An operadic view of Goncharov's bialgebra of iterated integrals

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work in progress with
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Outline

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Operads and cooperads in a monoidal category $(\mathbf{C}, \otimes, I)$

An non-symmetric operad $\mathscr{O}$ is a sequence of objects $\mathscr{O}(n)$, $n \geq 0$ with an identity $u : I \to \mathscr{O}(1)$ and composition structures

$$\circ_i \colon \mathscr{O}(k) \otimes \mathscr{O}(m) \longrightarrow \mathscr{O}(k+m-1), \qquad 1 \leq i \leq k,$$

that are unital, $\circ_1(u \otimes \mathrm{id}) = \mathrm{id} = \circ_i(\mathrm{id} \otimes u)$, and associative,

$$(- \circ_i -) \circ_j - = \begin{cases} - \circ_i (- \circ_{j-i+1} -) & \text{if } i \leq j < m+i \\ ((- \circ_j -) \circ_{i+n-1} -)(23) & \text{if } 1 \leq j < i, \end{cases}$$

where $(23) : \mathscr{O}(k) \otimes \mathscr{O}(m) \otimes \mathscr{O}(n) \cong \mathscr{O}(k) \otimes \mathscr{O}(n) \otimes \mathscr{O}(m)$.

Example: The endomorphism operad of an object A

If A is an object in a monoidal closed category $(\mathbf{C}, \otimes, I, \mathrm{Hom}_{\mathbf{C}})$, one has the endomorphism operad of all 'multilinear' maps on A,

$$\mathscr{E}nd_A(n) = \mathrm{Hom}_{\mathbf{C}}(A^{\otimes n}, A)$$

with the obvious identity,

$$u : I \to \mathrm{Hom}_{\mathbf{C}}(A, A)$$

and composition structures

$$\circ_i \colon \mathscr{E}nd(k) \otimes \mathscr{E}nd(m) \longrightarrow \mathscr{E}nd(k+m-1), \qquad 1 \le i \le k,$$

substituting an m-ary map as the ith argument of a k-ary map.

- The composition structures $\circ_i$, $i = 1, \ldots, k$, combine to form

$$\gamma : \mathscr{O}(k) \otimes \mathscr{O}(m_1) \otimes \cdots \otimes \mathscr{O}(m_k)$$
$$\xrightarrow{\circ_1 \otimes \mathrm{id}} \mathscr{O}(m_1 + k - 1) \otimes \mathscr{O}(m_2) \otimes \cdots \otimes \mathscr{O}(m_k) \xrightarrow{\circ_{m_1+1} \otimes \mathrm{id}} \cdots$$
$$\cdots \longrightarrow \mathscr{O}(m_1 + \cdots + m_{k-1} + 1) \otimes \mathscr{O}(m_k)$$
$$\longrightarrow \mathscr{O}(m_1 + \cdots + m_k)$$

satisfying unital and associative laws.

- The $\circ_i$ operations can be recovered from γ and the unit u,

$$\circ_i : \mathscr{O}(k) \otimes \mathscr{O}(m) \cong \mathscr{O}(k) \otimes I^{\otimes i-1} \mathscr{O}(m) \otimes I^{\otimes k-i}$$
$$\xrightarrow{\mathrm{id} \otimes u^{\otimes i-1} \otimes \mathrm{id} \otimes u^{k-i}} \mathscr{O}(k) \otimes \mathscr{O}(1)^{\otimes i-1} \mathscr{O}(m) \otimes \mathscr{O}(1)^{\otimes k-i}$$
$$\xrightarrow{\quad\gamma\quad} \mathscr{O}(k + m - 1)$$

A cooperad $\mathscr{C}$ is a sequence of objects $\mathscr{C}(n)$, $n \geq 0$, with (counital and coassociative) decomposition structures

$$\check{\circ}_i \colon \mathscr{C}(k+m-1) \longrightarrow \mathscr{C}(k) \otimes \mathscr{C}(m), \qquad 1 \leq i \leq k,$$

or, equivalently,

$$\check{\gamma} \colon \mathscr{C}(m_1 + \cdots + m_k) \longrightarrow \mathscr{C}(k) \otimes \mathscr{C}(m_1) \otimes \cdots \otimes \mathscr{C}(m_k).$$

for all $m_1, \ldots, m_k \geq 0$.

The simplicial category

- One denotes by $\mathbf{\Delta}$ the category whose objects are the finite non-empty standard ordinals

$$[n] := \{0, 1, \cdots, n\}, \qquad n \geq 0.$$

  Morphisms are the non-decreasing maps between them.

