Exponentially small splitting of separatrices for the pendulum with fast periodic or quasiperiodic meromorphic perturbation

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Integrable Hamiltonian Systems with a fast periodic or quasiperiodic perturbation

Consider a non-autonomous perturbation of a one degree of freedom hamiltonian

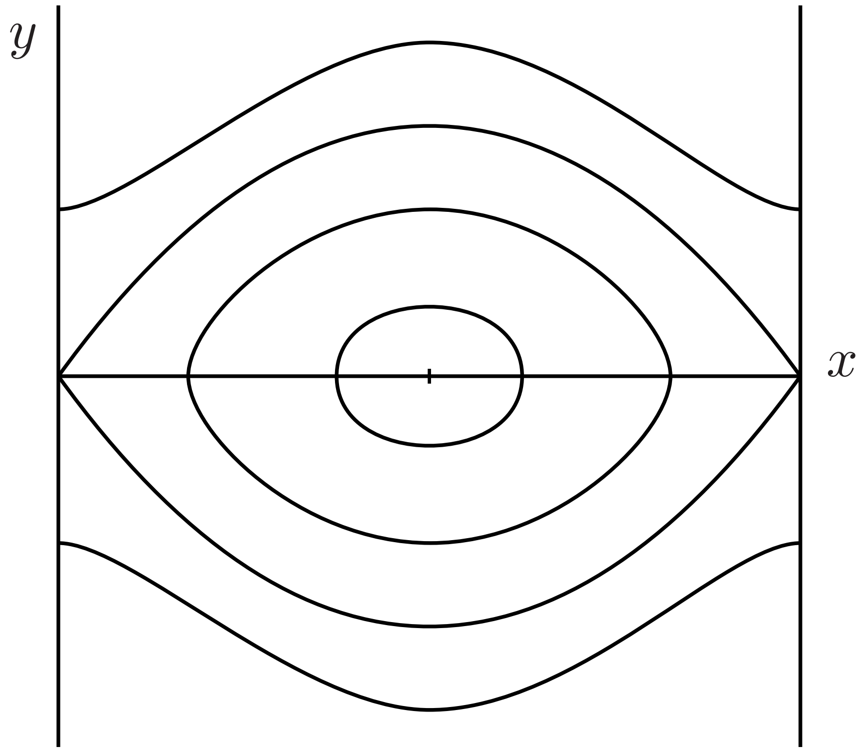
$$H\left(x, y, \frac{t}{\varepsilon}\right) = H_0(x, y) + \mu\varepsilon^\eta H_1\left(x, y, \frac{t}{\varepsilon}\right)$$

such that $\eta \geq 0$, $\varepsilon \ll 1$ and μ is a parameter not necessarily small.

Assume

- H is analytic.
- H_0 has a hyperbolic fixed point whose stable and unstable invariant manifolds coincide along a separatrix.
- H_1 depends periodically or quasiperiodically on t and has zero average

Unperturbed Hamiltonian H_0



- Phase space: $(x, y) \in \mathbb{T}^1 \times \mathbb{R}$.
- It has an hyperbolic fixed point at $(0, 0)$ with two separatrices.

- What happens to the (upper) separatrix when we add the perturbation?
- We know that the perturbed invariant manifolds are exponentially close (Neisthadt, Simó).
- We want
  - To obtain an asymptotic formula for the distance between the perturbed invariant manifolds.
  - This will allow us to see when the separatrix breaks down creating transversal intersections

- This problem has been studied thoroughly in the last decades.
- Most of the results in the literature assume that the Hamiltonian has trigonometric or algebraic polynomial dependence on the state variables (x, y).
- Nevertheless, in many models, for instance in Celestial Mechanics, the Hamiltonian functions are not entire but have a finite strip of analyticity.
- We consider H_1 meromorphic in x and we want to see how the distance between the perturbed invariant manifolds depends on the width of the analyticity strip.
- We focus our study in a particular example.

Model:

$$\ddot{x} = \sin x + \mu\varepsilon^{\eta} \frac{\sin x}{(1+\alpha \sin x)^2} f\left(\frac{t}{\varepsilon}\right),$$

where $f(\tau)$ is an analytic function which depends either periodically or quasiperiodically on τ.

