Groupoids and Faà di Bruno Formulae for Green Functions

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The classical Faà di Bruno formula for $\mathrm{Diff}(\mathbb{C}, 0)$

- The classical Faà di Bruno formula tells how to compute the higher derivatives of the composite of two functions in one variable.
- Consider a formal power series in one variable

$$f(z) = \sum_{n=1}^{\infty} a_n(f)\, z^n \in \mathbb{C}[[z]]$$

  with $f(0) = 0$ and coefficients given by higher derivatives

$$a_n(f) = \frac{f^{(n)}(0)}{n!}.$$

- Traditionally $f(z)$ is regarded as the germ of a C^∞ function satisfying $f(0) = 0$. It is a diffeomorphism if $a_1(f) \neq 0$, orientation-preserving if $a_1(f) > 0$ and tangent to the identity if $a_1(f) = 1$.

The classical Faà di Bruno formula (continued)

- Under composition of functions $\mathbb{C}[[z]]$ becomes a (noncommutative) monoid,

$$(g \circ f)(z) = \sum_{n=1}^{\infty} a_n(g) \left(\sum_{m=1}^{\infty} a_m(f)\, z^m \right)^n . \qquad (1)$$

  with unit the identity function $\mathbf{1}(z) = z$.

- The coordinate functions are elements of the linear dual of this monoid,

$$\langle a_n, f \rangle = a_n(f), \qquad a_n \in \mathbb{C}[[z]]^*.$$

- The polynomial ring in the a_n has a coalgebra structure, with counit $\varepsilon(a_n) = \langle a_n, \mathbf{1} \rangle$ and comultiplication defined by

$$\langle \Delta a_n, f \otimes g \rangle = \langle a_n, g \circ f \rangle$$

  which may be determined explicitly by expanding (1).

The Faà di Bruno bialgebra

- The *Faà di Bruno bialgebra*

$$\mathcal{F} = \mathbb{C}[[a_1, a_2, \ldots]]$$

  is the free commutative algebra on the symbols a_n, $n \geq 1$, with counit and comultiplication defined above.
- This is only a bialgebra. It is graded, but not connected: $\mathcal{F}_0$ is spanned by the powers of a_1, which are group-like.
- One may impose the relation $a_1 = 1$ (which is easily seen to generate a bi-ideal), to obtain the classical Hopf algebra

$$\mathcal{H} = \mathbb{C}[[a_2, a_3, \ldots]].$$

- The antipode is the classical Lagrange inversion formula.
- For our purposes it will be more important to consider the bialgebra $\mathcal{F}$, without the restriction $a_1 = 1$.

The Green function for Diff$(\mathbb{C}, 0)$

- The formula for Δ can be packaged into a single equation, We consider the formal series (the Green function)

$$A = \sum_{k \geq 1} \frac{A_k}{k!} = \sum_{k \geq 1} a_k \quad \in \mathbb{C}[[a_1, a_2, a_3, \ldots]]$$

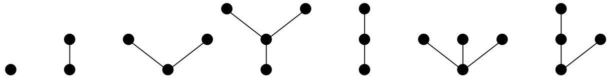
- The resulting form of the Faà di Bruno formula is the Leitmotiv of the present work:

$$\Delta(A) = \sum_{k \geq 1} A^k \otimes a_k.$$

- The values of Δ on the individual generators a_k can be extracted from this formula.
- With the the obvious convention $a_0 = 0$, we may allow the sum to start at $k = 0$

Bialgebras of trees

- **Connes-Kreimer bialgebra of rooted trees**:
  is the free $\mathbb{C}$-algebra $\mathcal{H}$ on the set of isomorphism classes of (combinatorial) trees.



- The comultiplication is given on generators by

$$\begin{aligned} \Delta : \mathcal{H} &\longrightarrow \mathcal{H} \otimes \mathcal{H} \\ T &\longmapsto \sum_c P_c \otimes S_c, \end{aligned}$$

- Here the sum is over all admissible cuts of T
- P_c is the forest (interpreted as a monomial) found above the cut.
- S_c is the subtree found below the cut (or the empty forest, in case the cut is below the root).

Bialgebras of trees (continued)

- $\mathcal{H}$ is a connected bialgebra: the grading is by the number of nodes, and $\mathcal{H}_0$ is spanned by the unit.
- Therefore, by general principles it acquires an antipode and becomes a Hopf algebra.

