Generalized Syzygies for Commutative Koszul Algebras

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Towards generalized syzygies

Definition: Koszul homology

Let $\{a_1, \ldots, a_n\}$ be a sequence of elements in a commutative $\mathbb{C}$-algebra A. Let W be an n-dimensional complex vector space with basis $\{\theta_1, \ldots, \theta_n\}$. The **Koszul homology** of A with respect to the sequence $\{a_1, \ldots, a_n\}$ is the homology of the complex

$$A \otimes \bigwedge W,$$

where A has homological degree zero, each θ_i has homological degree one, and the **Koszul differential** is given by the formula

$$d_K = \sum_{i=1}^{n} a_i \frac{\partial}{\partial \theta_i}.$$

Koszul Algebras (I)

Definition: Quadratic algebra

Let A be a positively graded connected algebra, locally finite-dimensional. A is called **quadratic** if it is determined by a vector space of generators $V = A_1$ and subspace of quadratic relations $I \subset A_1 \otimes A_1$

Definition: Koszul quadratic dual

The **Koszul dual** algebra $A^!$ associated with a quadratic algebra A is

$$A^! = T(V^*)/I^\perp,$$

where $I^\perp \subset V^* \otimes V^*$ is the annihilator of I. Clearly $A^{!!} = A$.

Koszul algebras (II)

Definition: Koszul algebra

A quadratic as above is a **Koszul algebra** iff

$$A^! \cong \mathrm{Ext}_A^*(\mathbb{k}, \mathbb{k})$$

Lie superalgebras (I)

Definition: Lie superalgebra

A **Lie superalgebra** over $\mathbb{C}$ is a $\mathbb{Z}/2$-graded vector space (over $\mathbb{C}$) $L = L_{(0)} \oplus L_{(1)}$ with a map $[\cdot,\cdot] : L \otimes L \to L$ of $\mathbb{Z}/2$-graded spaces, satisfying:

1. (anti-symmetry) $[x, y] = -(-1)^{|x||y|}[y, x]$ for all homogeneous $x, y \in L$,
2. (Jacobi identity) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ for all homogeneous $x, y, z \in L$.

Lie superalgebras (II)

Definition: Lie superalgebra (continued)

Here $|x|$ is the parity of x: $|x| = i$ when $x \in L_{(i)}$ for $i = 0, 1$. An element x in $L_{(0)}$ or $L_{(1)}$ is termed even or odd respectively. We recover the familiar definition of a Lie algebra (over $\mathbb{C}$) in the case $L = L_{(0)}$.

Definition: graded Lie superalgebra

A **graded Lie superalgebra** is a Lie superalgebra L together with a grading compatible with the bracket and supergrading. That is, $L = \bigoplus_{m \geq 1} L_m$ such that $[L_i, L_j] \subset L_{i+j}$ and $L_{(i)} = \bigoplus_{m \geq 1} L_{2m-i}$ for $i = 0, 1$.

Koszul dual as universal envelope

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Assume that $A = T(V)/I$ is commutative, hence $\bigwedge^2 V \subset I$. Therefore $I^\perp$ is contained in $S^2(V^*)$, and so is generated by certain linear combinations of anti-commutators $[a_i^*, a_j^*] = a_i^* a_j^* + a_j^* a_i^*$. As a consequence, the Koszul quadratic dual of A can be described as the universal envelope of a graded Lie superalgebra,

$$A^! = U(L), \qquad L = \bigoplus_{m \geq 1} L_m = \mathbb{L}(V^*)/J, \tag{1}$$

where $\mathbb{L}$ is the free Lie superalgebra functor, the space of (odd) generators V^* is concentrated in degree 1, and J is the Lie ideal with the same generators as $I^\perp$ but viewed as linear combinations of supercommutators.

Some Lie ideals

The following Lie ideals are main characters of our work.

Definition

For $k \geq 2$, we define the graded Lie superalgebras

$$L_{\geq k} = \bigoplus_{m \geq k} L_m.$$

The algebra of syzygies (I)

Definition: the algebra of syzygies

Let A be a commutative $\mathbb{C}$-algebra which is a module over $S(V)$, and suppose we have a minimal free resolution of A,

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0.$$

This is an exact sequence of graded free $S(V)$-modules,

$$F_p = \bigoplus_{q \geq m_p} R_{pq} \otimes S(V),$$

where R_{pq} is the finite dimensional vector spaces of p-**th order syzygies of degree** q for A, and m_p is the minimum degree among the p-th order syzygies.

