + \cdots + c_{n-1} x + c_n \Rightarrow \]

1. \[ c_1 = \text{tr}(A) = 0 \]

\[ A^k = (a^k_{i,j}), \ a^k_{i,j} = \text{number of walks of length } k \text{ from } i \text{ to } j; \]
\[ c := \text{number of closed walks of length } k \Rightarrow \]

\[ c = \text{tr}(A^k) = \sum_i \lambda^k_i \]

In particular, \[ \text{tr}(A^2) = \sum_i \lambda^2_i = 2 \cdot m \text{ and} \]
\[ \text{tr}(A^3) = \sum_i \lambda^3_i = 6 \cdot t, \text{ where } t = \text{number of triangles}. \]

2. \[ -c_2 = m \]

3. \[ -c_3 = 2 \cdot t \]
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The Hierarchical Product of Graphs

The hierarchical product

Definition and basic properties

Definition

For $i = 1, \ldots, N$, $G_i$ graph rooted at 0

$H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$

- vertices $x_N \ldots x_3 x_2 x_1$, $x_i \in V_i$
- if $x_j \sim y_j$ in $G_j$ then

$x_N \ldots x_{j+1} x_j 0 \ldots 0 \sim x_N \ldots x_{j+1} y_j 0 \ldots 0$

Example

The hierarchical products $K_2 \sqcap K_3$ and $K_3 \sqcap K_2$
The Hierarchical Product of Graphs

The hierarchical product

Definition and basic properties

\[ G_N \sqcap \cdots \sqcap G_2 \sqcap G_1 \text{ is a spanning subgraph of } G_N \sqcap \cdots \sqcap G_2 \sqcap G_1 \]

Example

*The hierarchical product* \( P_4 \sqcap P_3 \sqcap P_2 \)
The Hierarchical Product of Graphs

The hierarchical product

Definition and basic properties

\[ G^N = G \sqcap \cdots \sqcap G \] is the hierarchical \( N \)-power of \( G \)

Example

The hierarchical powers \( K_2^2 \), \( K_2^4 \) and \( K_2^5 \)
Example

*The hierarchical power $C_4^3$*
The Hierarchical Product of Graphs
The hierarchical product
Definition and basic properties

Order and size

\[ n_i = |V_i| \text{ and } m_i = |E_i| \]
\[ H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1 \]

- \[ n_H = n_N \cdots n_3 n_2 n_1 \]
- \[ m_H = \sum_{k=1}^{N} m_k n_{k+1} \cdots n_N \]

Properties of \( \sqcap \)

- Associativity. \( G_3 \sqcap G_2 \sqcap G_1 = G_3 \sqcap (G_2 \sqcap G_1) = (G_3 \sqcap G_2) \sqcap G_1 \)
- Right-distributivity. \( (G_3 \cup G_2) \sqcap G_1 = (G_3 \sqcap G_1) \cup (G_2 \sqcap G_1) \)
- Left-semi-distributivity. \( G_3 \sqcap (G_2 \cup G_1) = (G_3 \sqcap G_2) \cup n_3 G_1, \) where \( n_3 G_1 = \overline{K}_{n_3} \sqcap G_1 \) is \( n_3 \) copies of \( G_1 \)
- \( G \sqcap K_1 = K_1 \sqcap G = G \)

\((G, \sqcap) \) is a monoid
The Hierarchical Product of Graphs

The hierarchical product

Vertex hierarchy

Degrees

- If \( \delta_i = \delta_{G_i}(0) \), then
  
  \[
  \delta_{G}(0) = \sum_{i=1}^{N} \delta_i
  \]

  \[
  x = x_Nx_{N-1} \ldots x_k00 \ldots 0, \ x_k \neq 0 \Rightarrow
  \]

  \[
  \delta_{H}(x) = \sum_{i=1}^{k-1} \delta_i + \delta_{G_k}(x_k)
  \]

- If \( G \) is \( \delta \)-regular, the degrees of the vertices of \( G^N \) follow an exponential distribution, \( P(k) = \gamma^{-k} \), for some constant \( \gamma \)

