The Hierarchical Product of Graphs

Lali Barrière
Francesc Comellas
Cristina Dalfó
Miquel Àngel Fiol

Universitat Politècnica de Catalunya - DMA4

March 22, 2007
The Hierarchical Product of Graphs

Outline

1. Introduction

2. The hierarchical product
   - Definition and basic properties
   - Vertex hierarchy
   - Metric parameters

3. Algebraic properties
   - Spectral properties of $G \sqcap K_2^m$
   - The spectrum of the binary hypertree $T_m = K_2^m$
   - The spectrum of a generic two-term product $G_2 \sqcap G_1$

4. Generalization of the hierarchical product

5. Conclusions
Introduction

The hierarchical product

Definition and basic properties
Vertex hierarchy
Metric parameters

Algebraic properties
Spectral properties of $G \sqcap K_2^m$
The spectrum of the binary hypertree $T_m = K_2^m$
The spectrum of a generic two-term product $G_2 \sqcap G_1$

Generalization of the hierarchical product

Conclusions
Motivation

Complex networks: randomness, heterogeneity, modularity


Hierarchical networks: degree distribution, modularity

Our work

- Deterministic graphs
- Algebraic methods
Our work

- Deterministic graphs
- Algebraic methods
- Far from "real networks"

but a beautiful mathematical object !!!
Previous work

1 Introduction

2 The hierarchical product
   Definition and basic properties
   Vertex hierarchy
   Metric parameters

3 Algebraic properties
   Spectral properties of $G \sqcap K_2^m$
   The spectrum of the binary hypertree $T_m = K_2^m$
   The spectrum of a generic two-term product $G_2 \sqcap G_1$

4 Generalization of the hierarchical product

5 Conclusions
The Hierarchical Product of Graphs

The hierarchical product

Definition and basic properties

Definition

For $i = 1, \ldots, N$, $G_i$ graph rooted at 0
$H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$

- vertices $x_N \ldots x_3 x_2 x_1$, $x_i \in V_i$
- if $x_j \sim y_j$ in $G_j$ then
  $x_N \ldots x_{j+1} x_j 0 \ldots 0 \sim x_N \ldots x_{j+1} y_j 0 \ldots 0$

Example

The hierarchical products $K_2 \sqcap K_3$ and $K_3 \sqcap K_2$
$G_N \sqcap \cdots \sqcap G_2 \sqcap G_1$ is a spanning subgraph of $G_N \Box \cdots \Box G_2 \Box G_1$

**Example**

*The hierarchical product* $P_4 \sqcap P_3 \sqcap P_2$
The Hierarchical Product of Graphs
Definition and basic properties

\[ G^N = G \sqcap \cdots \sqcap G \] is the hierarchical \( N \)-power of \( G \)

Example

The hierarchical powers \( K_2^2, K_2^4 \) and \( K_2^5 \)
Example

The hierarchical power $C_4^3$
Order and size

\[ n_i = |V_i| \] and \( m_i = |E_i| \)

\[ H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1 \]

- \( n_H = n_N \cdots n_3 n_2 n_1 \)
- \( m_H = \sum_{k=1}^{N} m_k n_{k+1} \cdots n_N \)

Properties of \( \sqcap \)

- **Associativity.** \( G_3 \sqcap G_2 \sqcap G_1 = G_3 \sqcap (G_2 \sqcap G_1) = (G_3 \sqcap G_2) \sqcap G_1 \)
- **Right-distributivity.** \( (G_3 \cup G_2) \sqcap G_1 = (G_3 \sqcap G_1) \cup (G_2 \sqcap G_1) \)
- **Left-semi-distributivity.** \( G_3 \sqcap (G_2 \cup G_1) = (G_3 \sqcap G_2) \cup n_3 G_1 \), where \( n_3 G_1 = \overline{K}_{n_3} \sqcap G_1 \) is \( n_3 \) copies of \( G_1 \)
- \( G \sqcap K_1 = K_1 \sqcap G = G \)

\((\mathcal{G}, \sqcap)\) is a monoid
The Hierarchical Product of Graphs

The hierarchical product

Vertex hierarchy

**Degrees**

- If \( \delta_i = \delta_{G_i}(0) \), then
  - \( \delta_G(0) = \sum_{i=1}^{N} \delta_i \)
  - \( x = x Nx_{N-1} \ldots x_k 00 \ldots 0, \ x_k \neq 0 \Rightarrow \)
    \[
    \delta_H(x) = \sum_{i=1}^{k-1} \delta_i + \delta_{G_k}(x_k)
    \]

