# Central Cohomology Operations and $K$-theory

## Work in Progress

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Higher Homotopy in Barcelona 2012
IMUB, Universitat de Barcelona

March 24th 2012

Introduction

Cohomology operations

$BP\langle n\rangle$ theories

Main Theorem

Outline of proof

Open questions

# What is known about centres of cohomology operations rings

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- Ever since, we have been trying to fill the gap between the these two families of theories at both ends of the chromatic spectrum.
- We have now done it for unstable operations for $E = BP\langle n\rangle$ for all chromatic levels $n$.
- However, there are still many open questions...

# Rings of stable operations in generalized cohomology theories

- Stable operations are self maps of spectra up to homotopy. For $E$ a ring spectrum, the ring of stable operations in $E$-theory is given by

$$E^*(E) = [E, E]_*$$

A representative of $\varphi$ a degree operation $k$ will be a map of spectra, given levelwise by $f_m : E_m \to E_{m+k}$ commuting with suspensions and such that $\varphi = [f]$.

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- We will be interested in degree 0 operations.

$$E^0(E) = [E, E]_0$$

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- This ring acts on the cohomology of any space or spectrum.
- Classical examples: Steenrod algebra, Landweber-Novikov algebra, $K$-theory operations...

# Theories with good duality, stable case

- There is also a ring of cooperations, namely the $E$-homology of the spectrum $E$ itself.

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- If $E_*(E)$ is flat over the coefficient ring $E_*$, one can get a Hopf algebroid (or bilateral Hopf algebra) structure defined on it.

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- $p$-local $K$-theory and its split summands.

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- When there is good duality,

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- That means that in these good cases, operations are determined by the action they induce on coefficients.

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- Not all operations are now additive. Non additive operations are a very different business. We concentrate in *additive operations*, known as $PE^0(\underline{E}_0)$, which we will name $\mathcal{A}(E)$.
- Algebraic structure: products. Hopf ring structure.

# Generalized unstable cohomology operations, II

- Sending an operation to its action on coefficients gives

$$\begin{aligned} E^0(\underline{E}_0) &\to \mathrm{End}(\pi_*(\underline{E}_0)) \\ \phi &\mapsto \phi_*. \end{aligned}$$

- The restriction of this map to the additive $E$-operations $\mathcal{A}(E)$ is a ring homomorphism $\beta_E$:

$$\begin{aligned} \beta_E : \mathcal{A}(E) &\to \mathrm{End}(\pi_*(\underline{E}_0)) \\ \phi &\mapsto \phi_*. \end{aligned}$$

# Theories with good duality, unstable case

- The theories considered by [SW10a] all have good duality for the unstable case. Again, operations are dual to cooperations under the isomorphism of $E_*$-modules

$$E^*(\underline{E}_0) \cong \hom_{E_*}(E_*(\underline{E}_0), E_*)$$

On the left-hand side, there is the profinite topology, on the right the dual-finite topology.

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- This isomorphism restricted to the additive operations $\mathcal{A}(E)$ gives

$$\mathcal{A}(E) = PE^*(E_0) \cong \hom_{E_*}(QE_*(\underline{E}_0), E_*)$$

where $QE_*(\underline{E}_0)$ is the quotient of the indecomposable quotient of the cooperations for the $\star$-product in the Hopf ring $E_*(\underline{E}_0)$.

# Stable Adams operations in some complex oriented theories

## Proposition

*Let $E = MU, BP, KU$ or $G$. A stable multiplicative cohomology operation $\theta : E \to E$ is uniquely determined by its value $\theta(x_E)$ on the orientation class $x_E \in E^2(\mathbb{C}P^\infty)$.*

To have stable Adams operations, we need to work $p$-locally, for a prime $p$.

# Stable Adams operations in some complex oriented theories, II

## Definition

Let $E = MU_{(p)}$, $BP$, $KU_{(p)}$ or $G$. A stable Adams operation $\Psi^\alpha_E$ is defined for each $\alpha \in \mathbb{Z}^\times_{(p)}$ as the unique multiplicative operation given on the orientation class $x_E$ by

$$\Psi^\alpha_E(x_E) = \frac{[\alpha]_E(x_E)}{\alpha} \in E^*(\mathbb{C}P^\infty) = E_*[[x_E]]\,,$$

where $[\alpha]_E$ denotes the formal sum.

