An operadic view of Goncharov’s bialgebra of iterated integrals

Imma Gálvez
Universitat Politècnica de Catalunya

work in progress with
Ralph Kaufmann (Purdue) and Andy Tonks (London Metropolitan)
Outline

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Operads and cooperads in a monoidal category \((\mathcal{C}, \otimes, I)\)

An non-symmetric operad \(\mathcal{O}\) is a sequence of objects \(\mathcal{O}(n), n \geq 0\) with an identity \(u : I \rightarrow \mathcal{O}(1)\) and composition structures

\[
\circ_i : \mathcal{O}(k) \otimes \mathcal{O}(m) \longrightarrow \mathcal{O}(k + m - 1), \quad 1 \leq i \leq k,
\]

that are unital, \(\circ_1 (u \otimes \text{id}) = \text{id} = \circ_i (\text{id} \otimes u)\), and associative,

\[
(\neg \circ_i \neg) \circ j \neg = \begin{cases} 
\circ_i (- \circ_{j-i+1} \neg) & \text{if } i \leq j < m + i \\
((\neg \circ_j \neg) \circ_{i+n-1} \neg) & \text{if } 1 \leq j < i,
\end{cases} \quad \text{(23)}
\]

where (23) : \(\mathcal{O}(k) \otimes \mathcal{O}(m) \otimes \mathcal{O}(n) \cong \mathcal{O}(k) \otimes \mathcal{O}(n) \otimes \mathcal{O}(m)\).
Example: The endomorphism operad of an object $A$

If $A$ is an object in a monoidal closed category $(\mathcal{C}, \otimes, I, \text{Hom}_\mathcal{C})$, one has the endomorphism operad of all ‘multilinear’ maps on $A$,

$$\mathcal{E}nd_A(n) = \text{Hom}_\mathcal{C}(A \otimes^n, A)$$

with the obvious identity,

$$u : I \to \text{Hom}_\mathcal{C}(A, A)$$

and composition structures

$$\circ_i : \mathcal{E}nd(k) \otimes \mathcal{E}nd(m) \to \mathcal{E}nd(k + m - 1), \quad 1 \leq i \leq k,$$

substituting an $m$-ary map as the $i$th argument of a $k$-ary map.
The composition structures $\circ_i$, $i = 1, \ldots, k$, combine to form

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_k)$$

$$\xrightarrow{\circ_1 \otimes \text{id}} \mathcal{O}(m_1 + k - 1) \otimes \mathcal{O}(m_2) \otimes \cdots \otimes \mathcal{O}(m_k) \xrightarrow{\circ_{m_1 + 1} \otimes \text{id}} \cdots$$

$$\cdots \rightarrow \mathcal{O}(m_1 + \cdots + m_{k-1} + 1) \otimes \mathcal{O}(m_k)$$

$$\rightarrow \mathcal{O}(m_1 + \cdots + m_k)$$

satisfying unital and associative laws.

The $\circ_i$ operations can be recovered from $\gamma$ and the unit $u$,

$$\circ_i : \mathcal{O}(k) \otimes \mathcal{O}(m) \cong \mathcal{O}(k) \otimes I^{i-1} \mathcal{O}(m) \otimes I^{k-i}$$

$$\xrightarrow{\text{id} \otimes u^{i-1} \otimes \text{id} \otimes u^{k-i}} \mathcal{O}(k) \otimes \mathcal{O}(1)^{i-1} \mathcal{O}(m) \otimes \mathcal{O}(1)^{k-i}$$

$$\mathcal{O}(k + m - 1)$$
A cooperad \( C \) is a sequence of objects \( C(n), n \geq 0 \), with (counital and coassociative) decomposition structures

\[
\varsigma_i : C(k + m - 1) \longrightarrow C(k) \otimes C(m), \quad 1 \leq i \leq k,
\]

or, equivalently,

\[
\gamma : C(m_1 + \cdots + m_k) \longrightarrow C(k) \otimes C(m_1) \otimes \cdots \otimes C(m_k).
\]

for all \( m_1, \ldots, m_k \geq 0 \).
The simplicial category

- One denotes by $\Delta$ the category whose objects are the finite non-empty standard ordinals

$$[n] := \{0, 1, \ldots, n\}, \quad n \geq 0.$$ Morphisms are the non-decreasing maps between them.