The algebra of syzygies (II)

Syzygies and the Koszul complex

- Since the chosen resolution is minimal, the differential vanishes on tensoring this complex with the trivial $S(V)$-module $\mathbb{C}$.
- Hence
$$\mathrm{Tor}_p^{S(V)}(A, \mathbb{C}) \;=\; R_p := \bigoplus_{q \geq m_p} R_{pq}.$$

The algebra of syzygies (III)

- The functor $\mathrm{Tor}^{S(V)}$ may also be calculated by resolving the other argument $\mathbb{C}$.
- The Koszul complex $K(S(V))$ of the symmetric algebra $S(V) = \mathbb{C}[a_1, \ldots, a_n]$, is given by $(S(V) \otimes \bigwedge W, \, d_K)$, that is,

$$(\mathbb{C}[a_1, \ldots, a_n] \otimes \bigwedge(\theta_1, \ldots, \theta_n), \, d_K). \tag{2}$$

with the Koszul differential d_K of the sequence $\{a_1, \ldots, a_n\}$.

The algebra of syzygies (IV)

Since this complex is a resolution of $\mathbb{C}$ we can calculate the syzygies R_p of the quadratic algebra $A = T(V)/I$ by the homology of the complex

$$A \otimes_{S(V)} K(S(V)) = A \otimes_{S(V)} S(V) \otimes_{\mathbb{C}} \bigwedge W = A \otimes \bigwedge W.$$

This homology inherits a multiplication from $K(S(V))$ and becomes an associative algebra.

Cohomology of $L_{\geq 2}$

Theorem [4, 12, 9]

Let $A = T(V)/I$ be a commutative Koszul algebra, $R = \oplus_p R_p$ is its algebra of syzygies and $A^! = T(V^*)/I^\perp = U(L)$ is its Koszul dual.
Then $R_{pq} \cong H^{q-p}(L_{\geq 2}, \mathbb{C})_q$ as algebras.

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Lie ideals and the algebra of syzygies

- So $H^*(L_{\geq 2}, \mathbb{C})$ gives indeed the algebra of syzygies of A.

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- The interpretation of the algebras $L_{\geq k}$ for $k > 2$ was outlined by Berkovits in [4].

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- So $H^*(L_{\geq 2}, \mathbb{C})$ gives indeed the algebra of syzygies of A.
- The interpretation of the algebras $L_{\geq k}$ for $k > 2$ was outlined by Berkovits in [4].
- We concentrate on the case $k = 3$.

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Berkovits Complex

Let $A = \mathbb{C}[a_1, \ldots, a_n]/I$ be commutative Koszul, with minimal set of generators $\{\Gamma_1, \ldots, \Gamma_m\}$ of I representing lowest degree syzygies.

Lemma

If the quadratic relations for A are defined by the formulas

$$\Gamma_k = \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i a_j,$$

for $k = 1, \ldots, m$, then the representative for the homology class in the algebra of syzygies defined by the sequence $\{\Gamma_1, \ldots, \Gamma_m\}$ is

$$\tilde{\Gamma}_k = \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i \theta_j, \qquad k = 1, \ldots, m$$

Berkovits complex (II)

Definition

The **Berkovits complex** of a commutative Koszul algebra A is

$$C_B(A) := A \otimes \bigwedge(\theta_1, \ldots, \theta_n) \otimes \mathbb{C}[y_1, \ldots, y_m]$$

equipped with the **Berkovits differential**

$$d_B = d_K + d_{Ber} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial \theta_i} + \sum_{k=1}^{m} \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i \theta_j \frac{\partial}{\partial y_k},$$

where the y_k have homological degree two.

Generalized syzygies

Main Theorem [GGST 2011]

Let A be a commutative Koszul algebra and $A^! = U(L)$. Then,

$$H_*(C_B(A), d_B) \cong H^*(L_{\geq 3}, \mathbb{C}).$$

$H_*(C_B(A), d_B)$ is called the *algebra of generalized syzygies* of A

Tools for the proof: A result on DG algebras

Key Lemma

- Let (C, d) be a commutative DG algebra over $\mathbb{C}$, nonnegatively graded and finitely generated in each degree.