- If \( G \) is \( \delta \)-regular, the degrees of the vertices of \( G^N \) follow an exponential distribution, \( P(k) = \gamma^{-k} \), for some constant \( \gamma \)
  
  For \( k = 1, \ldots, N - 1 \), \( G^N \) contains \((n - 1)n^{N-k}\) vertices with degree \( k\delta \) and \( n \) vertices with degree \( N\delta \)
Example

\[ T_m = K^m_2 \text{ has } 2^{m-k} \text{ vertices of degree } k = 1, \ldots, m - 1, \text{ and two vertices of degree } m \]
Modularity

\[ H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1, \text{ } \mathbf{z} \text{ an appropriate string} \]
\[ H\langle \mathbf{z}x_k \ldots x_1 \rangle = H[\{\mathbf{z}x_k \ldots x_1| x_i \in V_i, 1 \leq i \leq k\}] \]
\[ H\langle x_N \ldots x_k \mathbf{z} \rangle = H[\{x_N \ldots x_k \mathbf{z}| x_i \in V_i, k \leq i \leq N\}] \]

Lemma

- \( H\langle \mathbf{z}x_k \ldots x_1 \rangle = G_k \sqcap \cdots \sqcap G_1, \text{ for any fixed } \mathbf{z} \)
- \( H\langle x_N \ldots x_k 0 \rangle = G_N \sqcap \cdots \sqcap G_k \)
- \( H\langle x_N \ldots x_k \mathbf{z} \rangle = (n_N \cdots n_k)K_1, \text{ for any fixed } \mathbf{z} \neq 0 \)

\( H^* = H - 0 \)

Lemma

- \( (G_N \sqcap \cdots \sqcap G_2 \sqcap G_1)^* = \bigcup_{k=1}^{N}(G_k^* \sqcap G_{k-1} \sqcap \cdots \sqcap G_1) \)
- \( (K_2^N)^* = \bigcup_{k=0}^{N-1} K_2^k \)
- \( K_2^N - \{\{0, 1\}\} = K_2^{N-1} \cup K_2^{N-1} \)
Example

Modularity and symmetry of $T_m = K_2^m$
The Hierarchical Product of Graphs

The hierarchical product

Vertex hierarchy

Example

Modularity and symmetry of $T_m = K_2^m$
Example

Modularity and symmetry of $T_m = K_2^m$
Example

Modularity and symmetry of $T_m = K_{2}^{m}$
Example

*Modularity and symmetry of $T_m = K_2^m$*
Example

Modularity and symmetry of $T_m = K_{2}^m$
Eccentricity, radius and diameter

\[ H = G_N \sqcap \cdots \sqcap G_2 \sqcap G_1 \]
\[ \varepsilon_i = \text{ecc}_{G_i}(0), \quad r_{G_N} = r_N \text{ and } D_{G_N} = D_N \]
\[ \rho_i \text{ shortest path routing of } G_i, \quad i = 1, \ldots, N \]

Proposition

- \( \{\rho_i\}_{i=1}^{N} \) induce a shortest path routing \( \rho \) in \( H \)
- The eccentricity, radius and diameter of \( H \) are

\[
\text{ecc}_H(0) = \sum_{i=1}^{N} \varepsilon_i, \quad r_H = r_N + \sum_{i=1}^{N-1} \varepsilon_i, \quad D_H = D_N + 2 \sum_{i=1}^{N-1} \varepsilon_i
\]
The Hierarchical Product of Graphs

The hierarchical product

Metric parameters

Proof.
Proof.
Mean distance

$G$ graph of order $n$

Mean distance. \( d_G = \frac{1}{n(n-1)} \sum_{v \neq w \in V} \text{dist}_G(v, w) \)

Local mean distance. \( d^0_G = \frac{1}{n} \sum_{v \in V} \text{dist}_G(0, v) \)

Proposition

\( H = G_2 \cap G_1 \Rightarrow \left\{ \begin{array}{l} d^{00}_H = d^0_1 + d^0_2 \\ d_H = \frac{1}{n-1} \left[ (n_1 - 1)d_1 + n_1(n_2 - 1)(d_2 + 2d^0_1) \right] \end{array} \)
Mean distance

$G$ graph of order $n$

Mean distance. $d_G = \frac{1}{n(n-1)} \sum_{v \neq w \in V} dist_G(v, w)$

Local mean distance. $d_G^0 = \frac{1}{n} \sum_{v \in V} dist_G(0, v)$

Proposition

$H = G_2 \sqcap G_1 \Rightarrow \left\{ \begin{array}{l} d_H^{00} = d_1^0 + d_2^0 \\ d_H = \frac{1}{n-1} [(n_1 - 1)d_1 + n_1(n_2 - 1)(d_2 + 2d_1^0)] \end{array} \right.$