- Among these maps one considers the following generators: for $n \geq 0$ and $i \in [n],$

$$\partial_n^i : [n - 1] \to [n]$$

is the only increasing injection which skips the value $i$, and

$$\sigma_n^i : [n + 1] \to [n]$$

is the only non-decreasing surjection that repeats $i$.

- One usually writes just $\partial^i$ and $\sigma^i$. 
Simplicial objects in a category $\mathbf{C}$

- A simplicial object is a (contravariant) functor
  \[ X : \Delta^{\text{op}} \to \mathbf{C}. \]

Common notation is
\[ X_n = X([n]), \quad X_\bullet = (X_n)_{n \geq 0} \]

- The maps $\partial^i$, $\sigma^i$ in $\Delta$ induce the so-called face maps
  \[ d_i^n := X(\partial^i_n) : X_n \to X_{n-1}, \quad i = 0, \ldots, n, \]
  and degeneracy maps
  \[ s_i^n := X(\sigma^i_n) : X_n \to X_{n+1}, \quad i = 0, \ldots, n. \]

One writes $d_i$ and $s_i$ if there is no danger of confusion.

- A simplicial map $X_\bullet \to Y_\bullet$ is a natural transformation, that is: a family of morphisms $(X_n \to Y_n)_{n \geq 0}$ in $\mathbf{C}$ that commute with the face and degeneracy maps.

- Simplicial objects in $\mathbf{C}$ form a category $\mathbf{C_\Delta} := \text{Fun}(\Delta^{\text{op}}, \mathbf{C})$. 
The standard simplices

- The **standard simplices** are the simplicial sets

\[ \Delta[n] = \text{Hom}_\Delta(\cdot, [n]) : \Delta^{\text{op}} \to \text{Set} \]

- Thus the \( k \)-simplices of \( \Delta[n] \) are non-decreasing functions

\[ [k] \to [n] \]

- By the Yoneda Lemma, for any simplicial set \( X \),

\[ \text{Hom}_{\text{Set}_\Delta}(\Delta[k], X) \cong X_k \]

- In particular

\[ \text{Hom}_{\text{Set}_\Delta}(\Delta[k], \Delta[n]) \cong \Delta[n]_k \cong \text{Hom}_\Delta([k], [n]) \]
The standard simplices as an operad

**Proposition**

The sequence \((\Delta[n])_{n \geq 0}\), forms an operad in \((\text{Set}_\Delta, \cup, \emptyset)\), with

\[
\emptyset \xrightarrow{u} \Delta[1] \quad \Delta[k] \cup \Delta[m] \xrightarrow{\circ_i} \Delta[k + m - 1]
\]

For \(i = 1, \ldots, k\), the partial composition \(\circ_i\) sends

- vertex \(a\) of \(\Delta[k]\) \(\mapsto\) vertex

\[
\begin{cases} 
    a\text{ of }\Delta[k + m - 1] & \text{if } a \leq i - 1 \\
    a + m - 1\text{ of }\Delta[k + m - 1] & \text{if } a \geq i
\end{cases}
\]

- vertex \(a\) of \(\Delta[m]\) \(\mapsto\) vertex \(a + i - 1\) of \(\Delta[k + m - 1]\).
Simplicial sets as cooperads

Let $X$ be any simplicial set.
Since $X_n \cong \text{Hom}_{\text{Set}_{\Delta}}(\Delta[n], X)$ and $(\Delta[n])$ is an operad, we have by adjunction:

**Proposition**

The sequence $(X_n)_{n \geq 0}$, forms a cooperad in $(\text{Set}, \times, \{*\})$, with the counit and cocomposition structures given by

![Diagram]

If $x \in X_m$ then we write $x_{(\alpha(0), \ldots, \alpha(r))}$ for $X \left( [r] \xrightarrow{\alpha} [m] \right) (x) \in X_r$. 
• As usual the partial cocompositions \( \tilde{\psi}_i \) determine, and are determined by, the cooperad structure \( \tilde{\gamma} \),

\[
X_n \xrightarrow{\tilde{\gamma}} \prod_{m_1 + \cdots + m_k = n} X_k \times X_{m_1} \times X_{m_2} \times \ldots X_{m_k}
\]

