Global phenomena in a neighborhood of codimension two local singularities of planar Filipov systems

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Main goal: Study planar Filippov Systems locally assuming that the discontinuity surface is a differentiable curve.

- Discontinuity curve given by $\Sigma = f^{-1}(0)$.

- A Filippov System is given by a piecewise $\mathcal{C}^r$ vector field:

$$Z(x,y) = \begin{cases} X(x,y) \ \text{ for } (x,y) \in \Sigma^+ = \{(x,y) \in U : f(x,y) > 0\} \\ Y(x,y) \ \text{ for } (x,y) \in \Sigma^- = \{(x,y) \in U : f(x,y) < 0\}. \end{cases}$$

- $\mathcal{Z}^r = \mathfrak{X}^r \times \mathfrak{X}^r$ with the $\mathcal{C}^r$ product topology.

Dynamics in the discontinuity curve (I)

Σ splits in

- Crossing region: $\Sigma^c = \{p \in \Sigma : Xf(p) \cdot Yf(p) < 0\}$
- Sliding region: $\Sigma^s = \{p \in \Sigma : Xf(p) < 0, Yf(p) > 0\}$
- Escaping region: $\Sigma^e = \{p \in \Sigma : Xf(p) > 0, Yf(p) < 0\}$

- Tangency points: $p \in \Sigma$ such that $Xf(p) = 0$ or $Yf(p) = 0$
  (where $Xf(p) = X(p) \cdot \text{grad} f(p)$)
  $\Longrightarrow p \in \partial\Sigma^c \cup \partial\Sigma^e \cup \partial\Sigma^s$

Orbits and singularities

- We establish rigorous definitions of orbit and singularity (Simic, Broucke and Pugh).
- We want two main features of the classical definition of orbit to persist:
  - Every point belongs to a unique orbit.
  - The phase space is the disjoint union of orbits.
- This approach seems the best to consider topological equivalences.

Simic, Broucke and Pugh approach

- Points $p \notin \Sigma$: take the classical orbit
- Crossing points $p \in \Sigma^c$: match the arriving and departing trajectories.
- Sliding and escaping points $p \in \Sigma^e \cup \Sigma^s$: orbit given by the Filippov convention

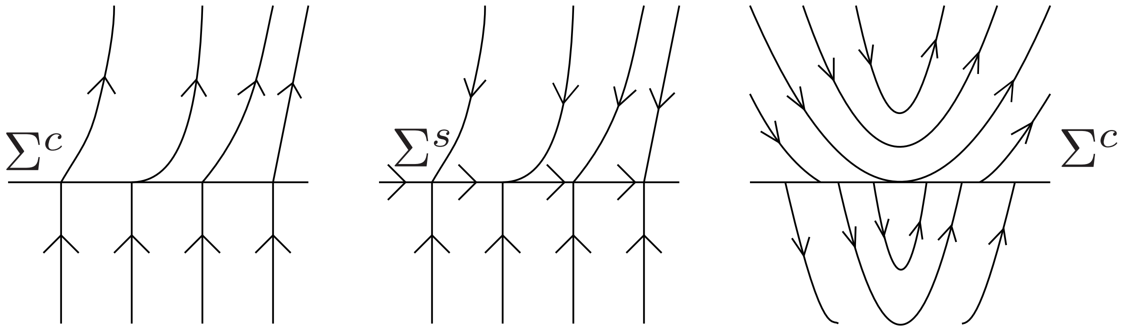
$$Z_s(p) = \frac{1}{Yf(p) - Xf(p)} \left(Yf(p)X(p) - Xf(p)Y(p) \right).$$

- Singular equilibrium: $p \in \Sigma^s \cup \Sigma^e$ critical point of Z_s.
- We have to establish separately the definition of orbit for the tangency points.
- We distinguish: $\begin{cases} \text{Regular tangency points} \\ \text{Singular tangency points} \end{cases}$

Regular tangency points

Points belonging exclusively to $\partial\Sigma^i$, $i = c, s, e$ for which the definition of orbit of Σ^i can be extended in a unique way.

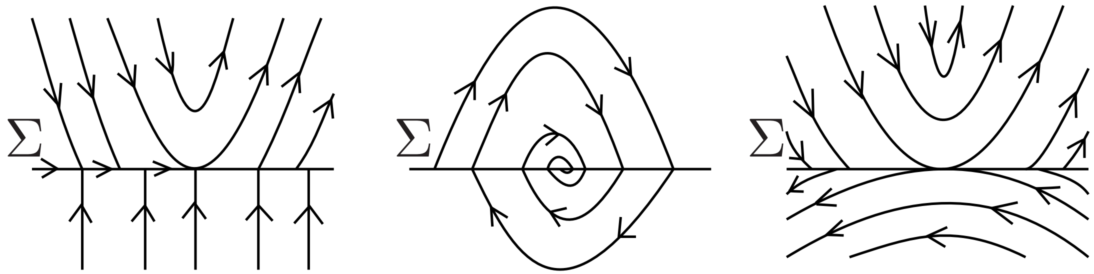
Examples:



Singular tangency points

All the tangency points for which can not be established an orbit following the same approach as before → their orbit is just themselves: $\varphi(t,p) = p$

Examples:



In the case that X or Y have a critical point, the orbit is also only themselves.