  For \( k = 1, \ldots, N - 1 \), \( G^N \) contains \((n - 1)n^{N-k} \) vertices with degree \( k\delta \) and \( n \) vertices with degree \( N\delta \)
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Vertex hierarchy

Example

\[ T_m = K_2^m \text{ has } 2^{m-k} \text{ vertices of degree } k = 1, \ldots, m - 1, \text{ and two vertices of degree } m \]
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Vertex hierarchy

Modularity

\[ H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1, \text{ } z \text{ an appropriate string} \]

\[ H\langle zx_k \ldots x_1 \rangle = H[\{zx_k \ldots x_1|x_i \in V_i, 1 \leq i \leq k\}] \]

\[ H\langle x_N \ldots x_k z \rangle = H[\{x_N \ldots x_k z|x_i \in V_i, k \leq i \leq N\}] \]

Lemma

- \[ H\langle zx_k \ldots x_1 \rangle = G_k \sqcap \cdots \sqcap G_1, \text{ for any fixed } z \]
- \[ H\langle x_N \ldots x_k 0 \rangle = G_N \sqcap \cdots \sqcap G_k \]
- \[ H\langle x_N \ldots x_k z \rangle = (n_N \cdots n_k)K_1, \text{ for any fixed } z \neq 0 \]

\[ H^* = H - 0 \]

Lemma

- \[ (G_N \sqcap \cdots \sqcap G_2 \sqcap G_1)^* = \bigcup_{k=1}^{N} (G_k^* \sqcap G_{k-1} \sqcap \cdots \sqcap G_1) \]
- \[ (K_2^N)^* = \bigcup_{k=0}^{N-1} K_2^k \]
- \[ K_2^N - \{0, 10\} = K_2^{N-1} \cup K_2^{N-1} \]
Example

*Modularity and symmetry of* $T_m = K^m_2$
Example

Modularity and symmetry of $T_m = K_2^m$
Example

*Modularity and symmetry of* $T_m = K_{2^m}$
Example

Modularity and symmetry of $T_m = K_{2^m}$
Example

Modularity and symmetry of $T_m = K_{2m}$
Example

*Modularity and symmetry of $T_m = K_2^m$*
Eccentricity, radius and diameter

$$H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$$

$$\varepsilon_i = \text{ecc}_{G_i}(0), \ r_{G_N} = r_N \text{ and } D_{G_N} = D_N$$

$$\rho_i$$ shortest path routing of $$G_i$$, $$i = 1, \ldots, N$$

Proposition

• $${\rho_i} \{i = 1 \ldots N\}$$ induce a shortest path routing $$\rho$$ in $$H$$

• The eccentricity, radius and diameter of $$H$$ are

$$\text{ecc}_H(0) = \sum_{i=1}^{N} \varepsilon_i, \quad r_H = r_N + \sum_{i=1}^{N-1} \varepsilon_i, \quad D_H = D_N + 2 \sum_{i=1}^{N-1} \varepsilon_i$$
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Metric parameters

Proof.
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Proof.
Mean distance

$G$ graph of order $n$

Mean distance. $d_G = \frac{1}{n(n-1)} \sum_{v \neq w \in V} \text{dist}_G(v, w)$

Local mean distance. $d_0^G = \frac{1}{n} \sum_{v \in V} \text{dist}_G(0, v)$

Proposition

$H = G_2 \cap G_1 \Rightarrow \begin{cases} d_H^{00} = d_1^0 + d_2^0 \\ d_H = \frac{1}{n-1} [(n_1 - 1)d_1 + n_1(n_2 - 1)(d_2 + 2d_1^0)] \end{cases}$
Mean distance

$G$ graph of order $n$

Mean distance. $d_G = \frac{1}{n(n-1)} \sum_{v \neq w \in V} \text{dist}_G(v, w)$

Local mean distance. $d^0_G = \frac{1}{n} \sum_{v \in V} \text{dist}_G(0, v)$

Proposition

$H = G_2 \sqcap G_1 \Rightarrow \left\{ \begin{array}{l}
  d^{00}_H = d^0_1 + d^0_2 \\
  d_H = \frac{1}{n-1} \left[ (n_1 - 1)d_1 + n_1(n_2 - 1)(d_2 + 2d^0_1) \right]
\end{array} \right.$

Proof.