- Taking $y = \dot{x}$ we have a Hamiltonian system with Hamiltonian

  $$H\left(x, y, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + \cos x - 1 + \mu\varepsilon^{\eta}\psi(x) f\left(\frac{t}{\varepsilon}\right),$$

  where $\psi(x)$ is a primitive of $-\sin x/(1+\alpha \sin x)^2$.

- $(0,0)$ is a hyperbolic critical point for the perturbed system.

- The perturbation is meromorphic in x and has analyticity strip of width

$$|\mathrm{Im}\, x| \le \ln\left(\frac{1+\sqrt{1-\alpha^2}}{\alpha}\right).$$

- Then,
  - When $\alpha = 0$ the perturbation is a trigonometric polynomial in x. This case has been studied by Treshev (periodic case) and Delshams, Gelfreich, Jorba and Seara (quasiperiodic case) for different choices of f.
  - When $\alpha = 1$ the perturbation is not defined in the whole real line.
  - We want to study the splitting for any $\alpha \in (0,1)$.
  - We first deal with periodic perturbations and later we will deal with quasiperiodic ones.

Periodic case

- Model:

$$\ddot{x} = \sin x + \mu\varepsilon^{\eta} \frac{\sin x}{(1+\alpha \sin x)^2} \sin \frac{t}{\varepsilon}$$

- We measure the maximal distance between the invariant manifolds in the section $x = \pi$, which we call d.

- First we review the previous results for $\alpha = 0$.

The polynomial case ($\alpha = 0$)

- If $\alpha = 0$,

$$H\left(x, y, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + \cos x - 1 + \mu\varepsilon^\eta(\cos x - 1)\sin\frac{t}{\varepsilon}.$$

- The unperturbed separatrix is $\gamma(u) = (x_0(u), \dot{x}_0(u))$ with

$$x_0(u) = 4\arctan(e^u)$$

which has singularities at $u = i\dfrac{\pi}{2} + ik\pi$, $k \in \mathbb{Z}$.

Classical Melnikov Theory

- We define the Melnikov function as:

$$M(t_0) = \int_{-\infty}^{+\infty} \{H_0, H_1\}(\gamma(u), (u + t_0)/\varepsilon)dt$$

where s corresponds to the time evolution through the separatrix.

- M can be computed through residuums method since H_0, H_1 and γ are known.

- It essentially gives the first order in μ of the distance between the perturbed invariant manifolds.

- In our example

$$d = 2\pi\mu\varepsilon^{\eta-2}e^{-\frac{\pi}{2\varepsilon}} + \mathcal{O}\left(\mu^2\varepsilon^{2\eta}\right).$$

- Therefore, in order to be valid the Melnikov prediction, μ has to be exponentially small with respect to ε.

Theorem (Treschev 97) For ε sufficiently small and $\mu < \mu_0$,

- If $\eta > 0$ (regular case), Melnikov predicts correctly the distance:

$$d = 2\pi\mu\varepsilon^{\eta-2}e^{-\frac{\pi}{2\varepsilon}}\left(1 + \mathcal{O}\left(\mu\varepsilon^{\eta}\right)\right)$$

- If $\eta = 0$ (singular case), Melnikov fails to predict correctly but the distance is exponentially small

$$d = \varepsilon^{-2}e^{-\frac{\pi}{2\varepsilon}}\left(|f(\mu)| + \mu\,\mathcal{O}\left(\varepsilon\ln(1/\varepsilon)\right)\right)$$

  where $f(\mu) = 2\pi i\mu + \mathcal{O}(\mu^3)$ is an analytic function.

- In both cases, the exponential small coefficient is given by $\pi/2$, which is the imaginary part of the singularity of the separatrix closest to the real axis.

This result has been generalized (Baldomá, Fontich, Guardia and Seara) to Hamiltonian Systems of the form

$$H(x, y, t/\varepsilon) = \frac{y^2}{2} + V(x) + \mu\varepsilon^\eta H_1(x, y, t/\varepsilon).$$

such that H has polynomial dependence on (x, y).

Under certain assumptions, there exists $\eta^* > 0$ such that,

- If $\eta > \eta^*$ (regular case), the Melnikov function predicts correctly the first order for the distance.
- If $\eta = \eta^*$ (singular case), the Melnikov function fails predicts correctly the first order for the distance.
- In both cases,

$$d \sim \varepsilon^\beta e^{-\frac{a}{\varepsilon}}$$

where a is the imaginary part of the singularities of the parameterization of the separatrix which are closest to the real axis.