Singularities in Σ

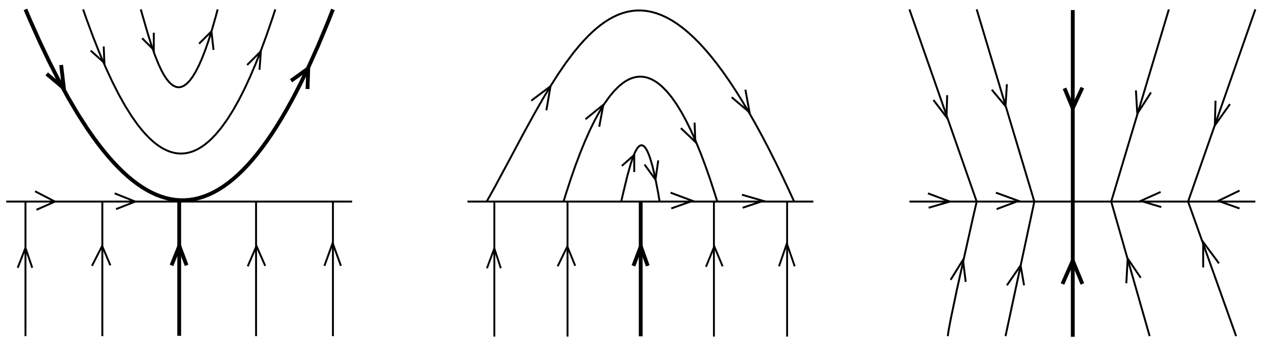
- Can be classified as
  - Singular tangency points
  - Singular equilibrium
- The orbit is only themselves: $\varphi(t, p) = p$
- Any other point is considered regular.

Separatrices and pseudoseparatrices

- An orbit $\gamma(t) = \varphi(t, p)$ departs from $q \in \overline{\Sigma^s}$ if $\lim_{t \to t_0^+} \gamma(t) = q$ (arrival is defined analogously).
- Unstable separatrix: regular orbit such that its α-limit set is a regular saddle point $p \in \Sigma^+ \cup \Sigma^-$.
- Unstable pseudoseparatrix: regular orbit which departs from a singularity $p \in \Sigma$.
- Stable separatrix and pseudoseparatrix defined analogously.
- If a separatrix or pseudoseparatrix is simultaneously stable and unstable $\to$ separatrix connection (global bifurcation).

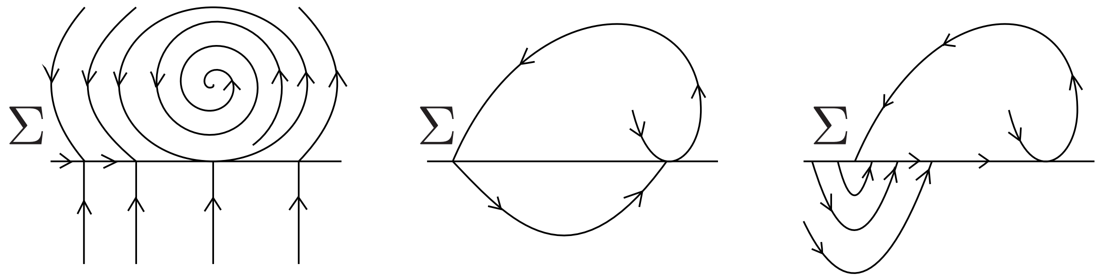
Examples of pseudoseparatrices (I)

- A visible tangency has three pseudoseparatrices and an invisible tangency just one.
- A node of the sliding vector field also has pseudoseparatrices.



Examples of pseudoseparatrices (II)

Notice that the separatrix connection of pseudoseparatrices of singular tangencies may lead to the classical discontinuity induced bifurcations of periodic orbits in Filippov Systems.



Σ-equivalence of Filippov vector fields

- Used in the classification of Filippov vector fields in the works by M. A. Teixeira, S. Simic, M. Broucke, C. Pugh, Y. Kuznetsov, S. Rinaldi, A. Gragnani,...
- $Z, Z' \in \mathcal{Z}^r$ are Σ-equivalent if there exists a homeomorphism $h : U, 0 \to V, 0$ which:
  - is orientation preserving
  - sends Σ to itself
  - sends orbits of Z to orbits of Z'
- $Z_0 \in \mathcal{Z}^r$ is Σ-structurally stable if there exists $Z_0 \in \mathcal{U} \in \mathcal{Z}^r$ such that for all $Z \in \mathcal{U}$, Z is Σ-equivalent to Z_0

Properties of the Σ-equivalence

It sends:

- Singularities to singularities.
- Separatrices and pseudoseparatrices to themselves.
- $\overline{\Sigma^c}$, $\overline{\Sigma^e}$, $\overline{\Sigma^s}$ to themselves.
- Σ^c does not play any role in the dynamics of the Filippov System

Why has Σ^c to be preserved by the equivalence?

Equivalence of Filippov vector fields

- Classical definition: $Z, Z' \in \mathcal{Z}^r$ are equivalent if there exists a homeomorphism $h : U, 0 \to V, 0$ which:
  - is orientation preserving.
  - sends orbits of Z to orbits of Z'
- Properties:
  - Sends singularities, separatrices and pseudoseparatrices to themselves.
  - May not preserve Σ but sends $\overline{\Sigma^s}$ and $\overline{\Sigma^e}$ to themselves.
  - Sends $\overline{\Sigma^c} \cup \Sigma^+ \cup \Sigma^-$ to $\overline{\Sigma^c} \cup \Sigma^+ \cup \Sigma^-$.
- $Z_0 \in \mathcal{Z}^r$ is structurally stable if there exists $Z_0 \in \mathcal{U} \in \mathcal{Z}^r$ such that for all $Z \in \mathcal{U}$, Z is equivalent to Z_0

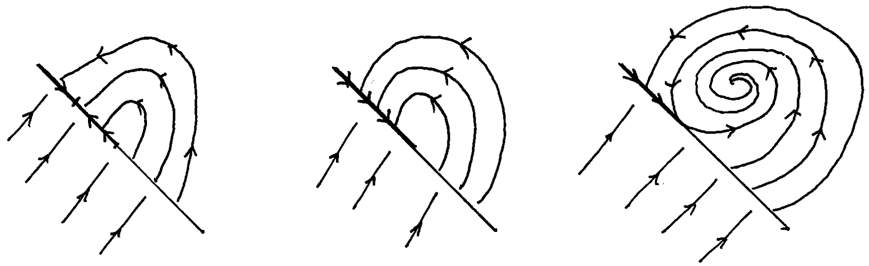
Low codimension local bifurcations

Using topological equivalences and Σ-equivalences one wants to classify the local behavior of $Z \in \mathcal{Z}^r$ around $p \in \Sigma$.