Just compute!
Corollary

\[ H = G^N, \ d = d_G, \ d^0 = d_G^0 \]

- \( \text{ecc}_N(0) = N\varepsilon, \ d_N^0 = Nd^0 \)
- \( r_N = r + (N - 1)\varepsilon, \ D_N = D + 2(N - 1)\varepsilon \)
- \( d_N = d + 2 \left( \frac{(N-1)n^N+1}{n^N-1} - \frac{1}{n-1} \right) d^0 \)

Asymptotically, \( d_N \sim d + 2d^0 \left( N - \frac{n}{n-1} \right) \sim d + 2Nd^0 \)

Example

\[ G = K_2 \Rightarrow \text{ecc} = r = D = 1, \ d^0 = 1/2 \text{ and } d = 1 \]

The metric parameters of \( T_m = K_2^m \) are

- \( \text{ecc}_m(0) = m, \ d_m^0 = m/2 \)
- \( r_m = m, \ D_m = 2m - 1 \)
- \( d_m = \frac{m^{2m}}{2^{m-1}} - 1 \sim m - 1 \)
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Background

Kronecker product $A \otimes B = (a_{ij}B)$
If $A$ and $B$ are square, $A \otimes B$ and $B \otimes A$ are permutation similar
Background

**Kronecker product** $A \otimes B = (a_{ij}B)$

If $A$ and $B$ are square, $A \otimes B$ and $B \otimes A$ are permutation similar

**Lemma**

$H = G_2 \cap G_1 \Rightarrow$

$$A_H = A_2 \otimes D_1 + I_2 \otimes A_1 \cong D_1 \otimes A_2 + A_1 \otimes I_2$$

*where* $D_1 = \text{diag}(1, 0, \ldots, 0)$
The Hierarchical Product of Graphs

Algebraic properties

Background

Kronecker product $A \otimes B = (a_{ij}B)$

If $A$ and $B$ are square, $A \otimes B$ and $B \otimes A$ are permutation similar

Lemma

$H = G_2 \cap G_1 \Rightarrow$

$$A_H = A_2 \otimes D_1 + I_2 \otimes A_1 \cong D_1 \otimes A_2 + A_1 \otimes I_2$$

where $D_1 = \text{diag}(1, 0, \ldots, 0)$

Example

$H = G \cap K_n$, $G$ of order $N \Rightarrow$

$$A_H = D_1 \otimes A_G + A_{K_n} \otimes I_N = \begin{pmatrix}
A_G & I_N & \cdots & I_N \\
I_N & 0 & \cdots & I_N \\
\vdots & \vdots & \ddots & \vdots \\
I_N & I_N & \cdots & 0
\end{pmatrix}$$
The Theorem (Silvester, 2000)

$R$ commutative subring of $F^{n \times n}$, the set of all $n \times n$ matrices over a field $F$ (or a commutative ring), and $M \in R^{m \times m}$. Then,

$$\det_F M = \det_F(\det_R M)$$

The Corollary (Silvester, 2000)

$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where $A$, $B$, $C$, $D$ commute with each other.

Then,

$$\det M = \det(AD - BC)$$
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Algebraic properties
Spectral properties of $G \sqcap K_2^m$

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Spectral properties of $G \sqcap K_2^m$

$G \sqcap K_2$

Example

*The Petersen graph, hierarchically multiplied by $K_2$*
$G \sqcap K_2$

$G$ graph of order $n$, 
$A$ adjacency matrix of $G$ and 
$\phi_G$ characteristic polynomial of $G$

- The adjacency matrix of $H = G \sqcap K_2$ is 

$$A_H = \begin{pmatrix} A & I_n \\ I_n & 0 \end{pmatrix}$$

- The characteristic polynomial of $H$ is 

$$\phi_H(x) = \det(xI_{2n} - A_H) = \det\left( \begin{pmatrix} xI_n - A & -I_n \\ -I_n & xI_n \end{pmatrix} \right) = \det((x^2 - 1)I_n - xA) = x^n \phi_G(x - \frac{1}{x})$$
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Spectral properties of $G \sqcap K_2^m$

\[ \phi_H(x) = x^n \phi_G(x - \frac{1}{x}) \]