Motivation from van Suijlekom's work

- **Walter D. van Suijlekom**.
  'The structure of renormalization Hopf algebras for gauge theories. I. Representing Feynman graphs on BV-algebras'. *Comm. Math. Phys., 290(1):291-319, 2009*
- There, the Connes–Kreimer Hopf algebra of Feynman graphs is considered.
- Trees encode nestings of Feynman graphs
- Individual graphs are not physically meaningful.
- But what is physically meaningful is to consider the *Green function* associated to them, that is, for a fixed kind of vertex v

$$G_v = 1 + \sum_{\mathrm{res}(\Gamma)=v} \Gamma / \operatorname{Aut}(\Gamma)$$

Motivation from van Suijlekom's work (continued)

- In van Suijlekom's work, a Faà di Bruno formula

$$\Delta(Y_v) = \sum_{n_1,\ldots,n_k} Y_v Y_{v_1}^{n_1} \ldots Y_{v_k}^{n_k} \otimes p_{n_1,\ldots,n_k}(Y_v)$$

  appears, with $p_{n_1,\ldots,n_k}$ is the projection onto graphs containing n_i vertices of type v_i, with

$$Y_v = \frac{G_v}{\prod_{e \in v} \sqrt{G_e}}$$

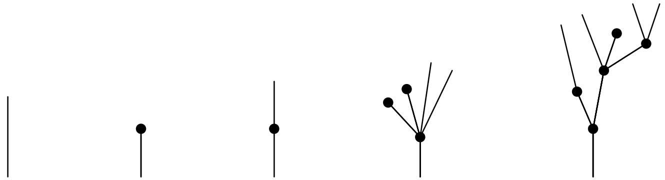
  where the product runs over the edges e of the vertex v.
- Note that for each type of edge, one has

$$G_e = 1 - \sum_{\mathrm{res}(\Gamma)=e} \Gamma / \mathrm{Aut}(\Gamma)$$

  for a fixed type of edge.
- Our aim was to prove a formula along the line of this one for *trees*. To do so, we need *operadic trees*.

Operadic trees

- In operad theory, the nodes represent operations, and trees are formal combinations of operations.
- These allow loose ends (leaves).
- Formal definition of operadic trees to be found in [Kock 2011,IMRN].
- Here are some examples (disregard the planar aspect)



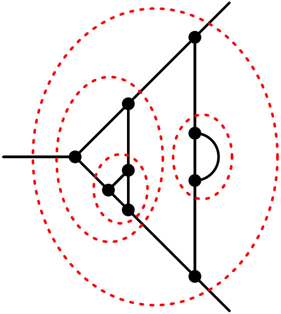
- The *leaves* are the edges that do not start in a node.
- The *root edge* does not end in a node.
- A node without incoming edge is a nullary operation.
- The (small) incoming edges drawn at every node serve to keep track of the arities of the operations.

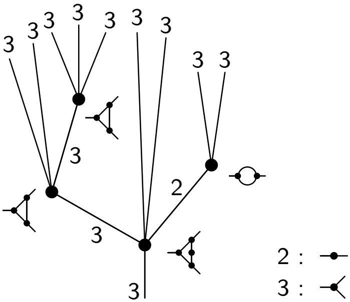
Operadic trees in pQFT

- Trees appearing in pQFT are naturally operadic.
- They encode nestings of 1PI Feynman graphs.
- Hence, have decorations by primitive 1PI graphs on nodes and by interaction labels on edges.
- So the graph can be recovered from the operadic (decorated) tree.
- Symmetries of the original Feynman graph are better dealt with by means of operadic trees.
- This is relevant for Green functions.

Correspondence between Feynman graphs and trees

- A Feynman graph can be reconstructed from the decorated tree.
- The decoration involves bijections encoding the exact way a small graph is substitued into the big graph.





- All this is taken care of by the theory of *polynomial functors* as described in last week's seminar (see also [Kock2011]).

The bialgebra of operadic trees

We will be considering

- the category of operadic trees and their morphisms.
- the category of forests and morphisms between them.