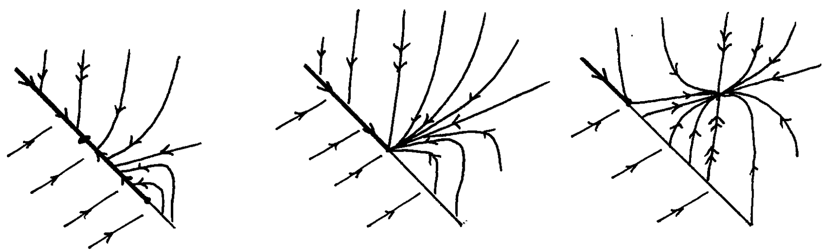
- The codimension-1 local bifurcations were studied by Kutznesov *et altrii*.
- We want to study the codimension-2 local bifurcations.
- In this talk we focus on two examples of codimension-2 local bifurcation.
- The first one has different unfoldings depending whether we use topological equivalence or Σ-equivalence.
- The second one presents infinitely many codimension-1 global bifurcation branches.

The boundary-focus and boundary-node bifurcations

- X has an attractor focus $p \in \Sigma$:



- X has an attractor node $p \in \Sigma$



Remarks on these singularities

- Hartmann theorem → these two singularities are topologically conjugated for smooth vector fields but not $\mathcal{C}^1$-conjugated.
- For non-smooth systems are non-equivalent singularities:
  - Focus: the fixed point is only arrival point of two orbits.
  - Node: the fixed point is the arrival point of infinitely many orbits.

A codimension 2 bifurcation: A Non-diagonalizable node in Σ

The singularity such that Y is regular and X has a hyperbolic fixed point with Jacobian

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$$

has codimension 2 and is arbitrarily close to the previous ones.

Normal form

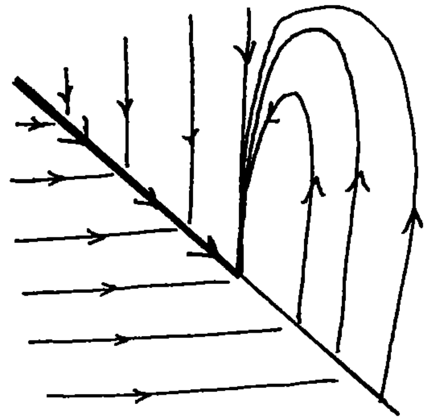
- Discontinuity curve:

$$\Sigma = \{y + x = 0\}$$

- Normal form: $Z_{0,0}(x, y) =$

$$\begin{cases} X = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} & \text{if } x + y > 0 \\ Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } y + x < 0 \end{cases}$$

with $a < 0$



Generic unfolding of the singularity

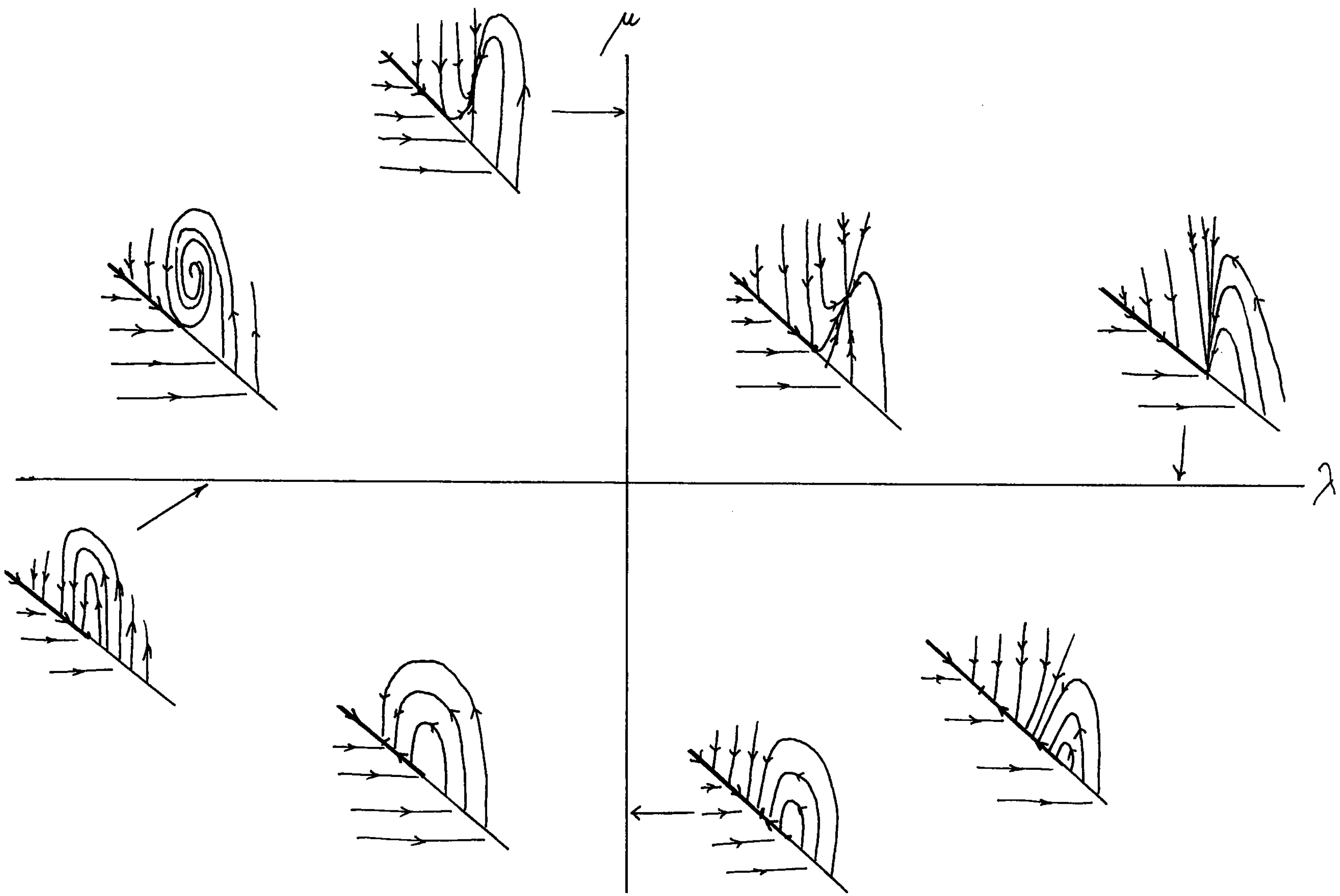
Generic unfolding:

$$Z_{\lambda,\mu}(x,y) = \begin{cases} X_{\lambda,\mu} = \begin{pmatrix} a & 1 \\ \lambda & a \end{pmatrix} \begin{pmatrix} x \\ y-\mu \end{pmatrix} & \text{if } x+y>0 \\ Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } y+x<0 \end{cases}$$

Therefore:

- The critical point of X is $p = (0, \mu)$.
- $\mu > 0 \longrightarrow p \in \Sigma^+$ (the critical point is visible).
- $\mu < 0 \longrightarrow p \in \Sigma^-$ (the critical point is non-visible).
- $\lambda > 0 \longrightarrow$ It is an atractor node.
- $\lambda < 0 \longrightarrow$ It is an atractor focus.

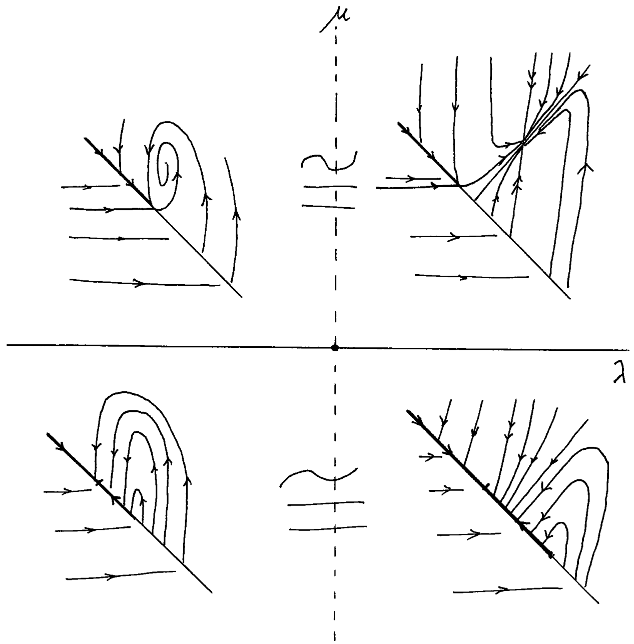
Unfolding using Σ-equivalences



Unfolding using equivalences

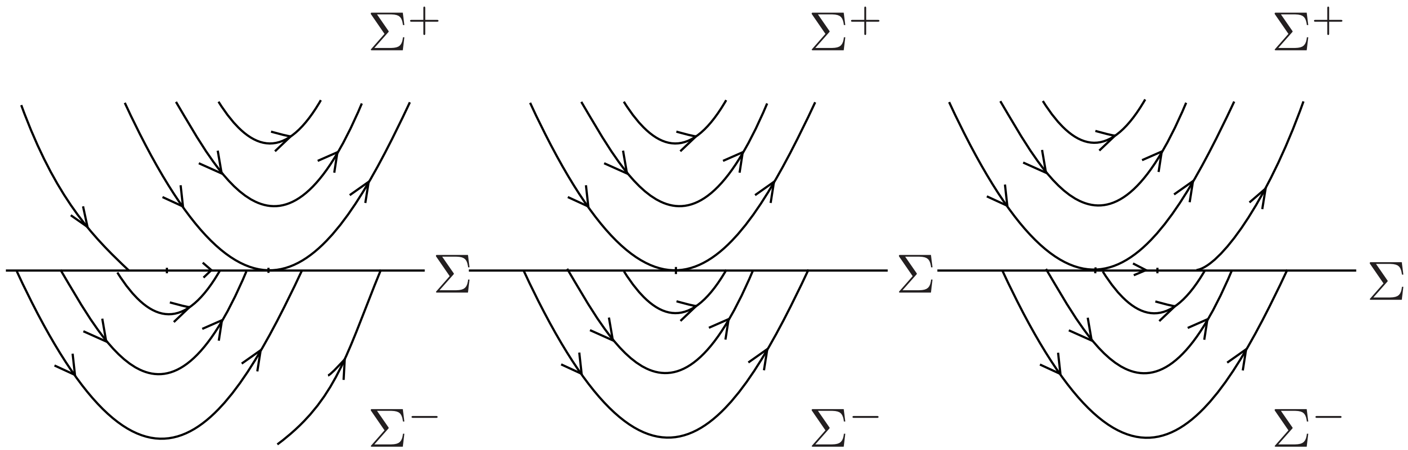
There are cases which are not Σ-equivalent that are equivalent:

- Apply Hartman Theorem in a neighbourhood of the fixed point.
- Define the equivalence between the sliding vector fields.
- Extend the equivalence using the flows of the vector fields.