**Proposition**

$H = G \sqcap K_2$ and \( \text{sp}G = \{ \lambda_{00}^m < \lambda_{11}^m < \ldots < \lambda_{dd}^m \} \) \implies

\[ \text{sp}H = \{ \lambda_{00}^m < \lambda_{01}^m < \ldots < \lambda_{0d}^m < \lambda_{10}^m < \lambda_{11}^m < \ldots < \lambda_{1d}^m \} \]

where \( \lambda_{0i} = f_0(\lambda_i) = \frac{\lambda_i - \sqrt{\lambda_i^2 + 4}}{2} \), \( \lambda_{1i} = f_1(\lambda_i) = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4}}{2} \)
\[ \phi_H(x) = x^n \phi_G(x - \frac{1}{x}) \]

**Proposition**

\( H = G \sqcap K_2 \) and \( \text{sp}G = \{ \lambda_0^{m_0} < \lambda_1^{m_1} < \ldots < \lambda_d^{m_d} \} \) ⇒

\[ \text{sp}H = \{ \lambda_{00}^{m_0} < \lambda_{01}^{m_1} < \ldots < \lambda_{0d}^{m_d} < \lambda_{10}^{m_0} < \lambda_{11}^{m_1} < \ldots < \lambda_{1d}^{m_d} \} \]

where \( \lambda_{0i} = f_0(\lambda_i) = \frac{\lambda_i - \sqrt{\lambda_i^2 + 4}}{2} \), \( \lambda_{1i} = f_1(\lambda_i) = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4}}{2} \)

**Proof.**

\( \lambda \in \text{sp}H \iff \phi_H(\lambda) = \lambda^n \phi_G(\lambda - \frac{1}{\lambda}) = 0 \iff \lambda - \frac{1}{\lambda} \in \text{sp}G \)

\[ \lambda_i \in \text{sp}G \implies \lambda^2 - \lambda_i \lambda - 1 = 0 \]
The Hierarchical Product of Graphs

Algebraic properties

Spectral properties of $G \sqcap K_2^m$

$$\phi_H(x) = x^n \phi_G(x - \frac{1}{x})$$
The Hierarchical Product of Graphs

Algebraic properties

Spectral properties of $G \square K_2^m$

$$H_m = G \square K_2^m$$

$H_m = H_{m-1} \square K_2$, $m \geq 1$. The adjacency matrix of $H_m$ is

$$A_m = \begin{pmatrix} A_{m-1} & I_{m-1} \\ I_{m-1} & 0 \end{pmatrix}$$

where $I_m$ denotes the identity matrix of size $n2^m$ (the same as $A_m$).

$H_0 = G$, $A_0 = A$ the adjacency matrix of $G$

Example
Let \( \{p_i, q_i\}_{i \geq 0} \) be the family of polynomials satisfying the recurrence equations
\[
\begin{align*}
    p_i &= p_{i-1}^2 - q_{i-1}^2 \\
    q_i &= p_{i-1} q_{i-1}
\end{align*}
\]
with initial conditions
\[
p_0 = x \text{ and } q_0 = 1
\]

**Proposition**

For every \( m \geq 0 \), the characteristic polynomial of \( H_m = G \sqcap K_2^m \) is
\[
\phi_m(x) = q_m(x)^n \phi_0 \left( \frac{p_m(x)}{q_m(x)} \right)
\]

**Lemma**

If \( p \) and \( q \) are arbitrary polynomials, then
\[
\det \begin{pmatrix} pI_n - qA & -qI_n \\ -qI_n & pI_n \end{pmatrix} = \det((p^2 - q^2)I_n - pqA)
\]
Proof of $\phi_m(x) = q_m(x)^n \phi_0 \left( \frac{p_m(x)}{q_m(x)} \right)$.