- A *cut* of an operadic tree is a subtree containing the root

$$c : S \subset T$$

- If a node is in a subtree, so are all its incident edges.
- For each edge e of T, there is an *ideal tree* consisting of e (as the new root) and all the descendent edges and nodes.
- P_c is the *forest* consisting of all the ideal trees generated by the leaves of S.

The bialgebra of operadic trees (continued)

- $\mathcal{B}$ is the free $\mathbb{C}$-algebra on the set of isomorphism classes of operadic trees.
- A comultiplication is defined on its generators by

$$\begin{aligned} \Delta : \mathcal{B} &\longrightarrow \mathcal{B} \otimes \mathcal{B} \\ T &\longmapsto \sum_{c : S \subset T} P_c \otimes S, \end{aligned}$$

- $\mathcal{B}$ becomes a graded bialgebra.

The bialgebra of operadic trees (end)

- $\mathcal{B}$ is not connected.
- $\mathcal{B}_0$ is spanned by the trivial tree $\Big|$ and all its powers.
- These are all group-like, so a connected bialgebra can be obtained by imposing the equation $1 = \Big|$.

From operadic trees to combinatorial trees

- The *core* of a non-trivial operadic tree T is the combinatorial tree $T^\bullet$ obtained by pruning off all leaves as well as the root edge.
- Taking core is functorial in root-preserving inclusions.
- Hence, it induces a bialgebra homomorphism from the bialgebra of operadic trees to the Hopf algebra of combinatorial trees à la Connes–Kreimer.

The Green function on the bialgebra of operadic trees

- Consider the power series ring completion of $\mathcal{B}$.
- The *Green function* in this ring is the series

$$G := \sum_{T} T / |\mathrm{Aut}(T)|$$

  where the sum runs over all isomorphism classes of operadic trees.
- It is analogous to the combinatorial Green function of Feynman graphs.
- If *decorated* trees are considered instead, there is a Green function for each possible decoration of the root edge.
- This is analogous to the Green function in QFT, where there is a Green function for each possible residue in the theory.

The Faà di Bruno formula for the Green function in $\mathcal{B}$

Theorem

Let g_n be the Green function of trees with n leaves in $\mathcal{B}$, so that

$$G = \sum_{n \in \mathbb{N}} g_n.$$

Then the following Faà di Bruno formula holds

$$\Delta(G) = \sum_{n \in \mathbb{N}} G^n \otimes g_n$$

- To prove this theorem soundly we will use *groupoids*.
- Here n is an isomorphism class of the groupoid of finite sets (of leaves). Our proof works for 'coloured' trees, over more general polynomial functors. The groupoid **FinSet** is then replaced by **FinSet**$/I$ where I is the set (or groupoid) of colours.

Groupoids

- A *groupoid* is a category in which every arrow is invertible.
- A *morphism* of groupoids is a functor.
- Intuitively, groupoids are 'fat sets with symmetries'.
- Instead of having just a few isolated points (elements in a set) we now have large chunks of points which are equivalent, with specific arrows linking them up.
- More than one arrow can exist between two given objects, and indeed a single object can have more than one arrow to itself — these are its symmetries.

Groupoids (continued)

- A set is considered a groupoid in which the only arrows are the identity arrows.
- Conversely, a groupoid X gives rise to a set by taking its set of connected components, i.e. the set of isomorphism classes in X, denoted $\pi_0(X)$.
- A group can be considered as a groupoid with only one object.
- Conversely, for each object x in a groupoid X there is associated a group, the *vertex group*, denoted $\pi_1(x)$ or $\mathrm{Aut}(x)$, which consists of all the arrows from x to itself.
- The homotopy notations π_0 and π_1 reflect the fact that groupoids are a model for certain topological spaces, the homotopy 1-types.

Equivalences of groupoids

- An *equivalence* of groupoids is just an equivalence of categories, i.e. a functor possessing a pseudo-inverse.
- This is the analogue of a homotopy equivalence in topology.
- Equivalent groupoids have the same properties, for example the same π_0, π_1, and the same *cardinality*.