Proof.
Just compute!
Corollary

$H = G^N, \ d = d_G, \ d^0 = d^0_G$

- $\text{ecc}_N(0) = N\epsilon, \ d^0_N = N d^0$
- $r_N = r + (N - 1)\epsilon, \ D_N = D + 2(N - 1)\epsilon$
- $d_N = d + 2 \left( \frac{(N-1)n^N+1}{n^{N-1}} - \frac{1}{n-1} \right) d^0$

Asymptotically, $d_N \sim d + 2d^0 \left( N - \frac{n}{n-1} \right) \sim d + 2Nd^0$

Example

$G = K_2 \Rightarrow \text{ecc} = r = D = 1, \ d^0 = 1/2 \ and \ d = 1$

The metric parameters of $T_m = K_2^m$ are

- $\text{ecc}_m(0) = m, \ d^0_m = m/2$
- $r_m = m, \ D_m = 2m - 1$
- $d_m = \frac{m2^m}{2^m-1} - 1 \sim m - 1$
The Hierarchical Product of Graphs
Algebraic properties

1 Introduction

2 The hierarchical product
   Definition and basic properties
   Vertex hierarchy
   Metric parameters

3 Algebraic properties
   Spectral properties of $G \sqcap K_2^m$
   The spectrum of the binary hypertree $T_m = K_2^m$
   The spectrum of a generic two-term product $G_2 \sqcap G_1$

4 Generalization of the hierarchical product

5 Conclusions
The Hierarchical Product of Graphs

Algebraic properties

Background

Kronecker product $\mathbf{A} \otimes \mathbf{B} = (a_{ij} \mathbf{B})$

If $\mathbf{A}$ and $\mathbf{B}$ are square, $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{B} \otimes \mathbf{A}$ are permutation similar
Background

Kronecker product $A \otimes B = (a_{ij}B)$

If $A$ and $B$ are square, $A \otimes B$ and $B \otimes A$ are permutation similar

Lemma

$H = G_2 \cap G_1 \Rightarrow$

$$A_H = A_2 \otimes D_1 + I_2 \otimes A_1 \cong D_1 \otimes A_2 + A_1 \otimes I_2$$

where $D_1 = \text{diag}(1, 0, \ldots, 0)$
Background

Kronecker product $A \otimes B = (a_{ij}B)$

If $A$ and $B$ are square, $A \otimes B$ and $B \otimes A$ are permutation similar

Lemma
$H = G_2 \sqcap G_1 \Rightarrow$

$A_H = A_2 \otimes D_1 + I_2 \otimes A_1 \cong D_1 \otimes A_2 + A_1 \otimes I_2$

where $D_1 = \text{diag}(1, 0, \ldots 0)$

Example
$H = G \sqcap K_n$, $G$ of order $N \Rightarrow$

$A_H = D_1 \otimes A_G + A_{K_n} \otimes I_N =$

$$
\begin{pmatrix}
A_G & I_N & \cdots & I_N \\
I_N & 0 & \cdots & I_N \\
\vdots & \vdots & \ddots & \vdots \\
I_N & I_N & \cdots & 0
\end{pmatrix}
$$
Theorem (Silvester, 2000)

$R$ commutative subring of $F^{n \times n}$, the set of all $n \times n$ matrices over a field $F$ (or a commutative ring), and $M \in R^{m \times m}$. Then,

$$\det_F M = \det_F(\det_R M)$$

Corollary (Silvester, 2000)

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ where } A, B, C, D \text{ commute with each other.}$$

Then,

$$\det M = \det(AD - BC)$$
The Hierarchical Product of Graphs

Algebraic properties
Spectral properties of $G \sqcap K_2^m$

1 Introduction

2 The hierarchical product
Definition and basic properties
Vertex hierarchy
Metric parameters

3 Algebraic properties
Spectral properties of $G \sqcap K_2^m$
The spectrum of the binary hypertree $T_m = K_2^m$
The spectrum of a generic two-term product $G_2 \sqcap G_1$

4 Generalization of the hierarchical product

5 Conclusions
The Hierarchical Product of Graphs
Algebraic properties
Spectral properties of $G \sqcap K_2^m$

$G \sqcap K_2$

Example

The Petersen graph, hierarchically multiplied by $K_2$
$G \sqcap K_2$

$G$ graph of order $n$,
$A$ adjacency matrix of $G$ and
$\phi_G$ characteristic polynomial of $G$