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Let $E = MU_{(p)}$, $BP$, $KU_{(p)}$ or $G$. A stable Adams operation $\Psi_E^\alpha$ is defined for each $\alpha \in \mathbb{Z}_{(p)}^\times$ as the unique multiplicative operation given on the orientation class $x_E$ by

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## Lemma

*$\Psi_E^\alpha$ acts as multiplication by $\alpha^n$ on $E_{2n}$.*

# The spectra $BP\langle n\rangle$( [Wil75], [JW73])

- For each $n \geq 0$, there is a connective commutative ring spectrum $BP\langle n\rangle$ with coefficient groups

  $$BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n] = BP_*/(v_{n+1}, v_{n+2}, \ldots) = BP_*/J_n$$

  $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, $v_i$ the Hazewinkel generators,
  $J_n = (v_{n+1}, v_{n+2}, \ldots)$.

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$J_n = (v_{n+1}, v_{n+2}, \dots)$.

- They form a tower of $BP$-module spectra:

$$BP \longrightarrow \dots \longrightarrow BP\langle n\rangle \longrightarrow BP\langle n-1\rangle \longrightarrow \dots$$

$$\dots \longrightarrow BP\langle 2\rangle \longrightarrow \dots \longrightarrow BP\langle 1\rangle \longrightarrow BP\langle 0\rangle$$

# The Wilson splitting of $BP\langle n\rangle$ from $BP$

- In [Wil75], Wilson constructs unstable splittings of the form

$$BP_k \cong BP\langle n\rangle_k \times \prod_{j>n} BP\langle j\rangle_{k+2(p^j-1)}$$

for $k \leq 2(p^n + \cdots + p + 1)$ and for $k < 2(p^n + \cdots + p + 1)$ this decomposition is as irreducibles.

# The Wilson splitting of $BP\langle n\rangle$ from $BP$: maps from it

- Let $n \geq 0$. The map above is one of $BP$-module spectra (indeed map of ring spectra):

$$\pi_n : BP \to BP\langle n\rangle$$

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and a ring map on homotopy groups which is the canonical projection

$$(\pi_n)_* : BP_* \to BP\langle n\rangle_*$$

# The Wilson splitting of $BP\langle n\rangle$ from $BP$: maps from it, II

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- Then $e_n = \theta_n \pi_n$ is the corresponding additive idempotent $BP$-operation.
- Choices can be made compatibly so that $e_n e_m = e_m e_n = e_m$ for $m < n$

# The Wilson splitting of $BP\langle n\rangle$ from $BP$: maps from it, III

**Lemma**

*We have maps*

$$i_n : \mathcal{A}(BP\langle n\rangle) \rightleftarrows \mathcal{A}(BP) : p_n$$

*such that*

1. $i_n \circ p_n : \mathcal{A}(BP) \to \mathcal{A}(BP)$ *is given by* $[f] \mapsto [e_n \circ f \circ e_n]$;
2. $p_n$ *splits* $i_n$ *(so* $i_n$ *is injective and* $p_n$ *is surjective);*
3. $i_n$ *is a ring homomorphism;*
4. $p_n$ *is an additive group homomorphism.*

# The Wilson splitting of $BP\langle n\rangle$ from $BP$: maps from it, IV

That is,

$$[\theta_n \circ - \circ \pi_n] : BP\langle n\rangle^0(\underline{BP\langle n\rangle}_0) \to BP^0(\underline{BP}_0)$$
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$\pi_n$ and $\theta_n$ are $H$-space maps, so they restrict to the additive operations, where we call them $i_n$ and $p_n$.

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$\pi_n$ and $\theta_n$ are $H$-space maps, so they restrict to the additive operations, where we call them $i_n$ and $p_n$.
(1) follows from $\theta_n\pi_n = e_n$.
(2),(3),(4) from $\pi_n\theta_n \simeq id$ and from $\theta_n$ being a map of $H$-spaces.

# The Wilson splitting of $BP\langle n\rangle$ from $BP$: maps from it, V

**Remark**

Hence we may identify as subrings of $\mathcal{A}(BP)$

$$\mathcal{A}(BP\langle n\rangle) \cong e_n \mathcal{A}(BP) e_n$$

# Theorem: the centre of unstable additive $(0,0)$-degree operations in $BP\langle n\rangle$ comes from $K$-theory

- Using the unstable $BP$ splittings $\theta_n$, we define an injective ring homomorphism

$$\hat{\iota}_n : \mathcal{A}(g) \to \mathcal{A}(BP\langle n\rangle)$$

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- This $\hat{\iota}$ is different from the one above $i_1 : \mathcal{A}(BP\langle 1\rangle) = \mathcal{A}(g) \to \mathcal{A}(BP)$.