Homotopy fibres

- We will need some homotopy universal constructions.
- The *(homotopy) fibre* of a morphism

$$E \xrightarrow{p} B$$

over $b \in B$ is the groupoid E_b with objects

$$(e, \phi), \qquad e \in E, \qquad \phi : pe \overset{\cong}{\rightarrow} b$$

and arrows

$$(\epsilon, \mathrm{Id}) : (e, \phi) \rightarrow (e', \phi')$$

with $\epsilon : e \rightarrow e'$ such that $\phi' \circ p\epsilon = \phi$

$$\begin{array}{ccc} pe & \xrightarrow[p\epsilon]{\cong} & pe' \\ {\scriptstyle \phi}\downarrow{\scriptstyle \cong} & & {\scriptstyle \phi'}\downarrow{\scriptstyle \cong} \\ b & \xrightarrow[=]{\mathrm{Id}} & b \end{array}$$

Weak quotients

- Whenever a group acts on a set or a groupoid X

$$G \times X \to X$$

  the *weak quotient* X/G is the groupoid obtained by gluing in a path between x and y for each $g \in G$ such that $gx = y$.
- The weak quotient is often denoted $X//G$ to distinguish it from the naïve quotient, but we don't need the latter here.
- If G acts on the set $\{x\}$, then the weak quotient $\{x\}/G$ is the groupoid with one object and vertex group G.
- For a groupoid X, we will be considering the groupoid $\{x\}/\operatorname{Aut}(x)$ for each object $x \in X$.

The equivalent skeleton of a groupoid

- Every groupoid X is equivalent to its skeleton:

$$X \simeq \sum_{x \in \pi_0 X} \{x\} / \operatorname{Aut}(x)$$

where the sum sign denotes disjoint union of groupoids.

Integration formula

- Let $f : X \to B$ be a morphism of groupoids.
- Consider the fibre over b for each $b \in \pi_0 B$.
- The weak quotient

$$X_b / \operatorname{Aut}(b)$$

gives rise to an equivalence of groupoids

$$X \simeq \sum_{b \in \pi_0 B} X_b / \operatorname{Aut}(b)$$

- We will denote

$$\int_{b \in B} X_b := \sum_{b \in \pi_0 B} X_b / \operatorname{Aut}(b)$$

Integration along the fibres (or: the Fubini Principle)

- Given morphisms of groupoids

$$X \xrightarrow{f} B \xrightarrow{t} I$$

we have

$$\sum_{b \in \pi_0 B} X_b / \operatorname{Aut}(b) \simeq \sum_{i \in \pi_0 I} \left(\sum_{b \in \pi_0 B_i} X_b / \operatorname{Aut}_i(b) \right) / \operatorname{Aut}(i)$$

- In integral notation,

$$\int_{b \in B} X_b \simeq \int_{i \in I} \left(\int_{b \in B_i} X_b \right) .$$

Double Counting Lemma

- Let A, B, U be groupoids, together with morphisms

$$B \longleftarrow U \longrightarrow A$$

  and write ${}_T U,\ U_S \subseteq U$ for the fibres over $T \in B,\ S \in A$ respectively.
- Then there are equivalences of groupoids

$$\int_{T \in B} {}_T U \;\simeq\; U \;\simeq\; \int_{S \in A} U_S .$$

Cardinality

- A groupoid X is called *compact* when $\pi_0 X$ is a finite set, and for each object $x \in X$ the fundamental group $\mathrm{Aut}(x)$ is a finite group.
- The *cardinality* of a compact groupoid (a.k.a. *groupoid cardinality* or *homotopy cardinality*) is the rational number

$$|X| := \sum_{x \in \pi_0 X} \frac{1}{|\mathrm{Aut}(x)|}$$

where $|\mathrm{Aut}(x)|$ denotes the order of the vertex group at x.

Cardinality-continued

- If X is a finite set considered as a groupoid, then the groupoid cardinality coincides with the set cardinality.
- If G is a group considered as a one-object groupoid, then the groupoid cardinality is the inverse of the order of the group.
- Groupoid cardinality is compatible with the sum, product and powers of groupoids:

$$|X + Y| = |X| + |Y|$$
$$|X \times Y| = |X| \times |Y|$$
$$|\mathbf{Grpd}(S, X)| = |X|^{|S|} \qquad (S \in \mathbf{FinSet})$$

just as for finite sets.