- The adjacency matrix of $H = G \sqcap K_2$ is

$$A_H = \begin{pmatrix} A & I_n \\ I_n & 0 \end{pmatrix}$$

- The characteristic polynomial of $H$ is

$$\phi_H(x) = \det(xI_{2n} - A_H) = \det\begin{pmatrix} xI_n - A & -I_n \\ -I_n & xI_n \end{pmatrix} = \det((x^2 - 1)I_n - xA) = x^n \phi_G(x - \frac{1}{x})$$
\[ \phi_H(x) = x^n \phi_G(x - \frac{1}{x}) \]

**Proposition**

\( H = G \sqcap K_2 \) and \( \text{sp} G = \{ \lambda_0^{m_0} < \lambda_1^{m_1} < \ldots < \lambda_d^{m_d} \} \Rightarrow \)

\[ \text{sp} H = \{ \lambda_0^{m_0} < \lambda_0^{m_1} < \ldots < \lambda_0^{m_d} < \lambda_1^{m_0} < \lambda_1^{m_1} < \ldots < \lambda_1^{m_d} \} \]

where \( \lambda_{0i} = f_0(\lambda_i) = \frac{\lambda_i - \sqrt{\lambda_i^2 + 4}}{2}, \lambda_{1i} = f_1(\lambda_i) = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4}}{2} \)
\[ \phi_H(x) = x^n \phi_G(x - \frac{1}{x}) \]

**Proposition**

\( H = G \sqcap K_2 \) and \( \text{sp} \ G = \{ \lambda_0^{m_0} < \lambda_1^{m_1} < \ldots < \lambda_d^{m_d} \} \) \( \Rightarrow \)

\[ \text{sp} \ H = \{ \lambda_{00}^{m_0} < \lambda_{01}^{m_1} < \ldots < \lambda_{0d}^{m_d} < \lambda_{10}^{m_0} < \lambda_{11}^{m_1} < \ldots < \lambda_{1d}^{m_d} \} \]

where \( \lambda_{0i} = f_0(\lambda_i) = \frac{\lambda_i - \sqrt{\lambda_i^2 + 4}}{2} \), \( \lambda_{1i} = f_1(\lambda_i) = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4}}{2} \)

**Proof.**

\[ \lambda \in \text{sp} \ H \Leftrightarrow \phi_H(\lambda) = \lambda^n \phi_G(\lambda - \frac{1}{\lambda}) = 0 \Leftrightarrow \lambda - \frac{1}{\lambda} \in \text{sp} \ G \]

\[ \lambda_i \in \text{sp} \ G \Rightarrow \lambda^2 - \lambda_i \lambda - 1 = 0 \]

\[ \square \]
The Hierarchical Product of Graphs

Algebraic properties

Spectral properties of $G \sqcap K_2^m$

$$\phi_H(x) = x^\phi G(x - \frac{1}{x})$$

$\lambda_d$

$\lambda_0$

$\lambda_1$

$\lambda_2$

$\lambda_{00}$ $\lambda_{01}$ $\lambda_{02}$ $\cdots$ $\lambda_{0d}$ $\cdots$ $\lambda_{1d}$

$f(x) = x - \frac{1}{x}$
The Hierarchical Product of Graphs

Algebraic properties

Spectral properties of $G \sqcap K_2^m$

\[ H_m = G \sqcap K_2^m \]

\[ H_m = H_{m-1} \sqcap K_2, \ m \geq 1. \] The adjacency matrix of $H_m$ is

\[ A_m = \begin{pmatrix} A_{m-1} & I_{m-1} \\ I_{m-1} & 0 \end{pmatrix} \]

where $I_m$ denotes the identity matrix of size $n2^m$ (the same as $A_m$)

$H_0 = G, \ A_0 = A$ the adjacency matrix of $G$

Example
The Hierarchical Product of Graphs

Algebraic properties

Spectral properties of $G \sqcap K_{2}^{m}$

Let $\{p_{i}, q_{i}\}_{i \geq 0}$ be the family of polynomials satisfying the recurrence equations

\[ p_{i} = p_{i-1}^{2} - q_{i-1}^{2} \]
\[ q_{i} = p_{i-1}q_{i-1} \]

with initial conditions

\[ p_{0} = x \text{ and } q_{0} = 1 \]

**Proposition**

For every $m \geq 0$, the characteristic polynomial of $H_{m} = G \sqcap K_{2}^{m}$ is

\[ \phi_{m}(x) = q_{m}(x)^{n} \phi_{0}\left(\frac{p_{m}(x)}{q_{m}(x)}\right) \]

**Lemma**

If $p$ and $q$ are arbitrary polynomials, then

\[
\det \left( \begin{array}{cc}
pI_{n} - qA & -qI_{n} \\
-qI_{n} & pI_{n}
\end{array} \right) = \det((p^{2} - q^{2})I_{n} - pqA)
\]
The Hierarchical Product of Graphs
Algebraic properties
Spectral properties of $G \sqcap K_2^m$

Proof of $\phi_m(x) = q_m(x)^n \phi_0 \left( \frac{p_m(x)}{q_m(x)} \right)$.