- Among these maps one considers the following generators: for $n \geq 0$ and $i \in [n]$,

$$\partial_n^i : [n-1] \to [n]$$

  is the only increasing injection which skips the value i, and

$$\sigma_n^i : [n+1] \to [n]$$

  is the only non-decreasing surjection that repeats i.

- One usually writes just ∂^i and σ^i.

Simplicial objects in a category $\mathbf{C}$

- A simplicial object is a (contravariant) functor

$$X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{C}.$$

Common notation is

$$X_n = X([n]), \qquad X_\bullet = (X_n)_{n \geq 0}$$

- The maps ∂^i, σ^i in $\mathbf{\Delta}$ induce the so-called face maps

$$d_i^n := X(\partial_n^i) : X_n \to X_{n-1}, \quad i = 0, \ldots, n,$$

and degeneracy maps

$$s_i^n := X(\sigma_n^i) : X_n \to X_{n+1}, \quad i = 0, \ldots, n.$$

One writes d_i and s_i if there is no danger of confusion.

- A simplicial map $X_\bullet \to Y_\bullet$ is a natural transformation, that is: a family of morphisms $(X_n \to Y_n)_{n \geq 0}$ in $\mathbf{C}$ that commute with the face and degeneracy maps.
- Simplicial objects in $\mathbf{C}$ form a category $\mathbf{C}_{\mathbf{\Delta}} := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{C})$.

The standard simplices

- The standard simplices are the simplicial sets

$$\Delta[n] = \mathrm{Hom}_{\mathbf{\Delta}}(-, [n]) : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$$

- Thus the k-simplices of $\Delta[n]$ are non-decreasing functions

$$[k] \to [n]$$

- By the Yoneda Lemma, for any simplicial set X,

$$\mathrm{Hom}_{\mathbf{Set}_{\mathbf{\Delta}}}(\Delta[k], X) \cong X_k$$

- In particular

$$\mathrm{Hom}_{\mathbf{Set}_{\mathbf{\Delta}}}(\Delta[k], \Delta[n]) \cong \Delta[n]_k \cong \mathrm{Hom}_{\mathbf{\Delta}}([k], [n])$$

The standard simplices as an operad

Proposition

The sequence $(\Delta[n])_{n\geq 0}$, forms an operad in $(\mathbf{Set_\Delta}, \cup, \varnothing)$, with

$$\varnothing \xrightarrow{u} \Delta[1]$$

$$\Delta[k] \cup \Delta[m] \xrightarrow{\circ_i} \Delta[k+m-1]$$

For $i = 1, \ldots, k$, the partial composition circle-i sends

- vertex a of $\Delta[k] \longmapsto$ vertex

$$\begin{cases} a \text{ of } \Delta[k+m-1] & \text{if } a \leq i-1 \\ a+m-1 \text{ of } \Delta[k+m-1] & \text{if } a \geq i \end{cases}$$

- vertex a of $\Delta[m] \longmapsto$ vertex $a+i-1$ of $\Delta[k+m-1]$.

Simplicial sets as cooperads

Let X be any simplicial set.
Since $X_n \cong \mathrm{Hom}_{\mathbf{Set}_{\boldsymbol{\Delta}}}(\Delta[n], X)$ and $(\Delta[n])$ is an operad, we have by adjunction:

Proposition

The sequence $(X_n)_{n\geq 0}$, forms a cooperad in $(\mathbf{Set}, \times, \{*\})$, with the counit and cocomposition structures given by

$$X_1 \xrightarrow{\quad \check{u} \quad} \{*\} \qquad \text{(counit)}$$

$$X_{k+m-1} \xrightarrow{\quad \check{\circ}_i \quad} X_k \times X_m \qquad (1 \leq i \leq k)$$

$$x \longmapsto \left(x_{(0,\ldots,i-1,\ i+m-1,\ldots,k+m-1)},\ x_{(i-1,\ldots,i+m-1)}\right)$$

If $x \in X_m$ then we write $x_{(\alpha(0),\ldots,\alpha(r))}$ for $X\Big([r] \xrightarrow{\alpha} [m]\Big)(x) \in X_r$.

