4-labelings and grid embeddings of plane quadrangulations

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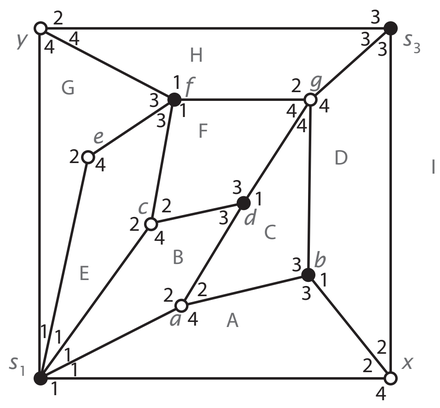
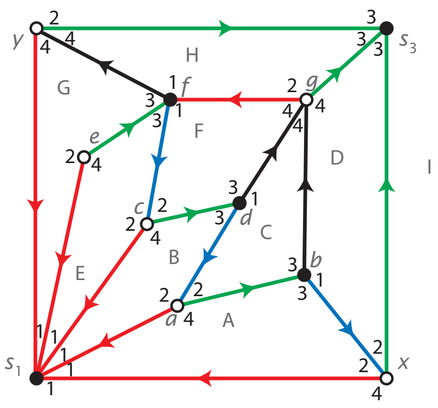
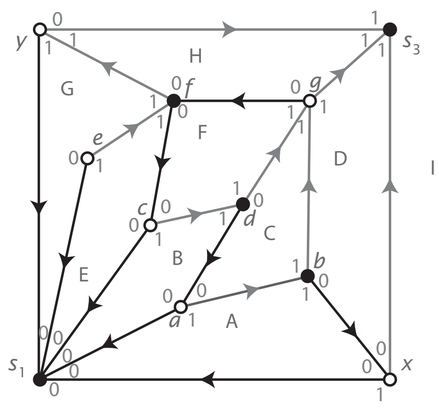
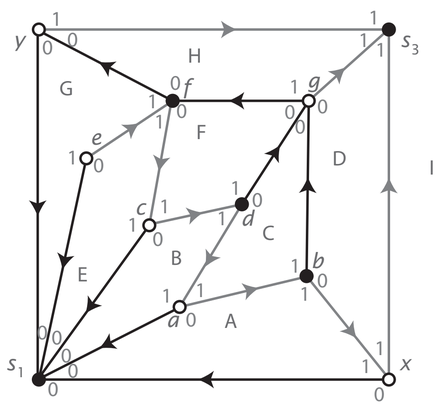
4-labelings and their properties

Definition (4-labeling of a quadrangulation Q)

Labeling of the angles of Q on $\{1,2,3,4\}$ satisfying:

- *Special vertices*: Black vertices of the external face, s_1 and s_3. All the angles incident to s_i are labeled i.
- Non-special black vertices: incident labels form a non-empty interval of 1s and a non-empty interval of 3s. White vertices satify the same property, with 2s and 4s.
- *Edges*: The incident labels coincide at one endpoint and differ at the other.
- *Internal faces*: The four angles are labeled consecutively counter-clockwise from 1 to 4.

Definition (4-coloring and 2-orientation associated to a 4-labeling)

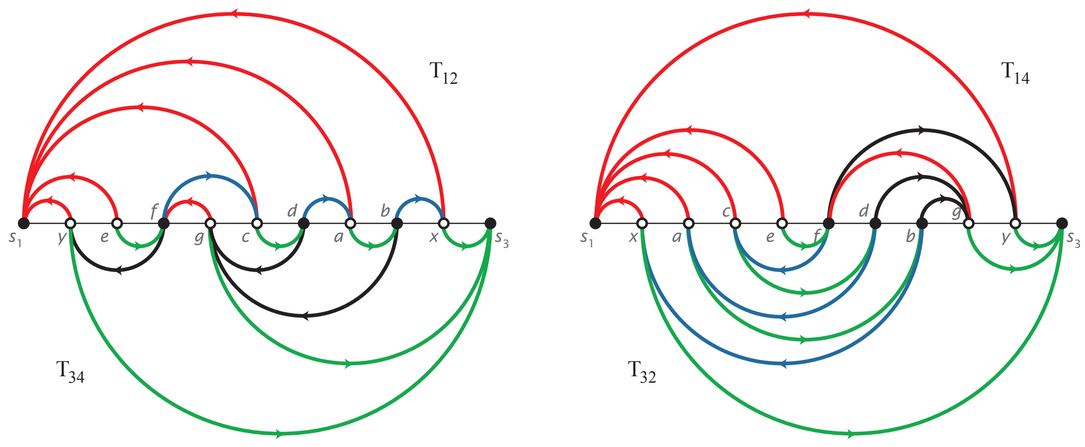
An edge uv is assigned color i and oriented from u to v if and only if the two angles incident to v are labeled i.