By induction on $m$, using the Lemma
Proof of $\phi_m(x) = q_m(x)^n \phi_0 \left( \frac{p_m(x)}{q_m(x)} \right)$.

By induction on $m$, using the Lemma

- **Case $m = 0$.** Trivially from $q_0(x) = 1$ and $p_0(x) = x$. 
Proof of $\phi_m(x) = q_m(x)^n \phi_0 \left( \frac{p_m(x)}{q_m(x)} \right)$.

By induction on $m$, using the Lemma

- Case $m = 0$. Trivially from $q_0(x) = 1$ and $p_0(x) = x$.
- $m \geq 1$. By induction on $i$, we prove that

  $\phi_m = \det(p_i I_{m-i} - q_i A_{m-i})$

  - $i = 0$: $\phi_m = \det(x I_m - A_m) = \det(p_0 I_m - q_0 A_m)$
  - $i - 1 \Rightarrow i$: $\phi_m = \det(p_{i-1} I_{m-i+1} - q_{i-1} A_{m-i+1}) = \det((p_{i-1}^2 - q_{i-1}^2) I_{m-i} - p_{i-1} q_{i-1} A_{m-i}) = \det(p_i I_{m-i} - q_i A_{m-i})$
Proof of $\phi_m(x) = q_m(x)^n \phi_0 \left( \frac{p_m(x)}{q_m(x)} \right)$.

By induction on $m$, using the Lemma

- **Case $m = 0$.** Trivially from $q_0(x) = 1$ and $p_0(x) = x$.
- **$m \geq 1$.** By induction on $i$, we prove that
  
  $$\phi_m = \det(p_i I_{m-i} - q_i A_{m-i})$$

  - $i = 0$ :  $\phi_m = \det(x I_m - A_m) = \det(p_0 I_m - q_0 A_m)$
  - $i-1 \Rightarrow i$ :  $\phi_m = \det(p_{i-1} I_{m-i+1} - q_{i-1} A_{m-i+1}) =
    \det((p_{i-1}^2 - q_{i-1}^2)I_{m-i} - p_{i-1}q_{i-1}A_{m-i}) =
    \det(p_i I_{m-i} - q_i A_{m-i})$

  - The case $i = m$ gives
    
    $$\phi_m(x) = \det(p_m(x) I_0 - q_m(x) A_0) =
    = \det \left( q_m(x) \left( \frac{p_m(x)}{q_m(x)} I_0 - A_0 \right) \right) = q_m(x)^n \phi_0 \left( \frac{p_m(x)}{q_m(x)} \right)$$
The Hierarchical Product of Graphs

Algebraic properties

1. Introduction

2. The hierarchical product
   Definition and basic properties
   Vertex hierarchy
   Metric parameters

3. Algebraic properties
   Spectral properties of \( G \sqcap K_2^m \)
   The spectrum of the binary hypertree \( T_m = K_2^m \)
   The spectrum of a generic two-term product \( G_2 \sqcap G_1 \)

4. Generalization of the hierarchical product

5. Conclusions
The Hierarchical Product of Graphs

Algebraic properties

The spectrum of the binary hypertree $T_m = K_2^m$

\[ T_m = K_2^m \]

\[ p_i = p_{i-1}^2 - q_{i-1}^2 \]
\[ q_i = p_{i-1}q_{i-1} \]

\[ p_0 = x, \quad q_0 = 1 \]

Corollary

- $\phi_{T_m}(x) = p_m(x)$
- $\phi_{T_m^*}(x) = q_m(x)$
The Hierarchical Product of Graphs

Algebraic properties

The spectrum of the binary hypertree \( T_m = K_2^m \)

\[
T_m = K_2^m
\]

\[p_i = p_{i-1}^2 - q_{i-1}^2\]
\[q_i = p_{i-1}q_{i-1}\]
\[p_0 = x, \quad q_0 = 1\]

Corollary

- \( \phi_{T_m}(x) = p_m(x) \)
- \( \phi_{T_m^*}(x) = q_m(x) \)

Proof.