- As usual the partial cocompositions $\check{\circ}_i$ determine, and are determined by, the cooperad structure $\check{\gamma}$,

$$X_n \xrightarrow{\quad\check{\gamma}\quad} \prod_{m_1+\cdots+m_k=n} X_k \times X_{m_1} \times X_{m_2} \times \ldots X_{m_k}$$

- Rewriting the indices as $i_0 = 0$, $i_1 = m_1$, and $i_{r+1} - i_r = m_{r+1}$

$$X_n \xrightarrow{\quad\check{\gamma}\quad} \prod_{0=i_0\leq i_1\leq\cdots\leq i_k=n} X_k \times X_{m_1} \times X_{m_2} \times \ldots X_{m_k}$$

$$x \longmapsto \big(x_{(i_0,i_1,\ldots,i_k)},$$

$$x_{(i_0,i_0+1,\ldots,i_1)}, x_{(i_1,i_1+1,\ldots,i_2)}, \ldots, x_{(i_{k-1},i_{k-1}+1,\ldots,i_k)}\big)$$

Bialgebras from cooperads with multiplication

Definition + Theorem

A (non-symmetric, unital) cooperad $(\mathscr{C}, \check{\gamma})$ with an associative compatible multiplication $\{\mu_{n,n'} : \mathscr{C}(n) \otimes \mathscr{C}(n') \to \mathscr{C}(n+n')\}$ determines a bialgebra structure $(\bigoplus \mathscr{C}(n),\ \mu, \delta)$.

Compatibility means that for all $n = \sum_{r=1}^{k} m_r$ and $n' = \sum_{r'=1}^{k'} m'_{r'}$,

$$\begin{array}{ccc}
\mathscr{C}(n) \otimes \mathscr{C}(n') & \xrightarrow{\check{\gamma}^{\otimes 2}} & \mathscr{C}(k) \otimes \bigotimes_{r=1}^{k} \mathscr{C}(m_r) \otimes \mathscr{C}(k') \otimes \bigotimes_{r'=1}^{k'} \mathscr{C}(m'_{r'}) \\
\downarrow{\scriptstyle \mu_{n,n'}} & & \downarrow{\scriptstyle \mu_{k,k'} \otimes \mathrm{id}} \\
\mathscr{C}(n+n') & \xrightarrow{\check{\gamma}} & \mathscr{C}(k+k') \otimes \bigotimes_{r=1}^{k} \mathscr{C}(m_r) \otimes \bigotimes_{r'=1}^{k'} \mathscr{C}(m'_{r'})
\end{array}$$

Definition of the comultiplication

The comultiplication δ on $\bigoplus \mathscr{C}(n)$ is defined as follows, using the associative multiplication μ and the cooperad structure $\check{\gamma}$

$$\mathscr{C}(n) \xrightarrow{\check{\gamma}} \bigoplus_{\substack{k\geq 1,\\ n=m_1+\cdots+m_k}} \left(\mathscr{C}(k) \otimes \bigotimes_{r=1}^{k} \mathscr{C}(m_r) \right)$$

$$\bigoplus_{\substack{k\geq 1,\\ n=m_1+\cdots+m_k}} \left(\mathscr{C}(k) \otimes \bigotimes_{r=1}^{k} \mathscr{C}(m_r) \right) \xrightarrow{\mathrm{id}\otimes\mu} \bigoplus_{k\geq 1} \mathscr{C}(k) \otimes \mathscr{C}(n)$$

$$\delta = (\mathrm{id} \otimes \mu)\check{\gamma} : \mathscr{C}(n) \to \bigoplus_{k\geq 1} \mathscr{C}(k) \otimes \mathscr{C}(n)$$

Check that $\bigoplus \mathscr{C}(n)$, $\mu, \delta = (\mathrm{id} \otimes \mu)\check{\gamma}$ forms a bialgebra

For example, the bialgebra axiom $\delta\mu = \mu^{\otimes 2}\delta^{\otimes 2}$ may be written:

$$\begin{array}{ccccc} \mathscr{C}(n) \otimes \mathscr{C}(n') & & \xrightarrow{\delta^{\otimes 2}} & & \begin{array}{c} \mathscr{C}(k) \otimes \mathscr{C}(n) \\ \otimes\, \mathscr{C}(k') \otimes \mathscr{C}(n') \end{array} \\ \Big\downarrow \mu & \text{compatibility} & \Big\downarrow & \text{associativity} & \Big\downarrow \mu^{\otimes 2} \\ \mathscr{C}(n+n') & & \xrightarrow[\delta]{} & & \mathscr{C}(k+k') \otimes \mathscr{C}(n+n'), \end{array}$$

by the compatibility square above and an associativity square for μ:

$$\begin{array}{ccc} \mathscr{C}(k) \otimes \bigotimes_{r=1}^{k} \mathscr{C}(m_r) \otimes \mathscr{C}(k') \otimes \bigotimes_{r'=1}^{k'} \mathscr{C}(m'_{r'}) & \xrightarrow{(\mathrm{id} \otimes \mu)^{\otimes 2}} & \begin{array}{c} \mathscr{C}(k) \otimes \mathscr{C}(n) \\ \otimes\, \mathscr{C}(k') \otimes \mathscr{C}(n') \end{array} \\ \Big\downarrow (\mu \otimes \mathrm{id})(\pi) & & \Big\downarrow \mu^{\otimes 2} \\ \mathscr{C}(k+k') \otimes \bigotimes_{r=1}^{k} \mathscr{C}(m_r) \otimes \bigotimes_{r'=1}^{k'} \mathscr{C}(m'_{r'}) & \xrightarrow{\mathrm{id} \otimes \mu} & \mathscr{C}(k+k') \otimes \mathscr{C}(n+n'). \end{array}$$

Simplicial monoids as algebras

- Let X be a simplicial set, and let A be the chain complex

$$A = C_*(X, \mathbb{Q})$$

  That is, A_n is the vector space with basis X_n.

- If X is a simplicial monoid then A becomes a differential graded algebra, using the shuffle product,

$$\mu : A \otimes A = C_*(X, \mathbb{Q}) \otimes C_*(X, \mathbb{Q}) \xrightarrow{\text{shuf}} C_*(X \times X, \mathbb{Q}) \xrightarrow{\text{mult}} C_*(X, \mathbb{Q}) = A.$$

Simplicial monoids as bialgebras

- The cooperad structure associated to X that we saw above induces a cooperad structure on the chain complex A,

$$\check{\gamma} : A_n \longrightarrow \bigoplus_{m_1+\cdots+m_k=n} A_k \otimes A_{m_1} \otimes \cdots \otimes A_{m_k}$$

- We may take the reduced complex $A = \overline{C}_*(X)$, $A_0 = 0$, so that the sum is finite.

- We may take the desuspended complex $A = \overline{C}_*(X)[1]$ so that $\check{\gamma}$ is a graded map of degree -1 for all n.

- The multiplication μ given by the shuffle map is compatible with the cooperad structure $\check{\gamma}$.

Proposition (From simplicial monoids to bialgebras)

Let A be the chain complex on a simplicial monoid X.
Then $A_\bullet = \bigoplus A_n$ has a bialgebra structure with the multiplication given by the shuffle product and the comultiplication given by

$$\delta : A_n \xrightarrow{\check{\gamma}} \bigoplus_{0=i_0<i_1<\cdots<i_k=n} A_k \otimes A_{m_1} \otimes A_{m_2} \otimes \cdots \otimes A_{m_k} \xrightarrow{\mathrm{id}\otimes\mu} \bigoplus_k A_k \otimes A_n$$

$$x \longmapsto \sum x_{(i_0,i_1,\ldots,i_k)} \otimes x_{(i_0,i_0+1,\ldots,i_1)} \cdot x_{(i_1,i_1+1,\ldots,i_2)} \cdots x_{(i_{k-1},i_{k-1}+1,\ldots,i_k)}$$

Generalisations and observations

- Note that this construction does not use the boundary maps of the chain complex A, only the fact that it is a family of vector spaces.
- The construction does not use all of the monoid structure of X, in fact the product xy is only needed if the final vertex of x coincides with the initial vertex of y.
- We could therefore state the proposition instead for $\mathbf{\Delta}_\bullet$-categories, where $\mathbf{\Delta}_\bullet$ is the subcategory of $\mathbf{\Delta}$ containing only the end-point-preserving maps.

Adams' cobar equivalence

- Let X be a 1-reduced simplicial set, $X_0 = X_1 = \{*\}$.
- The chain complex $A = (C_*(X, \mathbb{Q}), d)$ is a differential graded coalgebra, with the comultiplication given by

$$\Delta(x_{(0,1,\ldots,n)}) = \sum_{k=0}^{n} x_{(0,\ldots,k)} \otimes x_{(k,\ldots,n)} \quad \text{(Alexander-Whitney)}$$

- The cobar construction on a differential graded coalgebra (A, Δ) is the (free) differential graded algebra

$$\Omega A = T(\overline{A}[1]), \qquad d_\Omega = d + \Delta.$$

Theorem (Adams, 1956)

The homology groups of $\Omega(C_* X)$ are naturally isomorphic to those of the loop space on (the geometric realisation of) X