Meromorphic perturbation: $\alpha \in (0,1)$

Questions:

- Which is the size of the Melnikov function?
- Which is the size of the the splitting of separatrices?
- When the strip is of the form $\sigma \sim \ln(1/\alpha)$ with $\alpha \ll 1$, can we expand the perturbation in α and just consider the first order? or all the orders make a contribution to the first order of the difference between manifolds?

We focus our study in the Regular case, for which the Melnikov function predicts correctly the splitting.

Computation of the Melnikov function

- Model:

$$\ddot{x} = \sin x + \mu\varepsilon^{\eta} \frac{\sin x}{(1 + \alpha \sin x)^2} \sin \frac{t}{\varepsilon}$$

- Melnikov function:

$$\begin{aligned} M(t_0) &= \int_{-\infty}^{+\infty} y(u) \frac{\sin x(u)}{(1 + \alpha \sin x(u))^2} \sin \left(\frac{u + t_0}{\varepsilon} \right) du \\ &= 4 \int_{-\infty}^{+\infty} \frac{\sinh u \cosh u}{(\cosh^2 u - 2\alpha \sinh u)^2} \sin \left(\frac{u + t_0}{\varepsilon} \right) du \end{aligned}$$

- The first order of this integral can be computed using residues theorem.

Computation of the Melnikov function (II)

- If $\alpha = \mathcal{O}(\varepsilon^\nu)$ with $\nu > 2$: The integral is uniformly convergent in the reals.
- We expand $M(t_0)$ in power series of α:

$$M(t_0) = 4\sum_{k=0}^{\infty}(k+1)2^k\alpha^k \int_{-\infty}^{+\infty} \frac{\sinh^{k+1} u}{\cosh^{2k+3} u} \sin\left(\frac{u+t_0}{\varepsilon}\right) du.$$

- Its first term gives the bigger contribution to the Melnikov prediction for the distance

$$d \sim 2\pi\mu\varepsilon^{\eta-2}e^{-\frac{\pi}{2\varepsilon}}.$$

- The exponential coefficient $\pi/2$ is the imaginary part of the complex singularity of the separatrix.
- Conclusion: If the analyticity strip of the perturbation is big enough ($\alpha \ll \varepsilon^2$), the size of the Melnikov function is given as in the polynomial perturbation case.

Computation of the Melnikov function (III)

- If $\alpha = \mathcal{O}(\varepsilon^\nu)$ with $\nu \in [0, 2]$, the integral of the summands is bigger as k increases.

- We look for the singularities of the integrand of

$$M(t_0) = 4 \int_{-\infty}^{+\infty} \frac{\sinh u \cosh u}{\left(\cosh^2 u - 2\alpha \sinh u\right)^2} \sin\left(\frac{u + t_0}{\varepsilon}\right) du$$

- Consider $u^* = \zeta \pm i\rho$ singularities of the integrand closest to the reals.

- If α is small $\rho = \pm\left(\frac{\pi}{2} - \sqrt{\alpha} + \mathcal{O}(\alpha)\right)$.

- If α is fixed and independent of ε, ρ is also independent of ε and is unrelated to the singularities of the separatrix.

Computation of the Melnikov function (IV)

- Then, the Melnikov prediction for the distance is

$$d = C\mu \frac{\varepsilon^{\eta-1}}{\sqrt{\alpha}} e^{-\frac{\rho}{\varepsilon}}$$

- If, for instance, one takes $\alpha = \varepsilon$,

$$d = C\varepsilon^{\eta-\frac{3}{2}} e^{-\frac{\pi-2\sqrt{\varepsilon}}{2\varepsilon}}$$

- In these cases,
  - Even if α is small, the first order of the Melnikov function is given by the full jet in α of the perturbation.
  - The splitting has bigger size than in the polynomial case.

Validity of the Melnikov prediction

- If $\alpha = \mathcal{O}(\varepsilon^\nu)$ with $\nu > 2$ and ε is small enough: the Melnikov function predicts correctly the splitting provided $\eta > 0$.
- The limit case $\eta = 0$ (integrable system and perturbation of the same order) is expected to have exponentially small splitting of separatrices which is not well predicted by the Melnikov function.
- If $\alpha = \mathcal{O}(\varepsilon^\nu)$ with $\nu \in [0, 2]$, ε is small enough and $\alpha < 1$: Melnikov function predicts correctly the splitting provided $\eta + \dfrac{\nu}{2} - 1 > 0$.
- The limit case $\eta + \dfrac{\nu}{2} - 1 = 0$ is expected to have exponentially small splitting of separatrices which is not well predicted by the Melnikov function.