\( G = K_1 \Rightarrow \phi_0(x) = x \Rightarrow \phi_{T_m}(x) = q_m(x)^n \phi_0 \left( \frac{p_m(x)}{q_m(x)} \right) = p_m(x) \)

\( T_m^* = T_m - 0 = \bigcup_{i=0}^{m-1} T_i \Rightarrow \phi_{T_m^*}(x) = \prod_{i=0}^{m-1} p_i(x) = q_m(x) \)
Proposition

$T_m$, $m \geq 1$, has distinct eigenvalues $\lambda_0^m < \lambda_1^m < \cdots < \lambda_{n-1}^m$, with $n = 2^m$, satisfying the following recurrence relation:

$$\lambda_{\frac{n}{2}+k}^m = \frac{\lambda_{k-1}^{m-1} + \sqrt{(\lambda_{k-1}^{m-1})^2 + 4}}{2}$$

$$\lambda_{n-k-1}^m = -\lambda_k^m$$

for $m > 1$ and $k = \frac{n}{2}, \frac{n}{2} + 1, \ldots, n - 1$
Proposition

$T_m$, $m \geq 1$, has distinct eigenvalues $\lambda_0^m < \lambda_1^m < \cdots < \lambda_{n-1}^m$, with $n = 2^m$, satisfying the following recurrence relation:

$$
\lambda_{\frac{n}{2}+k}^m = \frac{\lambda_{k-1}^{m-1} + \sqrt{(\lambda_{k-1}^{m-1})^2 + 4}}{2}
$$

$$
\lambda_{n-k-1}^m = -\lambda_k^m
$$

for $m > 1$ and $k = \frac{n}{2}, \frac{n}{2} + 1, \ldots, n - 1$

Proof.

- $\lambda_{0i} = f_0(\lambda_i) = \frac{\lambda_i - \sqrt{\lambda_i^2 + 4}}{2}, \lambda_{1i} = f_1(\lambda_i) = \frac{\lambda_i + \sqrt{\lambda_i^2 + 4}}{2}$
- $T_m$ bipartite $\Rightarrow$ its spectrum is symmetric with respect to 0
- $sp\ T_0 = \{0^1\} \Rightarrow$ the multiplicity of every $\lambda_1^m$ is 1
Algebraic properties

The spectrum of the binary hypertree $T_m = K_2^m$

Properties of $\text{sp } T_m$

\[ \lambda_i \in \text{sp } G \Rightarrow \lambda^2 - \lambda_i \lambda - 1 = 0 \]

\[ f_0(x) = \frac{x - \sqrt{x^2 + 4}}{2}, \quad f_1(x) = \frac{x + \sqrt{x^2 + 4}}{2} \]

\[ m = 0 \Rightarrow \text{sp } T_0 = \{0\} \]

\[ m = 1 \Rightarrow \lambda_0 = f_0(0) = -1, \quad \lambda_1 = f_1(0) = 1 \]

\[ m = 2 \Rightarrow \]

\[ \begin{align*}
\lambda_0 &= f_0(-1) = f_0(f_0(0)) = -1.618 \\
\lambda_1 &= f_0(1) = f_0(f_1(0)) = -0.618 \\
\lambda_2 &= f_1(-1) = f_1(f_0(0)) = 0.618 \\
\lambda_3 &= f_1(1) = f_1(f_1(0)) = 1.618
\end{align*} \]

\[ \ldots \]

\[ m \text{ fixed, } i = i_{m-1} \ldots i_1 i_0 \in \mathbb{Z}_2^m \Rightarrow \]

\[ \Rightarrow \lambda_i = (f_{i_{m-1}} \circ \cdots \circ f_{i_1} \circ f_{i_0})(0) \]
The Hierarchical Product of Graphs

Algebraic properties

The spectrum of the binary hypertree $T_m = K_2^m$

The distinct eigenvalues of the hypertree $T_m$ for $0 \leq m \leq 6$. 
Proposition

The asymptotic behaviors of

- the spectral radius \( \rho_k = \max_{0 \leq i \leq n-1} \{ |\lambda_i| \} = \lambda_{111...1} \),
- the second largest eigenvalue \( \theta_k = \lambda_{111...10} \), and
- the minimum positive eigenvalue \( \sigma_k = \min_{0 \leq i \leq n-1} \{ |\lambda_i| \} = \lambda_{100...0} \)

of the hypertree \( T_m \) are:

\[
\rho_k \sim \sqrt{2k}, \quad \theta_k \sim \sqrt{2k}, \quad \sigma_k \sim 1/\sqrt{2k}
\]
The Hierarchical Product of Graphs

Algebraic properties

The spectrum of the binary hypertree $T_m = K_2^m$

Proof of $\rho_k \sim \sqrt{2k}$, $\theta_k \sim \sqrt{2k}$, $\sigma_k \sim 1/\sqrt{2k}$. 
Proof of $\rho_k \sim \sqrt{2k}$, $\theta_k \sim \sqrt{2k}$, $\sigma_k \sim 1/\sqrt{2k}$.