By induction on $m$, using the Lemma
Proof of $\phi_m(x) = q_m(x)^n \phi_0 \left( \frac{p_m(x)}{q_m(x)} \right)$.

By induction on $m$, using the Lemma

- Case $m = 0$. Trivially from $q_0(x) = 1$ and $p_0(x) = x$. 

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By induction on $m$, using the Lemma

- **Case $m = 0$.** Trivially from $q_0(x) = 1$ and $p_0(x) = x$.

- **$m \geq 1$.** By induction on $i$, we prove that
  \[ \phi_m = \det(p_i I_{m-i} - q_i A_{m-i}) \]

  - $i = 0$ :  $\phi_m = \det(x I_m - A_m) = \det(p_0 I_m - q_0 A_m)$
  - $i - 1 \Rightarrow i$ :  $\phi_m = \det(p_{i-1} I_{m-i+1} - q_{i-1} A_{m-i+1}) = \det((p_{i-1}^2 - q_{i-1}^2) I_{m-i} - p_{i-1} q_{i-1} A_{m-i})) = \det(p_i I_{m-i} - q_i A_{m-i})$
Proof of $\phi_m(x) = q_m(x)^n \phi_0 \left( \frac{p_m(x)}{q_m(x)} \right)$.

By induction on $m$, using the Lemma

- **Case $m = 0$.** Trivially from $q_0(x) = 1$ and $p_0(x) = x$.
- **$m \geq 1$.** By induction on $i$, we prove that
  \[
  \phi_m = \det(p_i I_{m-i} - q_i A_{m-i})
  \]
  - $i = 0$ : $\phi_m = \det(x I_m - A_m) = \det(p_0 I_m - q_0 A_m)$
  - $i - 1 \Rightarrow i$ : $\phi_m = \det(p_{i-1} I_{m-i+1} - q_{i-1} A_{m-i+1}) = $
  \[
  = \det((p_{i-1}^2 - q_{i-1}^2) I_{m-i} - p_{i-1} q_{i-1} A_{m-i}) = 
  \]
  \[
  = \det(p_i I_{m-i} - q_i A_{m-i})
  \]
  - The case $i = m$ gives
  \[
  \phi_m(x) = \det(p_m(x) I_0 - q_m(x) A_0) = 
  \]
  \[
  = \det \left( q_m(x) \left( \frac{p_m(x)}{q_m(x)} I_0 - A_0 \right) \right) = q_m(x)^n \phi_0 \left( \frac{p_m(x)}{q_m(x)} \right)
  \]
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  The spectrum of the binary hypertree $T_m = K_2^m$

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The spectrum of the binary hypertree $T_m = K_2^m$

\[ T_m = K_2^m \]

\[ p_i = p_{i-1}^2 - q_{i-1}^2 \]
\[ q_i = p_{i-1}q_{i-1} \]
\[ p_0 = x, \quad q_0 = 1 \]

Corollary

- $\phi_{T_m}(x) = p_m(x)$
- $\phi_{T_m^*}(x) = q_m(x)$
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Algebraic properties

The spectrum of the binary hypertree $T_m = K_2^m$

\[ T_m = K_2^m \]

\[ p_i = p_{i-1}^2 - q_{i-1}^2 \]
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\[ p_0 = x, \quad q_0 = 1 \]

Corollary

- $\phi_{T_m}(x) = p_m(x)$
- $\phi_{T_m^*}(x) = q_m(x)$

Proof.

$G = K_1 \implies \phi_0(x) = x \implies \phi_{T_m}(x) = q_m(x)^n \phi_0\left(\frac{p_m(x)}{q_m(x)}\right) = p_m(x)$

$T_m^* = T_m - 0 = \bigcup_{i=0}^{m-1} T_i \implies \phi_{T_m^*}(x) = \prod_{i=0}^{m-1} p_i(x) = q_m(x)$
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**Proposition**