• Rewriting the indices as \( i_0 = 0, \ i_1 = m_1, \) and \( i_{r+1} - i_r = m_{r+1} \)

\[
X_n \xrightarrow{\tilde{\gamma}} \prod_{0 = i_0 \leq i_1 \leq \cdots \leq i_k = n} X_k \times X_{m_1} \times X_{m_2} \times \ldots X_{m_k}
\]

\[
X \xrightarrow{\cdot} \left( X(i_0, i_1, \ldots, i_k), X(i_0, i_0 + 1, \ldots, i_1), X(i_1, i_1 + 1, \ldots, i_2), \ldots, X(i_{k-1}, i_{k-1} + 1, \ldots, i_k) \right)
\]
Bialgebras from cooperads with multiplication

**Definition + Theorem**

A (non-symmetric, unital) cooperad $(\mathcal{C}, \tilde{\gamma})$ with an associative compatible multiplication $\{\mu_{n,n'} : \mathcal{C}(n) \otimes \mathcal{C}(n') \to \mathcal{C}(n + n')\}$ determines a bialgebra structure $(\bigoplus \mathcal{C}(n), \mu, \delta)$.

Compatibility means that for all $n = \sum_{r=1}^{k} m_r$ and $n' = \sum_{r'=1}^{k'} m'_{r'}$,

\[
\mathcal{C}(n) \otimes \mathcal{C}(n') \xrightarrow{\tilde{\gamma} \otimes 2} \mathcal{C}(k) \otimes \bigotimes_{r=1}^{k} \mathcal{C}(m_r) \otimes \mathcal{C}(k') \otimes \bigotimes_{r'=1}^{k'} \mathcal{C}(m'_{r'})
\]

\[
\mathcal{C}(n + n') \xrightarrow{\tilde{\gamma}} \mathcal{C}(k + k') \otimes \bigotimes_{r=1}^{k} \mathcal{C}(m_r) \otimes \bigotimes_{r'=1}^{k'} \mathcal{C}(m'_{r'})
\]

\[
\mu_{n,n'} \Downarrow \quad \mu_{k,k'} \otimes \text{id} \Downarrow
\]

\[
\mathcal{C}(n + n') \xrightarrow{\tilde{\gamma}} \mathcal{C}(n + n') \xrightarrow{\tilde{\gamma}} \mathcal{C}(k + k') \otimes \bigotimes_{r=1}^{k} \mathcal{C}(m_r) \otimes \bigotimes_{r'=1}^{k'} \mathcal{C}(m'_{r'})
\]
Definition of the comultiplication

The comultiplication $\delta$ on $\bigoplus \mathcal{C}(n)$ is defined as follows, using the associative multiplication $\mu$ and the cooperad structure $\tilde{\gamma}$

\[ \delta = (\text{id} \otimes \mu)\tilde{\gamma} \]

\[
\begin{align*}
\mathcal{C}(n) & \xrightarrow{\tilde{\gamma}} \bigoplus_{k \geq 1, \ n = m_1 + \cdots + m_k} \left( \mathcal{C}(k) \otimes \bigotimes_{r=1}^{k} \mathcal{C}(m_r) \right) \\
& \xrightarrow{\text{id} \otimes \mu} \bigoplus_{k \geq 1} \mathcal{C}(k) \otimes \mathcal{C}(n)
\end{align*}
\]
Check that $\bigoplus C(n)$, $\mu$, $\delta = (\text{id} \otimes \mu)\gamma$ forms a bialgebra.

For example, the bialgebra axiom $\delta \mu = \mu \otimes^2 \delta \otimes^2$ may be written:

\[
\begin{array}{ccc}
C(n) \otimes C(n') & \xrightarrow{\delta \otimes^2} & C(k) \otimes C(n') \\
\downarrow{\mu} & & \downarrow{\mu \otimes^2} \\
C(n+n') & \xrightarrow{\delta} & C(k+k') \otimes C(n+n'),
\end{array}
\]

by the compatibility square above and an associativity square for $\mu$:

\[
\begin{array}{ccc}
C(k) \otimes \bigotimes_{r=1}^{k} C(m_r) \otimes C(k') \otimes \bigotimes_{r'=1}^{k'} C(m'_{r'}) & \xrightarrow{(\text{id} \otimes \mu) \otimes^2} & C(k) \otimes C(n) \\
\downarrow{\mu \otimes \text{id} \otimes \pi} & & \downarrow{\mu \otimes^2} \\
C(k+k') \otimes \bigotimes_{r=1}^{k} C(m_r) \otimes \bigotimes_{r'=1}^{k'} C(m'_{r'}) & \xrightarrow{\text{id} \otimes \mu} & C(k+k') \otimes C(n+n').
\end{array}
\]
Simplicial monoids as algebras