## Theorem: the centre of unstable additive $(0,0)$-degree operations in $BP\langle n\rangle$ comes from $K$-theory

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- Our main result: $\hat{\iota}_n(\mathcal{A}(g)) = Z(\mathcal{A}(BP\langle n\rangle))$
- This builds on the previous result:

**Theorem ( [SW10a])**

*There is an injective ring homomorphism $\hat{\iota} : \mathcal{A}(g) \to \mathcal{A}(BP)$ such that the image is precisely the centre of the ring $\mathcal{A}(BP)$.* □

# Outline of the proof: faithfulness

## Proposition

*For all $n \geq 0$, the ring homomorphism*

$$\beta_{BP\langle n\rangle} : \mathcal{A}(BP\langle n\rangle) \to \mathrm{End}(\pi_*(\underline{BP\langle n\rangle}_0))$$

*is injective.*

## Proof.

For $\phi \in \mathcal{A}(BP\langle n\rangle)$ such that $\beta_{BP\langle n\rangle}(\phi) = \phi_* = 0$,

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For $\phi \in \mathcal{A}(BP\langle n\rangle)$ such that $\beta_{BP\langle n\rangle}(\phi) = \phi_* = 0$, we have $\beta_{BP}(i_n(\phi)) = (i_n(\phi))_* = (\theta_n \phi \pi_n)_* = (\theta_n)_* \phi_* (\pi_n)_* = 0.$

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# Outline of the proof: Unstable Adams operations for $BP\langle n\rangle$

## Definition

The *unstable Adams operations for* $BP\langle n\rangle$ are the images of the corresponding $BP$ operations under the map $p_n$:

$$\Psi^k_{BP\langle n\rangle} := p_n(\Psi^k_{BP})$$

for $k \in \mathbb{Z}_{(p)}$.

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This gives unstable Adams operations for $BP\langle n\rangle$ such that $\Psi^k_{BP\langle n\rangle}(z) = k^{(p-1)r} z$, for $z \in BP\langle n\rangle_{2(p-1)r}$.

# Outline of the proof: the comparison map $\hat{\iota}$ is a ring map.

- We know from [SW10a] that

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- From $\Psi^k_{BP\langle n\rangle} = p_n(\Psi^k_{BP})$ and the description of $\mathcal{A}(g)$ in terms of Adams operations, one gets a map $\hat{\iota}_n$ determined by
  - mapping the $g$ Adams operations to the corresponding $BP\langle n\rangle$ Adams operations
  - extending to suitable infinite linear combinations.

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$$\hat{\iota}(\Psi_g^k) = \Psi_{BP}^k$$

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$$\hat{\iota}_n : \mathcal{A}(g) \to \mathcal{A}(BP\langle n\rangle)$$

- $\hat{\iota}_n$ is a ring homomorphism (even though $p_n$ is not).

# Outline of the proof: the comparison map $\hat{\iota}$ is a ring map,II.

### Proposition

*For all $n \geq 1$, the map $\hat{\iota}_n : \mathcal{A}(g) \to \mathcal{A}(BP\langle n\rangle)$ is an injective unital ring homomorphism whose image is contained in the centre of $\mathcal{A}(BP\langle n\rangle)$.*

# Outline of the proof: compatibility of maps.

We have this commutative diagram of abelian groups, for $m \leq n$.

$$\begin{array}{ccccc}
 & & \mathcal{A}(BP\langle n\rangle) & \xrightarrow[i_n]{\cong} & e_n\mathcal{A}(BP)e_n \\
 & \nearrow^{\hat{\iota}_m} & \uparrow{\scriptstyle p_n} & & \downarrow{\scriptstyle e_m\circ - \circ e_m} \\
\mathcal{A}(g) & \xrightarrow{\hat{\iota}} & \mathcal{A}(BP) & & \\
 & \searrow_{\hat{\iota}_n} & \downarrow{\scriptstyle p_m} & & \\
 & & \mathcal{A}(BP\langle m\rangle) & \xrightarrow[i_m]{\cong} & e_m\mathcal{A}(BP)e_m
\end{array}$$

# Outline of the proof: Diagonal operations

Unstable diagonal operations for $BP$ were defined in [SW10a]. The same can be done for $BP\langle n\rangle$.

**Definition**

Let $\mathcal{D}(BP\langle n\rangle)$ be the subring of $\mathcal{A}(BP\langle n\rangle)$ consisting of operations whose action on each $\pi_{2(p-1)r}(\underline{BP\langle n\rangle}_0)$ is multiplication by some $\mu_r$ of $\mathbb{Z}_{(p)}$.

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# Outline of the proof: Diagonal operations

**Lemma**

$\mathcal{D}(BP\langle n\rangle) \subseteq Z(\mathcal{A}(BP\langle n\rangle))$.

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We construct stable $BP$ operations that act as we want,
we then view these as additive unstable $BP$ operations
and then we project them to $\mathcal{A}(BP\langle n\rangle)$.