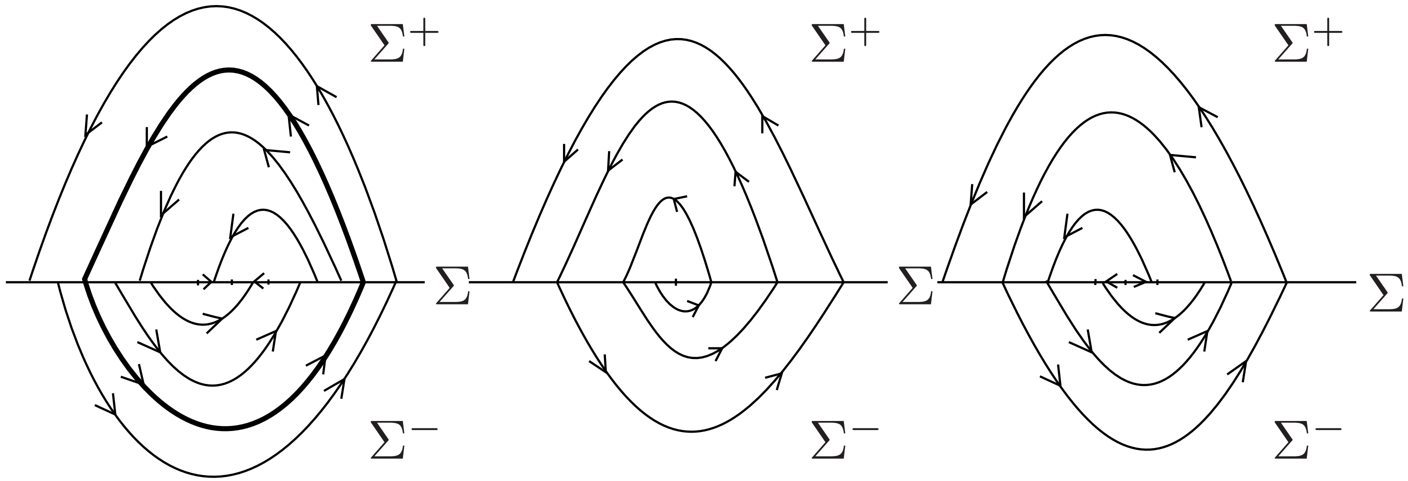
Fold-Fold bifurcation (I)

Occurs when both vector fields have a quadratic tangency in $p \in \Sigma$: one visible and one invisible.



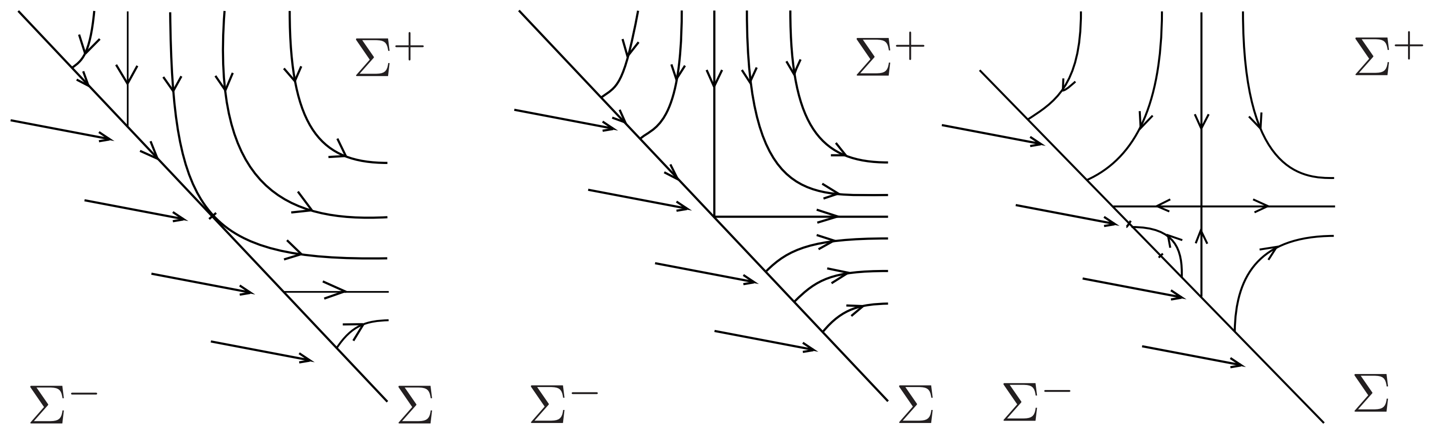
Fold-Fold bifurcation (II)

Hopf-like bifurcation: Occurs when both vector fields have an invisible quadratic tangency in $p \in \Sigma$.



Boundary-saddle bifurcation

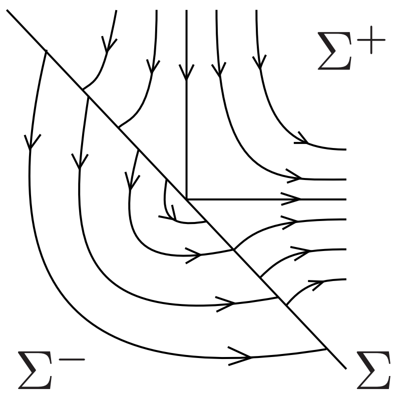
Ocurrs when X has a saddle in $p \in \Sigma$ and Y is transversal to Σ in p.



The saddle-fold bifurcation (I)

$Z = (X, Y)$ Filipov vector field and $p \in \Sigma$ such that

- X has a saddle at p such that
  - The eigenvalues have different modulus.
  - Both eigenspaces are transversal to Σ.
- Y has a quadratic tangency or fold at p.



Normal form

- Discontinuity curve: $\Sigma = \{x + y = 0\}$
- Normal form:

$$
Z(x,y) = \begin{cases} X(x,y) = \begin{pmatrix} \lambda_1 x \\ -\lambda_2 y \end{pmatrix} & \text{if } x + y > 0 \\ Y(x,y) = \begin{pmatrix} 1 + x - y \\ 1 + x - y \end{pmatrix} & \text{if } x + y < 0 \end{cases}
$$

with $\lambda_1, \lambda_2 > 0$ and $\lambda_1 > \lambda_2$.

