Generalized Syzygies for Commutative Koszul Algebras
Joint Work in Progress with Vassily Gorbounov (Aberdeen)
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Towards generalized syzygies

**Definition: Koszul homology**

Let \( \{a_1, \ldots, a_n\} \) be a sequence of elements in a commutative \( \mathbb{C} \)-algebra \( A \). Let \( W \) be an \( n \)-dimensional complex vector space with basis \( \{\theta_1, \ldots, \theta_n\} \). The **Koszul homology** of \( A \) with respect to the sequence \( \{a_1, \ldots, a_n\} \) is the homology of the complex

\[
A \otimes \bigwedge W,
\]

where \( A \) has homological degree zero, each \( \theta_i \) has homological degree one, and the **Koszul differential** is given by the formula

\[
d_K = \sum_{i=1}^{n} a_i \frac{\partial}{\partial \theta_i}.
\]
Definition: Quadratic algebra

Let $A$ be a positively graded connected algebra, locally finite-dimensional. $A$ is called quadratic if it is determined by a vector space of generators $V = A_1$ and subspace of quadratic relations $I \subset A_1 \otimes A_1$.

Definition: Koszul quadratic dual

The Koszul dual algebra $A^!$ associated with a quadratic algebra $A$ is

$$A^! = T(V^*)/I^\perp,$$

where $I^\perp \subset V^* \otimes V^*$ is the annihilator of $I$. Clearly $A^{!!} = A$. 

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Generalized Syzygies for Commutative Koszul Algebras
Definition: Koszul algebra

A quadratic as above is a **Koszul algebra** iff

\[ A^1 \cong \text{Ext}^*_A(k, k) \]
Definition: Lie superalgebra

A **Lie superalgebra** over $\mathbb{C}$ is a $\mathbb{Z}/2$-graded vector space (over $\mathbb{C}$) $L = L(0) \oplus L(1)$ with a map $[\cdot, \cdot] : L \otimes L \to L$ of $\mathbb{Z}/2$-graded spaces, satisfying:

1. (anti-symmetry) $[x, y] = -(-1)^{|x||y|}[y, x]$ for all homogeneous $x, y \in L$,

2. (Jacobi identity)

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$$

for all homogeneous $x, y, z \in L$. 

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Definition: Lie superalgebra (continued)

Here $|x|$ is the parity of $x$: $|x| = i$ when $x \in L(i)$ for $i = 0, 1$. An element $x$ in $L(0)$ or $L(1)$ is termed even or odd respectively. We recover the familiar definition of a Lie algebra (over $\mathbb{C}$) in the case $L = L(0)$.

Definition: graded Lie superalgebra

A **graded Lie superalgebra** is a Lie superalgebra $L$ together with a grading compatible with the bracket and supergrading. That is, $L = \bigoplus_{m \geq 1} L_m$ such that $[L_i, L_j] \subset L_{i+j}$ and $L(i) = \bigoplus_{m \geq 1} L_{2m-i}$ for $i = 0, 1$.
Assume that $A = T(V)/I$ is commutative, hence $\wedge^2 V \subset I$. Therefore $I^\perp$ is contained in $S^2(V^*)$, and so is generated by certain linear combinations of anti-commutators $[a^*_i, a^*_j] = a^*_i a^*_j + a^*_j a^*_i$. As a consequence, the Koszul quadratic dual of $A$ can be described as the universal envelope of a graded Lie superalgebra,

$$A^! = U(L), \quad L = \bigoplus_{m \geq 1} L_m = \mathbb{L}(V^*)/J,$$

where $\mathbb{L}$ is the free Lie superalgebra functor, the space of (odd) generators $V^*$ is concentrated in degree 1, and $J$ is the Lie ideal with the same generators as $I^\perp$ but viewed as linear combinations of supercommutators.
The following Lie ideals are main characters of our work.

**Definition**

For $k \geq 2$, we define the graded Lie superalgebras

$$L_{\geq k} = \bigoplus_{m \geq k} L_m.$$
The algebra of syzygies (I)

Definition: the algebra of syzygies

Let $A$ be a commutative $\mathbb{C}$-algebra which is a module over $S(V)$, and suppose we have a minimal free resolution of $A$,

$$\cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow A \rightarrow 0.$$ 

This is an exact sequence of graded free $S(V)$-modules,

$$F_p = \bigoplus_{q \geq m_p} R_{pq} \otimes S(V),$$

where $R_{pq}$ is the finite dimensional vector spaces of $p$-th order syzygies of degree $q$ for $A$, and $m_p$ is the minimum degree among the $p$-th order syzygies.
Since the chosen resolution is minimal, the differential vanishes on tensoring this complex with the trivial $S(V)$-module $\mathbb{C}$.