$T_m$, $m \geq 1$, has **distinct** eigenvalues $\lambda_0^m < \lambda_1^m < \cdots < \lambda_{n-1}^m$, with $n = 2^m$, **satisfying the following recurrence relation**:

$$
\lambda_{\frac{n}{2}+k}^m = \frac{\lambda_{k-1}^{m-1} + \sqrt{(\lambda_{k-1}^{m-1})^2 + 4}}{2}
$$

$$
\lambda_{n-k-1}^m = -\lambda_k^m
$$

for $m > 1$ and $k = \frac{n}{2}, \frac{n}{2} + 1, \ldots, n - 1$
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Proposition

$T_m, m \geq 1,$ has distinct eigenvalues $\lambda_0^m < \lambda_1^m < \cdots < \lambda_{n-1}^m,$ with $n = 2^m,$ satisfying the following recurrence relation:

$$
\lambda_{\frac{n}{2}+k}^m = \frac{\lambda_{k}^{m-1} + \sqrt{(\lambda_{k}^{m-1})^2 + 4}}{2}
$$

$$
\lambda_{n-k-1}^m = -\lambda_k^m
$$

for $m > 1$ and $k = \frac{n}{2}, \frac{n}{2} + 1, \ldots, n - 1$

Proof.

• $\lambda_0 i = f_0(\lambda_i) = \frac{\lambda_i - \sqrt{\lambda_i^2 + 4}}{2},$ $\lambda_1 i = f_1(\lambda_i) = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4}}{2}$

• $T_m$ bipartite $\Rightarrow$ its spectrum is symmetric with respect to 0

• $\text{sp } T_0 = \{0^1\} \Rightarrow$ the multiplicity of every $\lambda_1^m$ is 1
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**Properties of $sp \ T_m$**

$$\lambda_i \in spG \Rightarrow \lambda^2 - \lambda_i \lambda - 1 = 0$$

$$f_0(x) = \frac{x - \sqrt{x^2 + 4}}{2} \quad f_1(x) = \frac{x + \sqrt{x^2 + 4}}{2}$$

$m = 0 \Rightarrow sp T_0 = \{0\}$

$m = 1 \Rightarrow \lambda_0 = f_0(0) = -1, \ \lambda_1 = f_1(0) = 1$

$m = 2 \Rightarrow$

\[\begin{align*}
\lambda_0 &= f_0(-1) = f_0(f_0(0)) = -1.618 \\
\lambda_1 &= f_0(1) = f_0(f_1(0)) = -0.618 \\
\lambda_2 &= f_1(-1) = f_1(f_0(0)) = 0.618 \\
\lambda_3 &= f_1(1) = f_1(f_1(0)) = 1.618
\end{align*}\]

$\ldots$

$m$ fixed, $i = i_{m-1} \ldots i_1 i_0 \in \mathbb{Z}_2^m \Rightarrow$

$$\Rightarrow \lambda_i = (f_{i_{m-1}} \circ \ldots \circ f_{i_1} \circ f_{i_0})(0)$$
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The spectrum of the binary hypertree $T_m = K_2^m$

The distinct eigenvalues of the hypertree $T_m$ for $0 \leq m \leq 6$. 

![Graph Diagram]
Proposition

The asymptotic behaviors of

- the spectral radius $\rho_k = \max_{0 \leq i \leq n-1} \{|\lambda_i|^2\} = \lambda_{11\ldots1}$,
- the second largest eigenvalue $\theta_k = \lambda_{11\ldots10}$, and
- the minimum positive eigenvalue $\sigma_k = \min_{0 \leq i \leq n-1} \{|\lambda_i|^2\} = \lambda_{10\ldots0}$

of the hypertree $T_k$ are:

$$\rho_k \sim \sqrt{2^k}, \quad \theta_k \sim \sqrt{2^k}, \quad \sigma_k \sim 1/\sqrt{2^k}$$
Proof of $\rho_k \sim \sqrt{2k}$, $\theta_k \sim \sqrt{2k}$, $\sigma_k \sim \frac{1}{\sqrt{2k}}$. 
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Proof of $\rho_k \sim \sqrt{2k}, \theta_k \sim \sqrt{2k}, \sigma_k \sim 1/\sqrt{2k}$.