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- Consider the DG ideal $\langle \overline{B} \rangle$ of C generated by the augmentation ideal $\overline{B} = \ker(\varepsilon)$.
- If $\langle \overline{B} \rangle$ is freely generated as a $\overline{B}$-module by a graded basis of homogeneous elements $Z = \bigcup_{i \geq 0} Z_i$, then C is quasi-isomorphic to $C / \langle \overline{B} \rangle$.

The Chevalley complex

Assume that A is a commutative Koszul algebra with $A^! = U(L)$ as above. We construct a resolution of A in the category of DG algebras from the Chevalley complex of L.

Definition: the Chevalley complex

The **Chevalley complex** of L is the cochain complex with

$$\mathrm{Ch}^i(L) = \left(\bigwedge^i L\right)^*$$

and the differential $d_C : \mathrm{Ch}^k(L) \to \mathrm{Ch}^{k+1}(L)$

$$(d_C\varphi)(x_0, \ldots, x_k) = \sum_{i<j} (-1)^{j+\varepsilon(i,j)} \varphi(x_0, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \widehat{x_j}, \ldots, x_k)$$

Some properties of the Chevalley complex

- $d_C : \mathrm{Ch}^1(L) \to \mathrm{Ch}^2(L)$ is the map that is dual to the bracket,

$$(d_C f)(x_0, x_1) = -f[x_0, x_1].$$

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- The Chevalley complex is a cochain complex that calculates the cohomology of L with trivial coefficients, $H^*(L, \mathbb{C})$.
- This complex admits an algebra structure that descends to one on the cohomology of L so that the differential d_C is a derivation with respect to its product.
  That is, given $\varphi \in \left(\bigwedge^i L\right)^*$ and $\psi \in \left(\bigwedge^j L\right)^*$,
  $\varphi \odot \psi \in \left(\bigwedge^{i+j} L\right)^*$ and $d_C(\varphi \odot \psi) = d_C\varphi \odot \psi \pm \varphi \odot d_C\psi$.

The Chevalley complex as a chain complex

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- As well as being a *cochain complex* whose cohomology is that of L, the Chevalley complex may also be considered a *chain complex*.
- The chain complex is given by defining L^*_p to have homological grading $p-1$, so that the homological and cohomological gradings together give the total degree in $\bigwedge L^*$.

The Chevalley complex as a chain complex (II)

We illustrate the Chevalley complex as a chain complex with homological grading as follows:

$$\mathrm{Ch}_3(L) \xrightarrow{d_C} \mathrm{Ch}_2(L) \xrightarrow{d_C} \mathrm{Ch}_1(L) \xrightarrow{d_C} \mathrm{Ch}_0(L) \longrightarrow 0$$

$$0 \longrightarrow L_1^* \longrightarrow 0$$

$$0 \longrightarrow L_2^* \longrightarrow \bigwedge^2 L_1^* \longrightarrow 0$$

$$0 \longrightarrow L_3^* \longrightarrow L_2^* \wedge L_1^* \longrightarrow \bigwedge^3 L_1^* \longrightarrow 0$$

The original cohomological grading is seen on the diagonals.

The Chevalley complex as resolution

In our situation of a graded Lie superalgebra L with $A^! = U(L)$ we observe that

$$\mathrm{Ch}_0(L) = \bigwedge L_1^* = S(V).$$

Proposition

For a commutative Koszul algebra A with $A^! = U(L)$, the chain complex given by the Chevalley complex of L with homological grading is a resolution of A.

Lifting the Berkovits differential

The setup

We have a basis $\{q_1, \ldots, q_m\}$ of L_2^* such that

$$q_k = \sum_{i,j=1}^{n} \Gamma_{ij}^k \{a_i, a_j\},$$

and by construction,

$$d_C(q_k) = \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i a_j$$

for $k = 1, \ldots, m$.