# Outline of the proof: Congruences

$S_g$= subring of $\prod_{i=0}^{\infty} \mathbb{Z}_{(p)}$ of sequences $(\mu_i)_{i\geq 0}$ satisfying the system of congruences which characterizes the action on coefficient groups of an element of $\mathcal{A}(g)$.

# Outline of the proof: Congruences,II

Description of the congruences [SW10a, Section 4]:

- $G$ the periodic Adams summand, $G_* = \mathbb{Z}_{(p)}[\hat{u}^{\pm 1}]$.

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- The $n$-th congruence is the condition that the rational linear combination of the $\mu_i$ obtained from $\hat{f}_n$ by sending $\hat{w}^i$ to $\mu_i$ lies in $\mathbb{Z}_{(p)}$.
- Different choices of basis lead to equivalent systems of congruences with the same solution set $S_g$.

# Outline of the proof: Congruences, III

We have strenghtened the congruences result in of [SW10a] so that it goes down to $BP\langle n\rangle$.

## Proposition

*Fix $n \geq 1$. Suppose that an operation $\varphi \in \mathcal{A}(BP)$ is such that its action on homotopy $\varphi_* : BP_* \to BP_*$ satisfies the following conditions: for each $i \geq 0$, there is some $\mu_i \in \mathbb{Z}_{(p)}$ such that*

1. $\varphi_*(x) \equiv \mu_i x \mod J_n$ *if* $x \notin J_n$, $|x| = 2(p-1)i$.
2. $\varphi_*(x) = 0$ *if* $x \in J_n$.

*Then* $(\mu_i)_{i \geq 0} \in S_g$.

# Outline of the proof: Congruences,II

**Proposition**

*Let $n \geq 1$ and $\phi \in \mathcal{D}(BP\langle n\rangle)$. Then $i_n(\phi) \in \mathcal{A}(BP)$ satisfies the hypotheses of the proposition above.*

# End of the proof

### Theorem

*For all $n \geq 1$, the image of the injective ring homomorphism $\hat{\iota}_n : \mathcal{A}(g) \hookrightarrow \mathcal{A}(BP\langle n\rangle)$ is the centre $Z(\mathcal{A}(BP\langle n\rangle))$ of $\mathcal{A}(BP\langle n\rangle)$.*

$$
\begin{array}{ccccccc}
\mathcal{A}(g) & \xrightarrow[\hat{\iota}_n]{\cong} & \mathrm{Im}(\hat{\iota}_n) & \hookrightarrow & Z(\mathcal{A}(BP\langle n\rangle)) & \xrightarrow{=} & \mathcal{D}(BP\langle n\rangle) \\
\cong\downarrow & & & & & & \downarrow \\
S_g & & & \xrightarrow{=} & & & S_g
\end{array}
$$

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- For which theories is this true?
  Our guess is that, both in the stable and unstable case, the center of the ring of stable operations of degree 0 is the image of the one for $K$-theory.
  Similarly for the corresponding unstable version.
  But have not succeeded in proving it by a general argument for a wide enough class of theories.
- What is known for $n{=}2$ theories? We have some partial results for elliptic cohomology using Adams and Hecke operators, but we run into other problems.
- Relation to other $BP$-related work.

# More open questions

- What are Adams operations, really? The original Adams definitions for unstable ones in periodic $K$-theory, was followed by definitions for some stable ones in diverse contexts by Novikov and others. These ones got unstable versions as well. Despite some general arguments, construccions are very much *ad hoc*. One would like to have a general enough definition for them.

Imma Gálvez and Sarah Whitehouse, *Infinite sums of Adams operations and cobordism*, Math. Z. **251** (2005), no. 3, 475–489. MR 2190339 (2007k:55031)

David Copeland Johnson and W. Stephen Wilson, *Projective dimension and Brown-Peterson homology*, Topology **12** (1973), 327–353. MR 0334257 (48 #12576)

Haynes R. Miller and Douglas C. Ravenel, *Morava stabilizer algebras and the localization of Novikov's $E_2$-term*, Duke Math. J. **44** (1977), no. 2, 433–447. MR 0458410 (56 #16613)

M.-J. Strong and Sarah Whitehouse, *Infinite sums of unstable Adams operations and cobordism*, J. Pure Appl. Algebra **214** (2010), no. 6, 910–918. MR 2580668 (2011i:55024)

______, *Integer-valued polynomials and K-theory operations*, Proc. Amer. Math. Soc. **138** (2010), no. 6, 2221–2233. MR 2596063 (2011e:55022)

W. Stephen Wilson, *The Ω-spectrum for Brown-Peterson cohomology. II*, Amer. J. Math. **97** (1975), 101–123. MR 0383390 (52 #4271)