Hence

$$\text{Tor}^{S(V)}_p(A, \mathbb{C}) = R_p := \bigoplus_{q \geq m_p} R_{pq}.$$
The functor $\text{Tor}^S(V)$ may also be calculated by resolving the other argument $\mathbb{C}$.

The Koszul complex $K(S(V))$ of the symmetric algebra $S(V) = \mathbb{C}[a_1, \ldots, a_n]$, is given by $(S(V) \otimes \bigwedge W, d_K)$, that is,

$$(\mathbb{C}[a_1, \ldots, a_n] \otimes \bigwedge (\theta_1, \ldots, \theta_n), d_K).$$

with the Koszul differential $d_K$ of the sequence $\{a_1, \ldots, a_n\}$.
Since this complex is a resolution of \( \mathbb{C} \) we can calculate the syzygies \( R_p \) of the quadratic algebra \( A = T(V)/I \) by the homology of the complex

\[
A \otimes_{S(V)} K(S(V)) = A \otimes_{S(V)} S(V) \otimes_{\mathbb{C}} \bigwedge W = A \otimes \bigwedge W.
\]

This homology inherits a multiplication from \( K(S(V)) \) and becomes an associative algebra.
Theorem [4, 12, 9]

Let $A = T(V)/I$ be a commutative Koszul algebra, $R = \bigoplus_p R_p$ is its algebra of syzygies and $A^! = T(V^*)/I^\perp = U(L)$ is its Koszul dual.

Then $R_{pq} \cong H^{q-p}(L_{\geq 2}, \mathbb{C})_q$ as algebras.

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So $H^*(L_{\geq 2}, \mathbb{C})$ gives indeed the algebra of syzygies of $A$. 

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We concentrate on the case $k = 3$.

Let $A = \mathbb{C}[a_1, \ldots, a_n]/I$ be commutative Koszul, with minimal set of generators $\{\Gamma_1, \ldots, \Gamma_m\}$ of $I$ representing lowest degree syzygies.

**Lemma**

If the quadratic relations for $A$ are defined by the formulas

$$\Gamma_k = \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i a_j,$$

for $k = 1, \ldots, m$, then the representative for the homology class in the algebra of syzygies defined by the sequence $\{\Gamma_1, \ldots, \Gamma_m\}$ is

$$\tilde{\Gamma}_k = \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i \theta_j, \quad k = 1, \ldots, m$$
The Berkovits complex of a commutative Koszul algebra $A$ is

$$C_B(A) := A \otimes \bigwedge (\theta_1, \ldots, \theta_n) \otimes \mathbb{C}[y_1, \ldots, y_m]$$

equipped with the Berkovits differential

$$d_B = d_K + d_{Ber} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial \theta_i} + \sum_{k=1}^{m} \sum_{i,j=1}^{n} \Gamma_{ij}^{k} a_i \theta_j \frac{\partial}{\partial y_k},$$

where the $y_k$ have homological degree two.
Main Theorem [GGST 2011]

Let $A$ be a commutative Koszul algebra and $A^1 = U(L)$. Then,

$$H_*(C_B(A), d_B) \cong H^*(L_{\geq 3}, \mathbb{C}).$$

$H_*(C_B(A), d_B)$ is called the algebra of generalized syzygies of $A$. 
Key Lemma

Let \((C, d)\) be a commutative DG algebra over \(\mathbb{C}\), nonnegatively graded and finitely generated in each degree.
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Key Lemma

- Let \((C, d)\) be a commutative DG algebra over \(\mathbb{C}\), nonnegatively graded and finitely generated in each degree.
- Let \(B\) be a contractible DG subalgebra of \(C\), with quasi-isomorphism \(\varepsilon : B \to \mathbb{C}\).
- Consider the DG ideal \(\langle B \rangle\) of \(C\) generated by the augmentation ideal \(\overline{B} = \ker(\varepsilon)\).
Tools for the proof: A result on DG algebras

Key Lemma

Let $(C, d)$ be a commutative DG algebra over $\mathbb{C}$, nonnegatively graded and finitely generated in each degree.