- Let $X$ be a simplicial set, and let $A$ be the chain complex

$$A = C_*(X, \mathbb{Q})$$

That is, $A_n$ is the vector space with basis $X_n$.

- If $X$ is a simplicial monoid then $A$ becomes a differential graded algebra, using the shuffle product,

$$\mu : A \otimes A = C_*(X, \mathbb{Q}) \otimes C_*(X, \mathbb{Q}) \xrightarrow{\text{shuf}} C_*(X \times X, \mathbb{Q}) \xrightarrow{\text{mult}} C_*(X, \mathbb{Q}) = A.$$
Simplicial monoids as bialgebras

- The cooperad structure associated to $X$ that we saw above induces a cooperad structure on the chain complex $A$,

$$\tilde{\gamma} : A_n \rightarrow \bigoplus_{m_1 + \cdots + m_k = n} A_k \otimes A_{m_1} \otimes \cdots \otimes A_{m_k}$$

- We may take the reduced complex $A = \overline{C}_*(X)$, $A_0 = 0$, so that the sum is finite.

- We may take the desuspended complex $A = \overline{C}_*(X)[1]$ so that $\tilde{\gamma}$ is a graded map of degree $-1$ for all $n$.

- The multiplication $\mu$ given by the shuffle map is compatible with the cooperad structure $\tilde{\gamma}$. 
Proposition (From simplicial monoids to bialgebras)

Let $A$ be the chain complex on a simplicial monoid $X$. Then $A_\bullet = \bigoplus A_n$ has a bialgebra structure with the multiplication given by the shuffle product and the comultiplication given by

\[\delta : A_n \xrightarrow{\gamma} \bigoplus A_k \otimes A_{m_1} \otimes A_{m_2} \otimes \cdots \otimes A_{m_k} \xrightarrow{id \otimes \mu} \bigoplus_k A_k \otimes A_n\]

\[0 = i_0 < i_1 < \cdots < i_k = n\]

\[X \longmapsto \sum X(i_0, i_1, \ldots, i_k) \otimes X(i_0, i_0+1, \ldots, i_1) \cdot X(i_1, i_1+1, \ldots, i_2) \cdots X(i_{k-1}, i_{k-1}+1, \ldots, i_k)\]
Generalisations and observations

- Note that this construction does not use the boundary maps of the chain complex $A$, only the fact that it is a family of vector spaces.

- The construction does not use all of the monoid structure of $X$, in fact the product $xy$ is only needed if the final vertex of $x$ coincides with the initial vertex of $y$.

- We could therefore state the proposition instead for $\Delta_\bullet$-categories, where $\Delta_\bullet$ is the subcategory of $\Delta$ containing only the end-point-preserving maps.
Adams’ cobar equivalence

Let $X$ be a 1-reduced simplicial set, $X_0 = X_1 = \{\ast\}$.

The chain complex $A = (C_*(X, \mathbb{Q}), d)$ is a differential graded coalgebra, with the comultiplication given by

$$\Delta(x_{(0,1,\ldots,n)}) = \sum_{k=0}^{n} x_{(0,\ldots,k)} \otimes x_{(k,\ldots,n)} \quad \text{(Alexander-Whitney)}$$

The cobar construction on a differential graded coalgebra $(A, \Delta)$ is the (free) differential graded algebra

$$\Omega A = T(A[1]), \quad d_\Omega = d + \Delta.$$  

Theorem (Adams, 1956)

The homology groups of $\Omega(C_\ast X)$ are naturally isomorphic to those of the loop space on (the geometric realisation of) $X$. 