• $\rho_k \sigma_k = 1$
Proof of $\rho_k \sim \sqrt{2k}$, $\theta_k \sim \sqrt{2k}$, $\sigma_k \sim 1/\sqrt{2k}$.

- $\rho_k \sigma_k = 1$
- $\rho_k$ and $\theta_k$ verify the recurrence
  
  $\lambda_{k+1} = f_1(\lambda_k) = \frac{1}{2}(\lambda_k + \sqrt{\lambda_k^2 + 4})$
Proof of $\rho_k \sim \sqrt{2k}$, $\theta_k \sim \sqrt{2k}$, $\sigma_k \sim 1/\sqrt{2k}$.

- $\rho_k \sigma_k = 1$
- $\rho_k$ and $\theta_k$ verify the recurrence
  \[
  \lambda_{k+1} = f_1(\lambda_k) = \frac{1}{2}(\lambda_k + \sqrt{\lambda_k^2 + 4})
  \]
- Assuming $\lambda_k \sim \alpha k^\beta$
  \[
  \alpha (k + 1)^\beta \sim \frac{\alpha k^\beta + \sqrt{\alpha^2 k^{2\beta} + 4}}{2} \Rightarrow
  \Rightarrow \alpha^2 (k + 1)^\beta [(k + 1)^\beta - k^\beta] \sim 1
  \]
  \[
  2(k + 1)^{1/2} [(k + 1)^{1/2} - k^{1/2}] = \frac{2(k + 1)^{1/2}}{(k + 1)^{1/2} + k^{1/2}} \to 1
  \]
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The spectrum of a generic two-term product $G_2 \sqcap G_1$

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**Theorem**

Let $G_1$ and $G_2$ be two graphs on $n_i$ vertices, with adjacency matrix $A_i$ and characteristic polynomial $\phi_i(x)$, $i = 1, 2$.

Consider the graph $G_1^* = G_1 - 0$, with adjacency matrix $A_1^*$ and characteristic polynomial $\phi_1^*$.

Then the characteristic polynomial $\phi_H(x)$ of the hierarchical product $H = G_2 \sqcap G_1$ is:

$$\phi_H(x) = \phi_1^*(x)^{n_2} \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right)$$
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The spectrum of a generic two-term product \( G_2 \sqcap G_1 \)

Proof of \( \phi_H(x) = \phi_1^*(x)^n \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right). \)
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Proof of $\phi_H(x) = \phi_1^*(x)^{n_2} \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right)$.

- The adjacency matrix of $H$ is an $n_1 \times n_1$ block matrix, with blocks of size $n_2 \times n_2$

$$A_H = D_1 \otimes A_2 + A_1 \otimes I_2 = \begin{pmatrix} A_2 & B \\ B^\top & A_1^* \otimes I_2 \end{pmatrix}$$

where $B = \begin{pmatrix} I_2 & \ldots \ldots & I_2 \\ \end{pmatrix}$
Proof of $\phi_H(x) = \phi_1^*(x)^n_2 \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right)$.

- The adjacency matrix of $H$ is an $n_1 \times n_1$ block matrix, with blocks of size $n_2 \times n_2$

$$A_H = D_1 \otimes A_2 + A_1 \otimes I_2 = \begin{pmatrix} A_2 & B \\ B^\top & A_1^* \otimes I_2 \end{pmatrix}$$

where $B = \begin{pmatrix} I_2 & \cdots & (\delta) & I_2 & 0 & 0 & \cdots & 0 \end{pmatrix}$

- The characteristic polynomial of $H$ is

$$\phi_H(x) = \det(xI - A_H) = \det \begin{pmatrix} xI_2 - A_2 & -B \\ -B^\top & (xI_1^* - A_1^*) \otimes I_2 \end{pmatrix}$$
Proof of $\phi_H(x) = \phi_1^*(x)^n_2 \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right)$.