- $\rho_k \sigma_k = 1$
The Hierarchical Product of Graphs

Algebraic properties

The spectrum of the binary hypertree $T_m = K_2^m$

Proof of $\rho_k \sim \sqrt{2k}$, $\theta_k \sim \sqrt{2k}$, $\sigma_k \sim 1/\sqrt{2k}$.

• $\rho_k \sigma_k = 1$

• $\rho_k$ and $\theta_k$ verify the recurrence
  
  $$\lambda_{k+1} = f_1(\lambda_k) = \frac{1}{2}(\lambda_k + \sqrt{\lambda_k^2 + 4})$$
Proof of $\rho_k \sim \sqrt{2k}$, $\theta_k \sim \sqrt{2k}$, $\sigma_k \sim 1/\sqrt{2k}$.

- $\rho_k \sigma_k = 1$
- $\rho_k$ and $\theta_k$ verify the recurrence
  $\lambda_{k+1} = f_1(\lambda_k) = \frac{1}{2}(\lambda_k + \sqrt{\lambda_k^2 + 4})$
- Assuming $\lambda_k \sim \alpha k^\beta$
  $$\alpha(k + 1)^\beta \sim \frac{\alpha k^\beta + \sqrt{\alpha^2 k^{2\beta} + 4}}{2} \Rightarrow \alpha^2(k + 1)^\beta[(k + 1)^\beta - k^\beta] \sim 1$$
  $$2(k + 1)^{\frac{1}{2}}[(k + 1)^{\frac{1}{2}} - k^{\frac{1}{2}}] = \frac{2(k + 1)^{\frac{1}{2}}}{(k + 1)^{\frac{1}{2}} + k^{\frac{1}{2}}} \to 1$$
1 Introduction

2 The hierarchical product
   Definition and basic properties
   Vertex hierarchy
   Metric parameters

3 Algebraic properties
   Spectral properties of $G \sqcap K_2^m$
   The spectrum of the binary hypertree $T_m = K_2^m$
   The spectrum of a generic two-term product $G_2 \sqcap G_1$

4 Generalization of the hierarchical product

5 Conclusions
The Hierarchical Product of Graphs
Algebraic properties
The spectrum of a generic two-term product $G_2 \sqcap G_1$

**Theorem**

Let $G_1$ and $G_2$ be two graphs on $n_i$ vertices, with adjacency matrix $A_i$ and characteristic polynomial $\phi_i(x)$, $i = 1, 2$. Consider the graph $G_1^* = G_1 - 0$, with adjacency matrix $A_1^*$ and characteristic polynomial $\phi_1^*$.

Then the characteristic polynomial $\phi_H(x)$ of the hierarchical product $H = G_2 \sqcap G_1$ is:

$$
\phi_H(x) = \phi_1^*(x)^{n_2} \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right)
$$
Proof of \( \phi_H(x) = \phi_1^*(x)^n \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right) \).
Proof of $\phi_H(x) = \phi_1^*(x)^n_2 \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right)$.

- The adjacency matrix of $H$ is an $n_1 \times n_1$ block matrix, with blocks of size $n_2 \times n_2$

$$A_H = D_1 \otimes A_2 + A_1 \otimes I_2 = \begin{pmatrix} A_2 & B \\ B^\top & A_1^* \otimes I_2 \end{pmatrix}$$

where $B = \begin{pmatrix} I_2 & \cdots & \cdots & I_2 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$
The Hierarchical Product of Graphs

Algebraic properties

The spectrum of a generic two-term product $G_2 \sqcap G_1$

Proof of $\phi_H(x) = \phi_1^*(x)^n_2 \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right)$.

- The adjacency matrix of $H$ is an $n_1 \times n_1$ block matrix, with blocks of size $n_2 \times n_2$

$$A_H = D_1 \otimes A_2 + A_1 \otimes I_2 = \begin{pmatrix} A_2 & B \\ B^\top & A_1^* \otimes I_2 \end{pmatrix}$$

where $B = \begin{pmatrix} I_2 & \cdots & \cdots & I_2 & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$

- The characteristic polynomial of $H$ is

$$\phi_H(x) = \det(xI - A_H) = \det \begin{pmatrix} xI_2 - A_2 & -B \\ -B^\top & (xI_1^* - A_1^*) \otimes I_2 \end{pmatrix}$$
Proof of $\phi_H(x) = \phi_1^*(x)^n_2 \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right)$.