Baues' cobar comultiplication

- The cooperad structure on X induces a one on $\Omega(C_*X)$, and the (free) multiplication μ is compatible with it.
- We thus have a comultiplication δ on the cobar construction

$$\begin{aligned}\delta : T(\overline{C}_*X[1]) &\longrightarrow T(\overline{C}_*X[1]) \otimes T(\overline{C}_*X[1]) \\ x &\longmapsto \sum x_{(i_0,i_1,\ldots,i_k)} \otimes \\ &\qquad x_{(i_0,i_0+1,\ldots,i_1)} x_{(i_1,i_1+1,\ldots,i_2)} \cdots x_{(i_{k-1},i_{k-1}+1,\ldots,i_k)}\end{aligned}$$

- This coincides with the d.g. bialgebra structure of Baues.
- One can apply the cobar construction to the coalgebra ΩC_*X:

Theorem (Baues, 1981)

If X has trivial 2-skeleton then the homology groups of $\Omega\Omega C_*X$ are naturally isomorphic to those of the double loop space on X

Goncharov's bialgebra $\widetilde{\mathscr{I}}_{\bullet}(S)$

- Let S be any set, and let $\underline{S}$ be the trivial groupoid with object set S and exactly one arrow $s_1 \to s_2$ for all $(s_1, s_2) \in S^2$.
- E.g. $\underline{[n]}$ is the fundamental groupoid of the n-simplex,

$$S = [n] \quad \Rightarrow \quad \underline{S} = \pi_1(\Delta[n]),$$

- Let X be Ner($\underline{S}$), the simplical nerve of the groupoid $\underline{S}$.
- Explicitly, the $(n+1)$-simplices of X are tuples of elements

$$X_{n+1} \cong \{\mathbb{I}(s_0; s_1, \ldots, s_n; s_{n+1}) : s_r \in S\} \cong S^{n+2}.$$

- Let A be the free commutative algebra on $X = \mathrm{Ner}(\underline{S})$ modulo the 1-skeleton. The (free) multiplication μ is compatible with the cooperad structure $\check{\gamma}$ on A induced from X.

- We therefore have a comultiplication $(\mathrm{id} \otimes \mu)\check{\gamma} : A \to A \otimes A$, sending a generator $\mathbb{I}(s_0; s_1, \ldots; s_n)$ of A to

$$\sum_{\substack{k \geq 1 \\ 0 = i_0 \leq i_1 \leq \cdots \leq i_k = n}} \mathbb{I}(s_{i_0}; s_{i_1}, \ldots; s_{i_k})$$
$$\otimes\, \mathbb{I}(s_{i_0}; s_{i_0+1}, \ldots; s_{i_1})\mathbb{I}(s_{i_1}; s_{i_1+1}, \ldots; s_{i_2}) \cdots \mathbb{I}(s_{i_{k-1}}; s_{i_{k-1}+1}, \ldots; s_{i_k})$$

- The bialgebra $(A, \mu, \delta = (\mathrm{id} \otimes \mu)\check{\gamma})$ coincides with the bialgebra $(\widetilde{\mathscr{I}}_\bullet(S), \cdot, \Delta)$ of iterated integrals of Goncharov.

Cubical structure

- Baues' and Goncharov's comultiplications come from path or loop spaces and may be seen having natural cubical structure.
- The space of paths P from 0 to n in the n-simplex $|\Delta[n]|$ is a cell complex homeomorphic to the $(n-1)$-dimensional cube.
- Cubical complexes have a natural diagonal approximation,

$$\delta : P = [0,1]^{n-1} \xrightarrow{\cong} \bigcup_{K \cup L = \{1,\ldots,n-1\}} \partial_K^-[0,1]^{n-1} \times \partial_L^+[0,1]^{n-1} \xrightarrow{\subset} P \times P$$

- One can identify faces ∂_i^- of the cube P as the spaces of paths through the faces $x_{(0,\ldots,\hat{i},\ldots,n)}$ of the n-simplex x.
- Faces ∂_i^+ are products of a $(i-1)$-cube and $(n-i-1)$-cube: the spaces of paths through $x_{(0,\ldots,i)}$ and through $x_{(i,\ldots,n)}$.
- The term for $L = \{i_1, \ldots, i_{k-1}\}$ under this identification is

$$x_{(0,i_1,\ldots,i_{k-1},n)} \times x_{(0,1,\ldots,i_1)}\, x_{(i_1,i_1+1,\ldots,i_2)} \cdots x_{(i_{k-1},i_{k-1}+1,\ldots,n)}.$$