Properties

- Strong binary labelings and 4-labelings are in bijection.
- Around each vertex the colors of the edges appear in clockwise increasing order from 1 to 4, according to the following properties:
  - The vertex s_1 has only in-arcs, all of color 1; the vertex s_3 has only in-arcs, all of color 3.
  - At every non-special black vertex there are exactly one out-arc of color 2 and one of color 4, and two intervals of in-arcs of colors 1 and 3.
  - At every white vertex there are exactly one out-arc of colors 1 and one of color 3, and two intervals of in-arcs of colors 2 and 4.
- There is no vertex with an in-arc and an out-arc of the same color. This implies that the set of edges of color i is the union of disjoint stars, each with arcs pointing to its center.
- For every pair i, j, with $1 \le i, j \le 4$, i odd and j even, let us denote by T_{ij} the union of the sets of edges of colors i and j. Then, T_{ij} induces a directed tree with sink s_i, that spans all vertices but s_{4-i}.

Book embeddings of a quadrangulation

The two tree decompositions of Q: T_{12}, T_{34} and T_{14}, T_{32} allow us to define two different book embeddings of Q.



Properties

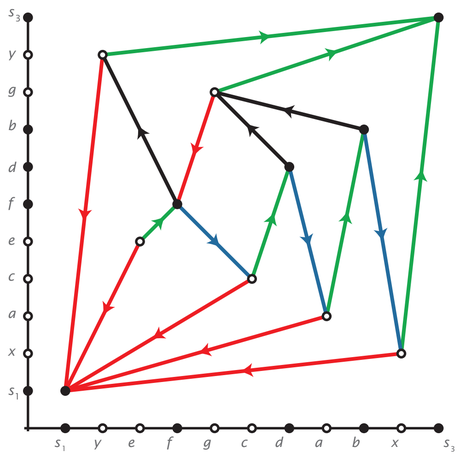
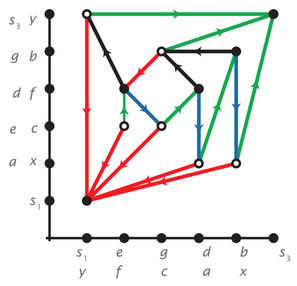
- The arcs of one page correspond to the edges of one of the trees in the tree decomposition of Q defined by the 4-labeling. More precisely, in the first book embedding, the tree T_{12} is in one page, while T_{34} is in the other. In the second book embedding, the tree T_{14} is in one page, while T_{32} is in the other.
- In both book embeddings, two edges of different color in the same page, have opposite orientations.

Embedding on the grid

Theorem

Place vertex v in the point of coordinates (i, j), with $0 \le i, j \le n-1$, if v is in position i in the first book embedding and j in the second one. Then:

1. No two vertices are in the same vertical line or horizontal line.
2. Arcs of color 1 point in direction south-west, arcs of color 2 in direction south-east, arcs of color 3 in direction north-east, and arcs of color 4 in direction north-west.
3. No two edges cross and any two disjoint edges are separated by a vertical line or a horizontal line.

Corollary

This grid embedding is a (closed) rectangle of influence drawing.

Reducing the grid size

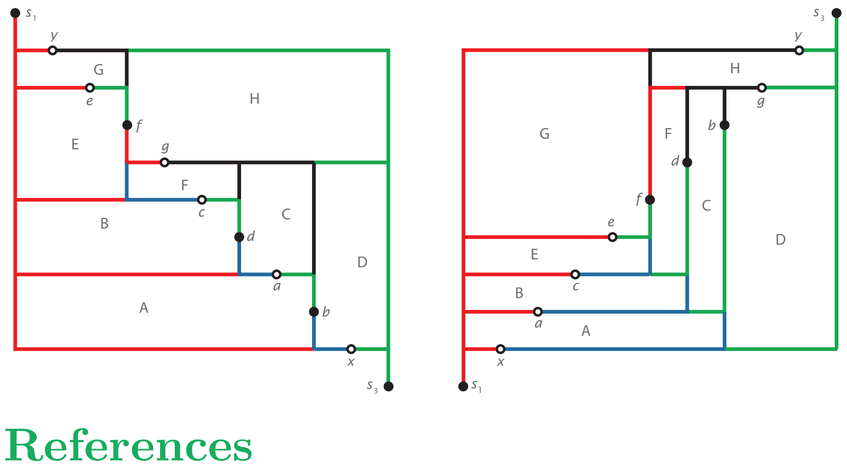
Corollary

The grid embedding can be reduced to a grid of size at most

$$\left\lceil \frac{n}{2} \right\rceil \times \left\lceil \frac{3n}{4} \right\rceil$$

such that each row and each column contains at most two vertices and no two edges cross.

Rectangulations



Theorem

Every quadrangulation (with outer face s_1, y, s_3, x) has a representation by a pair of rectangulations, such that each edge intersects the line $y = x$ in one rectangulation and intersects the line $y = n - x$ in the other one, and such that the two rectangulations are isomorphic drawings with fixed outer face s_1, y, s_3, x in clockwise order and s_1 being the left side of the bounding rectangle.

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