Cardinality and actions

- Let S be a compact groupoid and G a finite group. Given any action of G on S, we have

$$|S/G| = |S|/|G|.$$

Formal cardinality

- Let B be a groupoid such that $\mathrm{Aut}(b)$ is finite for each $b \in B$.
- Let $X \to B$ be a groupoid morphism with compact fibres.
- Consider the completed vector space spanned by the symbols δ_b for $b \in \pi_0(B)$.
- The *formal cardinality of X over B* is the element in that space given by

$$|X|_B := \sum_{b \in \pi_0 B} |X_b| \,/\, |\mathrm{Aut}(b)| \cdot \delta_b.$$

- If $B = B_1 \times B_2$ is a product groupoid we write the symbol

$$\delta_{(b_1, b_2)} \quad \text{as} \quad b_1 \otimes b_2$$

The groupoids of trees, of forests and of trees with cuts

- Let $\mathbf{T}$ and $\mathbf{F}$ be the groupoids of trees and forests respectively.
- Taking the set of leaves or the set of roots gives morphisms

$$\mathbf{FinSet} \xleftarrow{L} \mathbf{F} \xrightarrow{R} \mathbf{FinSet} \xleftarrow{L} \mathbf{T} \xrightarrow{R} *$$

- The fibres (of the first three) are denoted ${}_n\mathbf{F}$, $\mathbf{F}_n$, ${}_n\mathbf{T}$ as usual.
- Let $\mathbf{C}$ be the *groupoid of trees and cuts*, with
  - *Objects*: root preserving inclusions $S \rightarrowtail T$ of trees
  - *Morphisms:* isomorphisms of such arrows

$$\begin{array}{ccc} T & \xrightarrow[\cong]{\tau} & T' \\ \uparrow & & \uparrow \\ S & \xrightarrow[\cong]{\rho} & S' \end{array}$$

Double Counting Lemma for Trees and Cuts

- There are two projections $\mathbf{T} \xleftarrow{m} \mathbf{C} \xrightarrow{r} \mathbf{T}$

$$\left(T \xrightarrow{\cong} T' \right) \xleftarrow{m} \left(\begin{array}{ccc} T & \xrightarrow{\cong} & T' \\ \uparrow & & \uparrow \\ S & \xrightarrow[\cong]{} & S' \end{array} \right) \xmapsto{r} \left(S \xrightarrow{\cong} S' \right)$$

- There are equivalences of groupoids

$$\int_{T \in \mathbf{T}} {}_T\mathbf{C} \;\simeq\; \mathbf{C} \;\simeq\; \int_{S \in \mathbf{T}} \mathbf{C}_S$$

where ${}_T\mathbf{C}$, $\mathbf{C}_S$ are the fibres of m, r over $T, S \in \mathbf{T}$.

- For each tree T the fibre ${}_T\mathbf{C}$ is a *discrete* groupoid: it is equivalent to the set $\mathrm{cut}(T)$ of cuts of T.

The pullback groupoid $\mathbf{F} \times_{\mathbf{FinSet}} \mathbf{T}$

- Recall the groupoid homomorphisms

$$R : \mathbf{F} \to \mathbf{FinSet}, \qquad\qquad L : \mathbf{T} \to \mathbf{FinSet}$$

$R(P)$ = set of roots of a forest P, $L(S)$ = set of leaves of a tree S.

- We consider the (homotopy) pullback groupoid

$$\begin{array}{ccccc}
(\mathbf{F} \times_{\mathbf{FinSet}} \mathbf{T})_S & \longrightarrow & \mathbf{F} \times_{\mathbf{FinSet}} \mathbf{T} & \longrightarrow & \mathbf{T} \\
{\scriptstyle \cong}\downarrow & & \downarrow & & \downarrow{\scriptstyle L} \\
\mathbf{F}_{LS} & \longrightarrow & \mathbf{F} & \xrightarrow{R} & \mathbf{FinSet}
\end{array}$$

- *Objects*: $(P,\ S,\ R(P) \overset{\lambda}{\cong} L(S)\ )$ with P a forest and S a tree,
- *Morphisms* $(P, S, \lambda) \to (P', S', \lambda')$: pairs $(P \overset{\pi}{\cong} P', S \overset{\sigma}{\cong} S')$ compatible with the bijections λ, λ', that is, $L\sigma \circ \lambda = \lambda' \circ R\pi$.
- We will also need the **fibres** over a fixed tree S.