Narrow strip of analyticity

Recall

$$\ddot{x} = \sin x + \mu \frac{\sin x}{(1 + \alpha \sin x)^2} \sin \frac{t}{\varepsilon}$$

If we take $\alpha = 1 - \mathcal{O}(\varepsilon^r)$:

- The strip of analyticity in x of the Hamiltonian is $\mathcal{O}\left(\varepsilon^{\frac{r}{2}}\right)$.
- If $r \in (0, 2)$, the distance between the invariant manifolds increases as r increases.
- If $r > 2$, the distance becomes non-exponentially small

$$d \sim \mu \varepsilon^{\eta - 3r/2}$$

Quasiperiodic case

- Model

$$\ddot{x} = \sin x + \mu\varepsilon^{\eta} \frac{\sin x}{(1+\alpha \sin x)^2} F\left(\frac{\gamma t}{\varepsilon}, \frac{t}{\varepsilon}\right),$$

  where $F : \mathbb{T}^2 \to \mathbb{R}$ and $\gamma = \frac{1+\sqrt{5}}{2}$ is the golden mean.

- For $\alpha = 0$, this model was studied by Delshams, Gelfreich, Jorba and Seara (1997) assuming certain conditions on F.

- We assume the same hypotheses as them.
- Take

$$F(\theta_1, \theta_2) = \sum_{k \in \mathbb{Z}^2} F^{[k]} e^{ik\theta}$$

- Hypotheses:
  - There exists constant $r_1, r_2 > 0$ such that

$$\sup_{k \in \mathbb{Z}} \left| F^{[k]} e^{r_1|k_1| + r_2|k_2|} \right| < \infty.$$

  - There exists $m, k_0 > 0$ such that

$$\left| F^{[k]} \right| > m e^{-r_1|k_1| - r_2|k_2|}$$

    for all $|k_1|/|k_2|$ which are continuous fraction convergents of γ and $|k_2| > k_0$.

Results for $\alpha = 0$

- If $\eta > 1$ the Melnikov function predicts correctly the splitting between the invariant manifolds.
- The distance can be bounded as

$$C_1 \mu \varepsilon^{\eta-1} e^{-c(\varepsilon)\sqrt{\frac{\pi}{2\varepsilon}}} < d < C_2 \mu \varepsilon^{\eta-1} e^{-c(\varepsilon)\sqrt{\frac{\pi}{2\varepsilon}}}$$

  where c is a function which has upper and lower bounds independent of ε.

- These bounds only depend on r_1 and r_2 and the Diophantine constants of γ.

Results for $\alpha \in (0, 1)$

- If $\eta > \eta^*$, the Melnikov function predicts correctly the splitting.
- If $\alpha = \varepsilon^\nu$ with $\nu > 1$, the distance between manifolds coincides with the $\alpha = 0$ case.
- If $\alpha = \varepsilon^\nu$ with $\nu \in (0, 1)$ or α is independent of ε,

$$\frac{C_1 \mu \varepsilon^{\eta - \frac{1}{2}}}{\sqrt{\alpha}} e^{-c(\varepsilon)\sqrt{\frac{\rho}{\varepsilon}}} \leq d \leq \frac{C_2 \mu \varepsilon^{\eta - \frac{1}{2}}}{\sqrt{\alpha}} e^{-c(\varepsilon)\sqrt{\frac{\rho}{\varepsilon}}}.$$

- As in the periodic case, for α independent of ε the singularity of the separatrix does not play any role in the size of the splitting.

Final remarks for the quasiperiodic case

- As in the periodic case,
  - If $\alpha \to 1$, the splitting increases.
  - If the strip of analyticity is very small (for instance $\alpha = 1 - \varepsilon^2$) the splitting is non-exponentially small.
- This agrees with what Nekhorosev Theory says: the wider the strip of analyticity of the Hamiltonian the longer is the stability time.

Conclusions

- The size of the splitting of separatrices behaves differently for meromorphic and polynomial perturbations both in the periodic and quasiperiodic cases.
- The size of the splitting increases as the width of the analyticity strip narrows.
- When the strip of analyticity is very big, it is dangerous to expand the perturbation since it can give the wrong answer.