Baues’ cobar comultiplication

- The cooperad structure on $X$ induces a one on $\Omega(C_*X)$, and the (free) multiplication $\mu$ is compatible with it.
- We thus have a comultiplication $\delta$ on the cobar construction

$$\delta : T(C_*X[1]) \longrightarrow T(C_*X[1]) \otimes T(C_*X[1])$$

$$x \longmapsto \sum x(i_0,i_1,\ldots,i_k) \otimes$$

$$x(i_0,i_0+1,\ldots,i_1)x(i_1,i_1+1,\ldots,i_2) \cdots x(i_{k-1},i_{k-1}+1,\ldots,i_k)$$

- This coincides with the d.g. bialgebra structure of Baues.
- One can apply the cobar construction to the coalgebra $\Omega C_*X$:

**Theorem (Baues, 1981)**

If $X$ has trivial 2-skeleton then the homology groups of $\Omega \Omega C_*X$ are naturally isomorphic to those of the double loop space on $X$
Goncharov’s bialgebra $\mathfrak{I}(S)$

- Let $S$ be any set, and let $\mathfrak{S}$ be the trivial groupoid with object set $S$ and exactly one arrow $s_1 \to s_2$ for all $(s_1, s_2) \in S^2$.

- E.g. $[n]$ is the fundamental groupoid of the $n$-simplex,

$$S = [n] \Rightarrow \mathfrak{S} = \pi_1(\Delta[n]),$$

- Let $X$ be $\text{Ner}(S)$, the simplicial nerve of the groupoid $S$.

- Explicitly, the $(n + 1)$-simplices of $X$ are tuples of elements

$$X_{n+1} \cong \{ \Pi(s_0; s_1, \ldots, s_n; s_{n+1}) : s_r \in S \} \cong S^{n+2}.$$
Let $A$ be the free commutative algebra on $X = \text{Ner}(S)$ modulo the 1-skeleton. The (free) multiplication $\mu$ is compatible with the cooperad structure $\tilde{\gamma}$ on $A$ induced from $X$.

We therefore have a comultiplication $(\text{id} \otimes \mu) \tilde{\gamma} : A \to A \otimes A$, sending a generator $\mathbb{I}(s_0; s_1, \ldots; s_n)$ of $A$ to

$$\sum_{k \geq 1} \mathbb{I}(s_{i_0}; s_{i_1}, \ldots; s_{i_k})$$

$$0 = i_0 \leq i_1 \leq \cdots \leq i_k = n$$

$$\otimes \mathbb{I}(s_{i_0}; s_{i_0+1}, \ldots; s_{i_1}) \mathbb{I}(s_{i_1}; s_{i_1+1}, \ldots; s_{i_2}) \cdots \mathbb{I}(s_{i_{k-1}}; s_{i_{k-1}+1}, \ldots; s_{i_k})$$

The bialgebra $(A, \mu, \delta = (\text{id} \otimes \mu) \tilde{\gamma})$ coincides with the bialgebra $(\mathcal{F}(S), \cdot, \Delta)$ of iterated integrals of Goncharov.
Cubical structure

- Baues’ and Goncharov’s comultiplications come from path or loop spaces and may be seen having natural cubical structure.
- The space of paths $P$ from 0 to $n$ in the $n$-simplex $|\Delta[n]|$ is a cell complex homeomorphic to the $(n - 1)$-dimensional cube.
- Cubical complexes have a natural diagonal approximation,

$$
\delta : P = [0, 1]^{n-1} \overset{\sim}{\longrightarrow} \bigcup_{K \cup L = \{1, \ldots, n-1\}} \partial^- K [0, 1]^{n-1} \times \partial^+ L [0, 1]^{n-1} \subseteq P \times P
$$

- One can identify faces $\partial^-_i$ of the cube $P$ as the spaces of paths through the faces $x_{(0, \ldots, \hat{i}, \ldots, n)}$ of the $n$-simplex $x$.
- Faces $\partial^+_i$ are products of a $(i - 1)$-cube and $(n - i - 1)$-cube: the spaces of paths through $x_{(0, \ldots, i)}$ and through $x_{(i, \ldots, n)}$.
- The term for $L = \{i_1, \ldots, i_{k-1}\}$ under this identification is

$$
X_{(0,i_1,\ldots,i_{k-1},n)} \times X_{(0,1,\ldots,i_1)} X_{(i_1,i_1+1,\ldots,i_2)} \cdots X_{(i_{k-1},i_{k-1}+1,\ldots,n)}.
$$