- The adjacency matrix of $H$ is an $n_1 \times n_1$ block matrix, with blocks of size $n_2 \times n_2$
  \[ A_H = D_1 \otimes A_2 + A_1 \otimes I_2 = \begin{pmatrix} A_2 & B \\ B^\top & A_1^* \otimes I_2 \end{pmatrix} \]
  where $B = \begin{pmatrix} I_2 & \ddots & I_2 \\ I_2 & \ddots & I_2 \\ & & 0 & \ddots & 0 \end{pmatrix}$
- The characteristic polynomial of $H$ is
  \[ \phi_H(x) = \det(xI - A_H) = \det \begin{pmatrix} xI_2 - A_2 & -B \\ -B^\top & (xI_1^* - A_1^*) \otimes I_2 \end{pmatrix} \]
- Computing the determinant in $\mathbb{R}^{n_2 \times n_2}$:
  \[ \phi_H(x) = \det([xI_2 - A_2]\phi_1^*(x)I_2 + \phi_1(x)I_2 - xI_2\phi_1^*(x)) = \]
  \[ = \det(\phi_1(x)I_2 - \phi_1^*(x)A_2) = \det \left( \phi_1^*(x) \left[ \frac{\phi_1(x)}{\phi_1^*(x)}I_2 - A_2 \right] \right) = \]
  \[ = \phi_1^*(x)^n_2 \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right) \]
Corollary

$G_1$ walk-regular $\Rightarrow \phi_H(x) = \left(\frac{\phi'_1(x)}{n_1}\right)^{n_2} \phi_2 \left(\frac{n_1\phi_1(x)}{\phi'_1(x)}\right)$

Proof.

$\phi^*_1(x) = \frac{1}{n_1}\phi'_1(x)$

Corollary

$G$ graph of order $n_2 = N$ and characteristic polynomial $\phi_G \Rightarrow$ the characteristic polynomial of $H = G \sqcap K_n$ is

$\phi_H(x) = (x + 1)^{N(n-2)}(x - n + 2)^N \phi_G \left(\frac{(x + 1)(x - n + 1)}{(x - n + 2)}\right)$

Proof.

$K_n$ is walk-regular, $\phi_{K_n} = (x - n + 1)(x + 1)^{n-1}$ and

$\phi'_{K_n} = (x + 1)^{n-1} + (n - 1)(x - n + 1)(x + 1)^{n-2}$
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Definition of the generalized hierarchical product

\[ G_i = (V_i, E_i), \emptyset \neq U_i \subseteq V_i, \ i = 1, 2, \ldots, N - 1 \]

\[ H = G_N \boxtimes G_{N-1}(U_{N-1}) \boxtimes \cdots \boxtimes G_1(U_1) \] is the graph:

- vertices \( V_N \times \cdots V_2 \times V_1 \)
- if \( x_j \sim y_j \) in \( G_j \) and \( u_i \in U_i, \ i = 1, 2, \ldots, j - 1 \) then
  \[ x_N \cdots x_{j+1}x_ju_{j-1} \cdots u_1 \sim x_N \cdots x_{j+1}y_ju_{j-1} \cdots u_1 \]

Example

- For every \( i, U_i = V_i \Rightarrow \)
  \[ G_N \boxtimes G_{N-1}(U_{N-1}) \boxtimes \cdots \boxtimes G_1(U_1) = G_N \boxtimes G_{N-1} \boxtimes \cdots \boxtimes G_1 \]

- For every \( i, U_i = \{0\} \Rightarrow \)
  \[ G_N \boxtimes G_{N-1}(U_{N-1}) \boxtimes \cdots \boxtimes G_1(U_1) = G_N \boxtimes G_{N-1} \boxtimes \cdots \boxtimes G_1 \]
Example

Two views of a generalized hierarchical product $K_3^3$ with $U_1 = U_2 = \{0, 1\}$. 
Summary

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3. The particular case of $T_m$
4. Properties of $T_r^m$, $\text{sp} T_r^m$ and $\bigcup_m \text{sp} T_r^m$
5. Definition and properties of the generalized hierarchical product
Publications


• On the spectra of hypertrees, BCDF, Linear Algebra and its Applications, 428(7):1499–1510

• On the hierarchical product of graphs and the generalized binomial tree, BCDF, Linear and Multilinear Algebra, submitted (September 2007).

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Gracias !!!