- The adjacency matrix of $H$ is an $n_1 \times n_1$ block matrix, with blocks of size $n_2 \times n_2$

$$A_H = D_1 \otimes A_2 + A_1 \otimes I_2 = \left( \begin{array}{cc} A_2 & B \\ B^\top & A_1^* \otimes I_2 \end{array} \right)$$

where $B = \left( \begin{array}{cccccc} I_2 & \cdots & \cdots & I_2 & 0 & 0 & \cdots & \cdots & 0 \end{array} \right)$

- The characteristic polynomial of $H$ is

$$\phi_H(x) = \det(xI - A_H) = \det \left( \begin{array}{cc} xI_2 - A_2 & -B \\ -B^\top & (xI_1^* - A_1^*) \otimes I_2 \end{array} \right)$$

- Computing the determinant in $\mathbb{R}^{n_2 \times n_2}$:

$$\phi_H(x) = \det([xI_2 - A_2] \phi_1^*(x)I_2 + \phi_1(x)I_2 - xI_2 \phi_1^*(x)) =$$

$$= \det(\phi_1(x)I_2 - \phi_1^*(x)A_2) = \det \left( \phi_1^*(x) \left[ \frac{\phi_1(x)}{\phi_1^*(x)}I_2 - A_2 \right] \right) =$$

$$= \phi_1^*(x)^n_2 \phi_2 \left( \frac{\phi_1(x)}{\phi_1^*(x)} \right)$$
The Hierarchical Product of Graphs

Algebraic properties

The spectrum of a generic two-term product $G_2 \sqcap G_1$

**Corollary**

If $G_1$ is walk-regular, then

$$\phi_H(x) = \left( \frac{\phi_1'(x)}{n_1} \right)^{n_2} \phi_2 \left( \frac{n_1\phi_1(x)}{\phi_1'(x)} \right)$$

**Proof.**

$$\phi^*_1(x) = \frac{1}{n_1} \phi_1'(x)$$

**Corollary**

If $G$ is a graph of order $n_2 = N$ and characteristic polynomial $\phi_G \Rightarrow$ the characteristic polynomial of $H = G \sqcap K_n$ is

$$\phi_H(x) = (x + 1)^N(n-2)(x - n + 2)^N \phi_G \left( \frac{(x + 1)(x - n + 1)}{x - n + 2} \right)$$

**Proof.**

$K_n$ is walk-regular, $\phi_{K_n} = (x - n + 1)(x + 1)^{n-1}$ and

$$\phi'_{K_n} = (x + 1)^{n-1} + (n - 1)(x - n + 1)(x + 1)^{n-2}$$
The Hierarchical Product of Graphs

Generalization of the hierarchical product

1 Introduction

2 The hierarchical product
   Definition and basic properties
   Vertex hierarchy
   Metric parameters

3 Algebraic properties
   Spectral properties of $G \sqcap K_2^m$
   The spectrum of the binary hypertree $T_m = K_2^m$
   The spectrum of a generic two-term product $G_2 \sqcap G_1$

4 Generalization of the hierarchical product

5 Conclusions
Definition of the generalized hierarchical product

\[ G_i = (V_i, E_i), \quad \emptyset \neq U_i \subseteq V_i, \quad i = 1, 2, \ldots, N - 1 \]

\[ H = G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \cdots \sqcap G_1(U_1) \] is the graph:

- vertices \( V_N \times \cdots V_2 \times V_1 \)
- if \( x_j \sim y_j \) in \( G_j \) and \( u_i \in U_i, \quad i = 1, 2, \ldots, j - 1 \) then
  \[ x_N \cdots x_{j+1}x_j u_{j-1} \cdots u_1 \sim x_N \cdots x_{j+1}y_j u_{j-1} \cdots u_1 \]

Example

- For every \( i \), \( U_i = V_i \) ⇒
  \[ G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \cdots \sqcap G_1(U_1) = G_N \Box G_{N-1} \Box \cdots \Box G_1 \]
- For every \( i \), \( U_i = \{0\} \) ⇒
  \[ G_N \sqcap G_{N-1}(U_{N-1}) \sqcap \cdots \sqcap G_1(U_1) = G_N \sqcap G_{N-1} \sqcap \cdots \sqcap G_1 \]
Example

*Two views of a generalized hierarchical product $K_3^3$ with $U_1 = U_2 = \{0, 1\}$.***
Summary

1 Definition of the hierarchical product of graphs
2 Spectral properties
3 The particular case of $T_m$
4 Definition of the generalized hierarchical product

Further work

5 $T_m$, $\text{sp } T_m$ and $\bigcup_m \text{sp } T_m$ are structures with nice properties
6 Properties of the generalized hierarchical product
Thank you !!!