Let $B$ be a contractible DG subalgebra of $C$, with quasi-isomorphism $\varepsilon : B \rightarrow \mathbb{C}$.

Consider the DG ideal $\langle B \rangle$ of $C$ generated by the augmentation ideal $\overline{B} = \ker(\varepsilon)$.

If $\langle B \rangle$ is freely generated as a $\overline{B}$-module by a graded basis of homogeneous elements $Z = \bigcup_{i \geq 0} Z_i$, then $C$ is quasi-isomorphic to $C/\langle B \rangle$. 
Assume that $A$ is a commutative Koszul algebra with $A^! = U(L)$ as above. We construct a resolution of $A$ in the category of DG algebras from the Chevalley complex of $L$.

**Definition: the Chevalley complex**

The **Chevalley complex** of $L$ is the cochain complex with

$$\text{Ch}^i(L) = \left( \bigwedge^i L \right)^*$$

and the differential $d_C : \text{Ch}^k(L) \rightarrow \text{Ch}^{k+1}(L)$

$$(d_C \varphi)(x_0, \ldots, x_k) = \sum_{i<j} (-1)^{j+\varepsilon(i,j)} \varphi(x_0, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_k)$$
Some properties of the Chevalley complex

- $d_C : \text{Ch}^1(L) \to \text{Ch}^2(L)$ is the map that is dual to the bracket,

$$(d_C f)(x_0, x_1) = -f[x_0, x_1].$$
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- The Chevalley complex is a cochain complex that calculates the cohomology of \( L \) with trivial coefficients, \( H^*(L, \mathbb{C}) \).

- This complex admits an algebra structure that descends to one on the cohomology of \( L \) so that the differential \( d_C \) is a derivation with respect to its product.

That is, given \( \varphi \in \left( \bigwedge^i L \right)^* \) and \( \psi \in \left( \bigwedge^j L \right)^* \),

\[ \varphi \otimes \psi \in \left( \bigwedge^{i+j} L \right)^* \] and \( d_C(\varphi \otimes \psi) = d_C \varphi \otimes \psi \pm \varphi \otimes d_C \psi. \)
The Chevalley complex as a chain complex

As well as being a cochain complex whose cohomology is that of $L$, the Chevalley complex may also be considered a chain complex.
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- As well as being a *cochain complex* whose cohomology is that of $L$, the Chevalley complex may also be considered a *chain complex*.

- The chain complex is given by defining $L^*_p$ to have homological grading $p - 1$, so that the homological and cohomological gradings together give the total degree in $\wedge L^*$. 
We illustrate the Chevalley complex as a chain complex with homological grading as follows:

\[ \begin{array}{c}
\text{Ch}_3(L) \xrightarrow{dC} \text{Ch}_2(L) \xrightarrow{dC} \text{Ch}_1(L) \xrightarrow{dC} \text{Ch}_0(L) \rightarrow 0 \\
0 \rightarrow L_1^* \rightarrow 0 \\
0 \rightarrow L_2^* \rightarrow \wedge^2 L_1^* \rightarrow 0 \\
0 \rightarrow L_3^* \rightarrow L_2^* \wedge L_1^* \rightarrow \wedge^3 L_1^* \rightarrow 0
\end{array} \]

The original cohomological grading is seen on the diagonals.
In our situation of a graded Lie superalgebra $L$ with $A^1 = U(L)$ we observe that

$$\text{Ch}_0(L) = \bigwedge L_1^* = S(V).$$

**Proposition**

For a commutative Koszul algebra $A$ with $A^1 = U(L)$, the chain complex given by the Chevalley complex of $L$ with homological grading is a resolution of $A$. 
Lifting the Berkovits differential

The setup

We have a basis \( \{q_1, \ldots, q_m\} \) of \( L_2^* \) such that

\[
q_k = \sum_{i,j=1}^{n} \Gamma_{ij}^k \{a_i, a_j\},
\]

and by construction,

\[
d_C(q_k) = \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i a_j
\]

for \( k = 1, \ldots, m \).
The Berkovits differential

Define $Y$ as:

$$\text{Ch}(L) \otimes \bigwedge(\theta_1, \ldots, \theta_n) \otimes \mathbb{C}[y_1, \ldots, y_m],$$

with an obvious graded algebra structure.