Unfolding of the saddle-fold bifurcation

$$Z_{\mu,\varepsilon}(x,y) = \begin{cases} X_\mu(x,y) = \begin{pmatrix} \lambda_1 x - \mu \\ -\lambda_2 y + \mu \end{pmatrix} & \text{if } x+y>0 \\ Y_\varepsilon(x,y) = \begin{pmatrix} 1+x-y-\varepsilon \\ -1+x-y-\varepsilon \end{pmatrix} & \text{if } x+y<0 \end{cases}$$

Singularities of $Z_{\mu,\varepsilon}$:

- The saddle is $S = (\mu/\lambda_1, \mu/\lambda_2)$
- When $\mu \neq 0$, X has a fold $F_+ = (0,0)$
- Y has a fold at $F_- = (\varepsilon/2, -\varepsilon/2)$

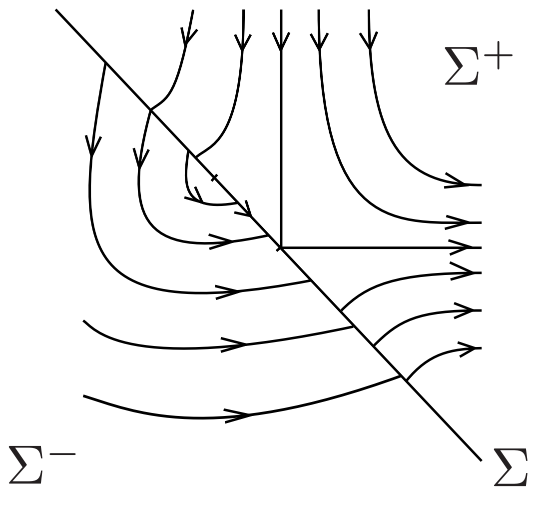
Creation of sliding or escaping regions

- For $\varepsilon > 0$: $\Sigma^e = \{(x,y) \in \Sigma : 0 < x < \varepsilon/2\}$
  $\Sigma^c = \{(x,y) \in \Sigma : x < 0 \text{ or } x > \varepsilon/2\}$
- For $\varepsilon < 0$: $\Sigma^s = \{(x,y) \in \Sigma : \varepsilon/2 < x < 0\}$
  $\Sigma^c = \{(x,y) \in \Sigma : x < \varepsilon/2 \text{ or } x > 0\}$
- For $\varepsilon = 0$: $\Sigma^c = \Sigma \setminus \{(0,0)\}$
- For $\mu > 0, \varepsilon \neq 0$, the sliding vector field has a pseudo-node P.

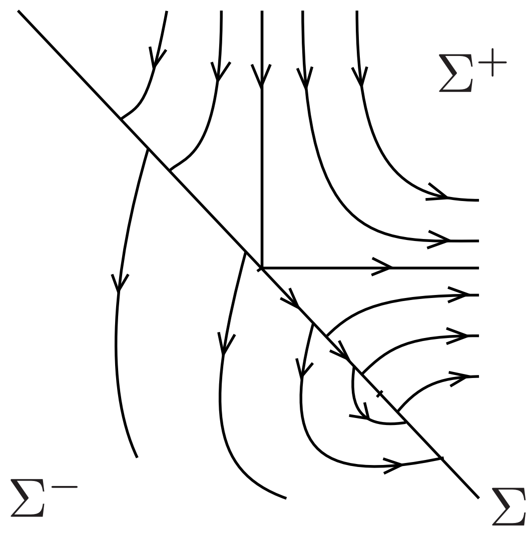
Codimension-1 local bifurcations in the unfolding (I)

- For $\mu > 0$ the saddle is visible.
- For $\mu < 0$ the saddle is invisible.
- For $\mu = 0, \varepsilon \neq 0$ there is a boundary-saddle bifurcation.

For $\mu = 0, \varepsilon < 0$



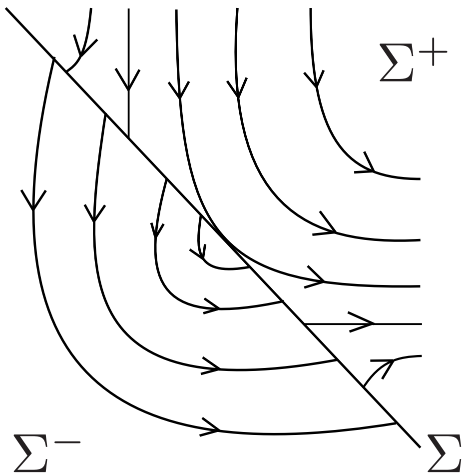
For $\mu = 0, \varepsilon > 0$



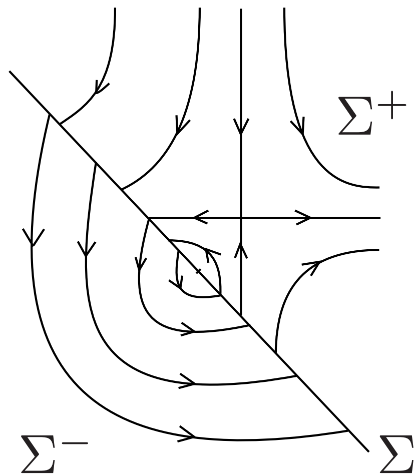
Codimension-1 local bifurcations in the unfolding (II)

- For $\mu \neq 0$ there exist two folds $F_+ = (0,0)$ and $F_- = (\varepsilon/2, -\varepsilon/2)$.
- For $\varepsilon > 0$: $F_- < F_+$.
- For $\varepsilon < 0$: $F_+ < F_-$.
- For $\varepsilon = 0, \mu \neq 0$ we have two different fold-fold bifurcations