Lifting the Berkovits differential (II)

The Berkovits differential

Define Y as:

$$\mathrm{Ch}(L) \otimes \bigwedge(\theta_1, \ldots, \theta_n) \otimes \mathbb{C}[y_1, \ldots, y_m],$$

with an obvious graded algebra structure.
We can try to lift the Berkovits differential to Y as follows:
$d_C + d_K + d_{Ber}$ where

$$d_{Ber} = \sum_{i,j=1}^{n} \sum_{k=1}^{m} \Gamma_{ij}^{k} a_i \theta_j \frac{\partial}{\partial y_k}.$$

Lifting the Berkovits differential (III)

The correction term

However, one checks that

$$(d_C + d_K + d_{Ber})^2 = \sum_{i,j=1}^{n} \sum_{k=1}^{m} \Gamma_{ij}^{k} a_i a_j \frac{\partial}{\partial y_k} \neq 0.$$

In order for the differential to square to zero, we define a correction to the differential as

$$d_S = -\sum_{k=1}^{m} q_k \frac{\partial}{\partial y_k}.$$

Lifting the Berkovits differential (IV)

Proposition

$(d_C + d_K + d_{Ber} + d_S)^2 = 0.$

The Resolution of $\mathbb{C}$ inside Y (I)

Proposition

The subalgebra T of Y given by:

$$\mathrm{Ch}(L_1, L_2) \otimes \bigwedge(\theta_1, \ldots, \theta_n) \otimes \mathbb{C}[y_1, \ldots, y_m]$$

is a resolution of $\mathbb{C}$.

This subcomplex is equipped with the differential

$$\sum_{i=1}^{n} a_i \frac{\partial}{\partial \theta_i} + \sum_{k=1}^{m} \left(\sum_{i,j=1}^{n} \Gamma_{ij}^{k} a_i \theta_j - q_k \right) \frac{\partial}{\partial y_k} + \sum_{k=1}^{m} \sum_{i,j=1}^{n} \Gamma_{ij}^{k} a_i a_j \frac{\partial}{\partial q_k}.$$

The Resolution of $\mathbb{C}$ inside Y (II)

Step 1

The Koszul complex (P, d_K) of the sequence $\theta_1, \ldots, \theta_n$ in $\mathrm{Ch}(L_1)$ and the Koszul complex (Q, d_S) of $y_1, \ldots, y_m$ in $\mathrm{Ch}(L_2)$ are contractible, as is the product $(P \otimes Q, d_K + d_S)$.

Step 2

We perturb the differential on $P \otimes Q$ to the differential on T by using the homological perturbation lemma.

Step 3

The perturbed homotopy is well-defined, so T has a strong deformation retraction to $(\mathbb{C}, 0)$ concentrated in degree zero.

Proof of Main Theorem (I)

- We have that T is a subalgebra in Y.

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- We have that T is a subalgebra in Y.
- We showed in a Proposition above that T is a resolution of $\mathbb{C}$. This satisfies the conditions of the Key Lemma and Y is quasi-isomorphic to $Y/\langle \overline{T} \rangle$, where $\langle \overline{T} \rangle$ is the DG ideal in Y generated by the augmentation ideal of T. Hence, $H_i(Y) = H_i(Y/\langle \overline{T} \rangle) = H_i(\mathrm{Ch}(L_{\geq 3}))$.

Proof of Main Theorem (II)

- Now, consider the filtration of Y

$$\{0\} \subset F_0 Y \subset F_1 Y \subset \cdots \subset F_n Y \subset \dots,$$

given by

$$F_p Y_q := \sum_{j \leq p} \sum_{i+j=p+q} \mathrm{Ch}_i(L) \otimes \Big(\bigwedge (\theta_1, \dots, \theta_n) \otimes \mathbb{C}[y_1, \dots, y_m] \Big)_j .$$

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- The differential on the E_0-term of the spectral sequence associated to this filtration is d_C.
- Since, $\bigwedge(\theta_1, \ldots, \theta_n) \otimes \mathbb{C}[y_1, \ldots, y_m]$ is a vector space over $\mathbb{C}$, it is flat as a $\mathbb{C}$-module.

Proof of main theorem (III)

- As $\mathrm{Ch}(L)$ is a resolution for A, we can conclude that the E_1-term of the spectral sequence is contained in one line and is given by,

$$A \otimes \bigwedge(\theta_1, \ldots, \theta_n) \otimes \mathbb{C}[y_1, \ldots, y_m]$$

with precisely the Berkovits differential d_B.

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  with precisely the Berkovits differential d_B.
- Hence, the homology of this complex is also $H_*(C_b(A), d_B)$ the homology of the Berkovits complex.

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