We can try to lift the Berkovits differential to $Y$ as follows:

$$d_C + d_K + d_{Ber}$$

where

$$d_{Ber} = \sum_{i,j=1}^{n} \sum_{k=1}^{m} \Gamma_{ij}^k a_i \theta_j \frac{\partial}{\partial y_k}.$$
The correction term

However, one checks that

$$(d_C + d_K + d_{Ber})^2 = \sum_{i,j=1}^{n} \sum_{k=1}^{m} \Gamma_{ij}^k a_i a_j \frac{\partial}{\partial y_k} \neq 0.$$ 

In order for the differential to square to zero, we define a correction to the differential as

$$d_s = -\sum_{k=1}^{m} q_k \frac{\partial}{\partial y_k}.$$
Proposition

\[(d_C + d_K + d_{Ber} + d_S)^2 = 0.\]
The Resolution of $\mathbb{C}$ inside $Y$ (I)

**Proposition**

The subalgebra $T$ of $Y$ given by:

$$\text{Ch}(L_1, L_2) \otimes \bigwedge (\theta_1, \ldots, \theta_n) \otimes \mathbb{C}[y_1, \ldots, y_m]$$

is a resolution of $\mathbb{C}$.

This subcomplex is equipped with the differential

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial \theta_i} + \sum_{k=1}^{m} \left( \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i \theta_j - q_k \right) \frac{\partial}{\partial y_k} + \sum_{k=1}^{m} \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i a_j \frac{\partial}{\partial q_k}.$$
The Resolution of $\mathbb{C}$ inside $Y$ (II)

**Step 1**

The Koszul complex $(P, d_K)$ of the sequence $\theta_1, \ldots, \theta_n$ in $\text{Ch}(L_1)$ and the Koszul complex $(Q, d_S)$ of $y_1, \ldots, y_m$ in $\text{Ch}(L_2)$ are contractible, as is the product $(P \otimes Q, d_K + d_S)$.

**Step 2**

We perturb the differential on $P \otimes Q$ to the differential on $T$ by using the homological perturbation lemma.

**Step 3**

The perturbed homotopy is well-defined, so $T$ has a strong deformation retraction to $(\mathbb{C}, 0)$ concentrated in degree zero.
We have that $T$ is a subalgebra in $Y$. 
Proof of Main Theorem (I)

- We have that $T$ is a subalgebra in $Y$.
- We showed in a Proposition above that $T$ is a resolution of $C$. This satisfies the conditions of the Key Lemma and $Y$ is quasi-isomorphic to $Y/\langle T \rangle$, where $\langle T \rangle$ is the DG ideal in $Y$ generated by the augmentation ideal of $T$. Hence, $H_i(Y) = H_i(Y/\langle T \rangle) = H_i(Ch(L_{\geq 3}))$. 
Proof of Main Theorem (II)

Now, consider the filtration of $Y$

$$\{0\} \subset F_0 Y \subset F_1 Y \subset \cdots \subset F_n Y \subset \ldots,$$

given by

$$F_p Y_q := \sum_{j \leq p} \sum_{i+j=p+q} \text{Ch}_i(L) \otimes \left( \bigwedge (\theta_1, \ldots, \theta_n) \otimes \mathbb{C}[y_1, \ldots, y_m] \right)_j.$$
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- The differential on the $E_0$-term of the spectral sequence associated to this filtration is $d_C$. 
Proof of Main Theorem (II)

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- The differential on the $E_0$-term of the spectral sequence associated to this filtration is $d_C$.

- Since, $\bigwedge (\theta_1, \ldots, \theta_n) \otimes \mathbb{C}[y_1, \ldots, y_m]$ is a vector space over $\mathbb{C}$, it is flat as a $\mathbb{C}$-module.
Proof of main theorem (III)

As $\text{Ch}(L)$ is a resolution for $A$, we can conclude that the $E_1$-term of the spectral sequence is contained in one line and is given by,

$$A \otimes \bigwedge(\theta_1, \ldots, \theta_n) \otimes \mathbb{C}[y_1, \ldots, y_m]$$

with precisely the Berkovits differential $d_B$. 

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Generalized Syzygies for Commutative Koszul Algebras
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with precisely the Berkovits differential $d_B$.

Hence, the homology of this complex is also $H_*(C_b(A), d_B)$ the homology of the Berkovits complex.