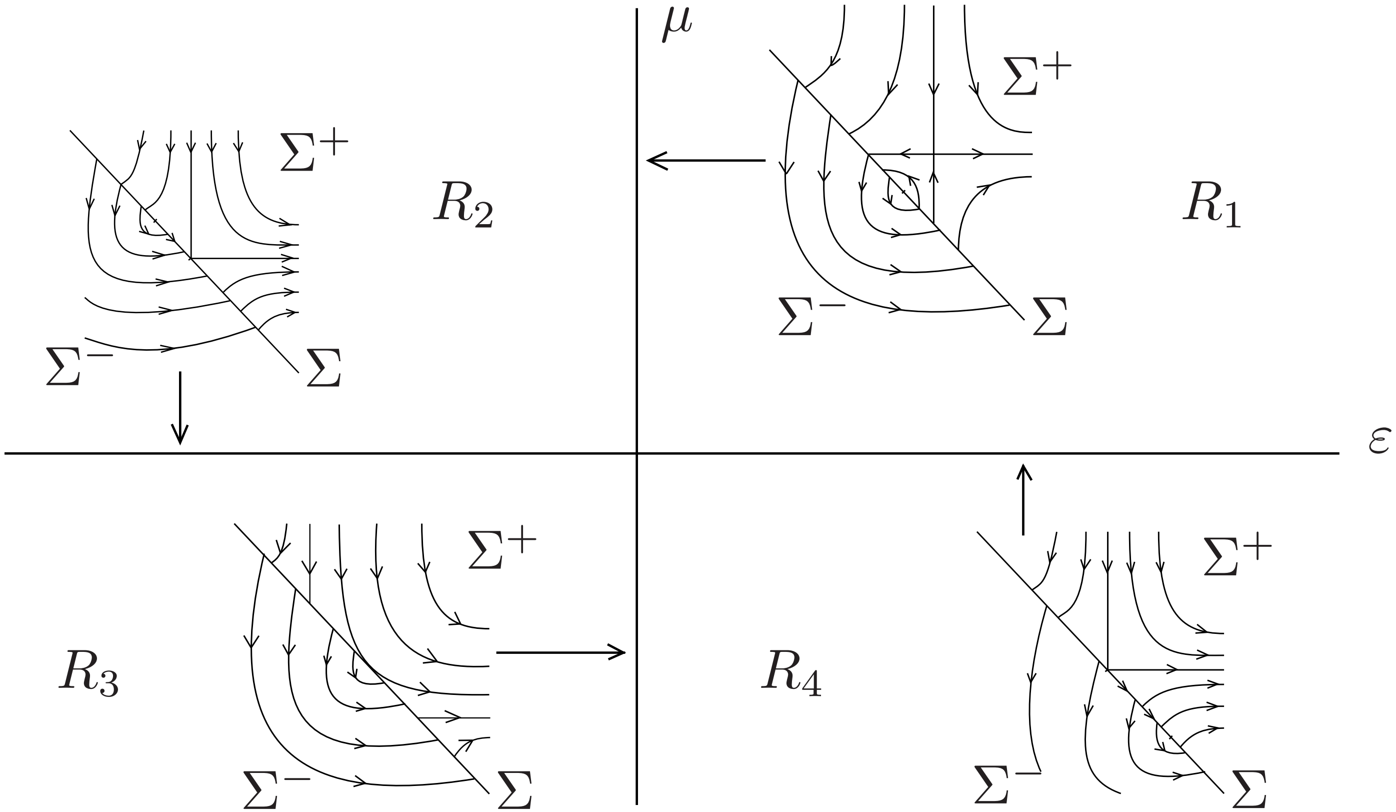
For $\varepsilon = 0, \mu < 0$



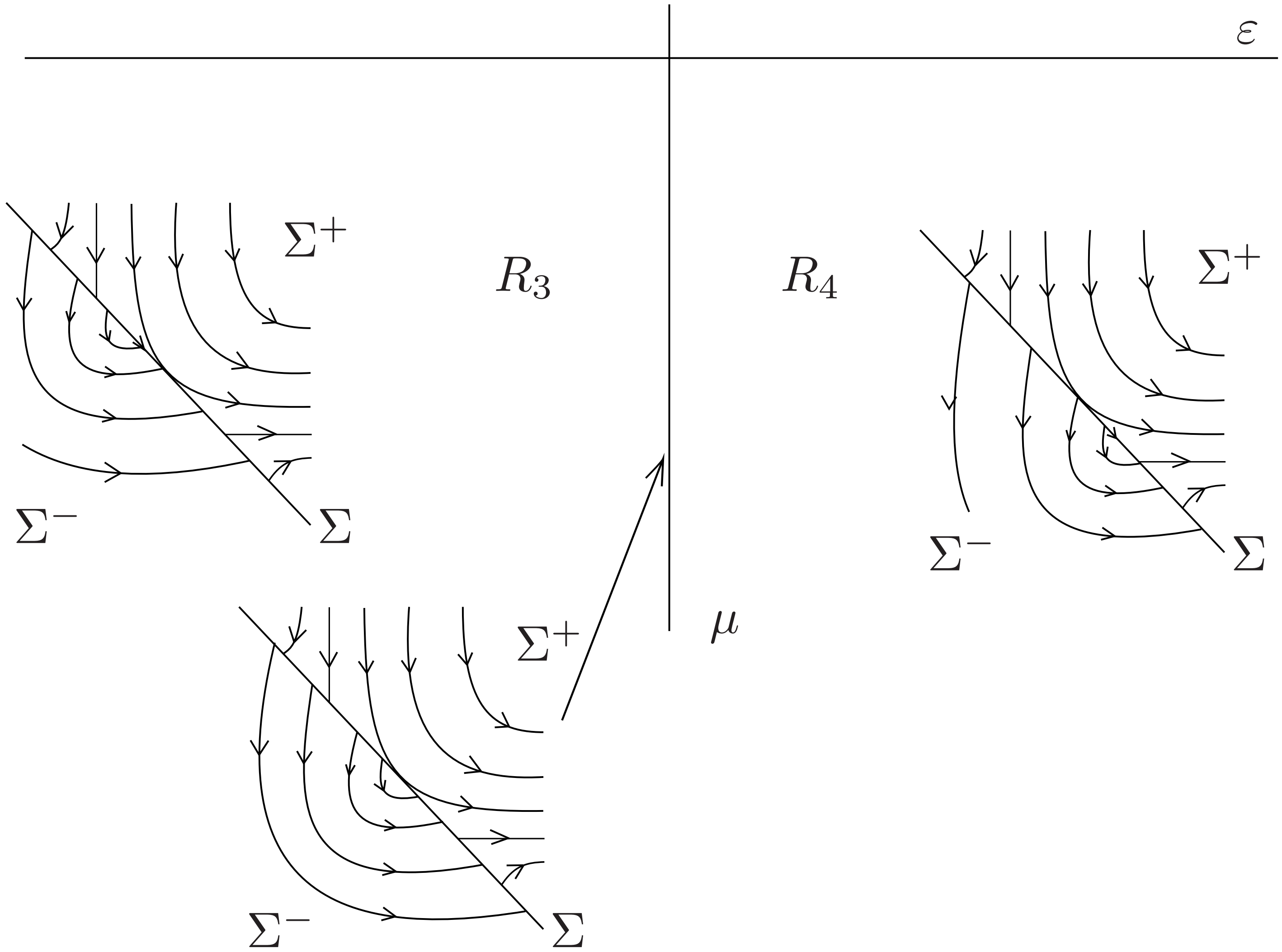