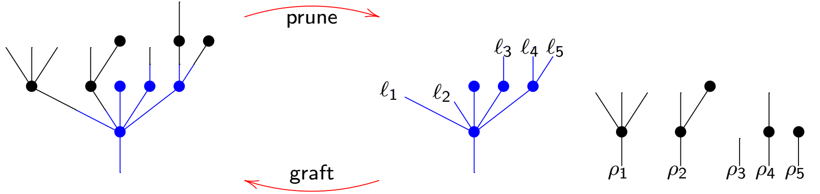
Lemma

There is an equivalence of groupoids

$$\mathbf{C} \xrightarrow{\simeq} \mathbf{F} \times_{\mathbf{FinSet}} \mathbf{T}$$

$$(S \rightarrowtail T) \underset{\text{graft}}{\overset{\text{prune}}{\rightleftarrows}} (P = (P_\rho)_{\rho \in RP},\ S,\ \lambda : RP \cong LS)$$

Idea of Proof



$S \rightarrowtail T$ $\qquad$ S $\qquad$ $P = (P_\rho) : \ \rho_i \overset{\lambda}{\longleftrightarrow} \ell_i$

The fibres over a fixed subtree S

- We have identified the fibres of the (homotopy) pullback

$$(\mathbf{F} \times_{\mathbf{FinSet}} \mathbf{T})_S \simeq \mathbf{F}_{LS}$$

- Hence, by the Key Lemma,

$$\mathbf{C}_S \simeq \mathbf{F}_{LS}$$

- By the double counting lemma,

$$\int_T \mathrm{cut}(T) \simeq \int_T {}_T\mathbf{C} \simeq \mathbf{C} \simeq \int_{S \in \mathbf{T}} \mathbf{C}_S \simeq \int_{S \in \mathbf{T}} \mathbf{F}_{LS}$$

Application of the Fubini principle

- Now we can split into different fibres, according to the composition

$$\mathbf{C} \to \mathbf{T} \xrightarrow{L} \mathbf{FinSet}$$

- Therefore one has

$$\begin{aligned} \int_T \mathrm{cut}(T) \;\simeq\; \mathbf{C} \;\simeq\; & \int_S \mathbf{C}_S && \text{double counting} \\ \simeq\; & \int_S \mathbf{F}_{LS} && \text{Key Lemma} \\ \simeq\; & \int_n \int_{S \in_n \mathbf{T}} \mathbf{F}_n && \text{Fubini} \\ \simeq\; & \int_n \mathbf{F}_n \times {}_n\mathbf{T} && \text{integration of constant} \end{aligned}$$

- This is the groupoid version of the Faà di Bruno Theorem.
- It is an equivalence of groupoids over $\mathbf{F} \times \mathbf{T}$.

Towards the Faà di Bruno Formula

Hence, we have proved the following

Theorem

$$\int_{T \in \mathbf{T}} \mathrm{cut}(T) \simeq \int_{n \in \mathbf{FinSet}} \mathbf{F}_n \times {}_n\mathbf{T}$$

- Both sides are groupoids over $\mathbf{F} \times \mathbf{T}$.
- The formal cardinality of the set $\mathrm{cut}(T)$ is

$$|\mathrm{cut}(T)| = \sum_{c \in \mathrm{cut}(T)} P_c \otimes S_c$$

- On the other hand, the formal cardinality of $\mathbf{F}_n \times {}_n\mathbf{T}$ is

$$|\mathbf{F}_n \times {}_n\mathbf{T}| = |\mathbf{T}|^n \otimes |{}_n\mathbf{T}| = G^n \otimes G_n$$

Theorem: The Faà di Bruno Formula

Therefore

$$\sum_{T\in\pi_0\mathbf{T}}\ \sum_{c\in\mathrm{cut}(T)} P_c \otimes S_c/|\,\mathrm{Aut}(T)| = \sum_{n\in\pi_0\mathbf{FinSet}} G^n \otimes G_n/|\,\mathrm{Aut}(n)|$$

That is,

$$\sum_{T\in\pi_0\mathbf{T}} \Delta(T)/|Aut(T)| = \sum_n G^n \otimes g_n$$

So that we have proved

Theorem: The Faà di Bruno Formula

$$\Delta(G) = \sum_n G^n \otimes g_n$$