For $\varepsilon = 0, \mu > 0$



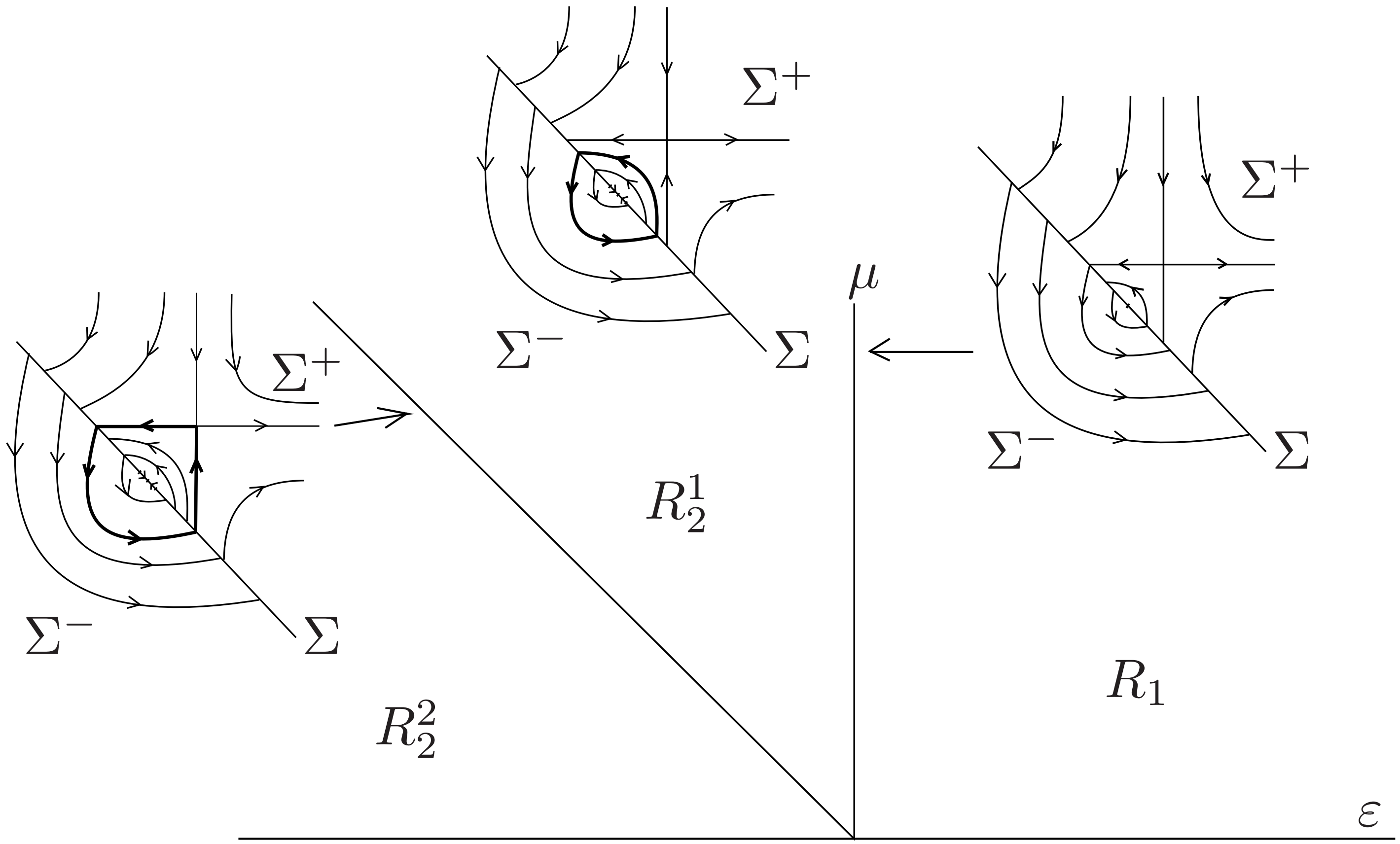
Codimension-1 local bifurcations in the unfolding (III)



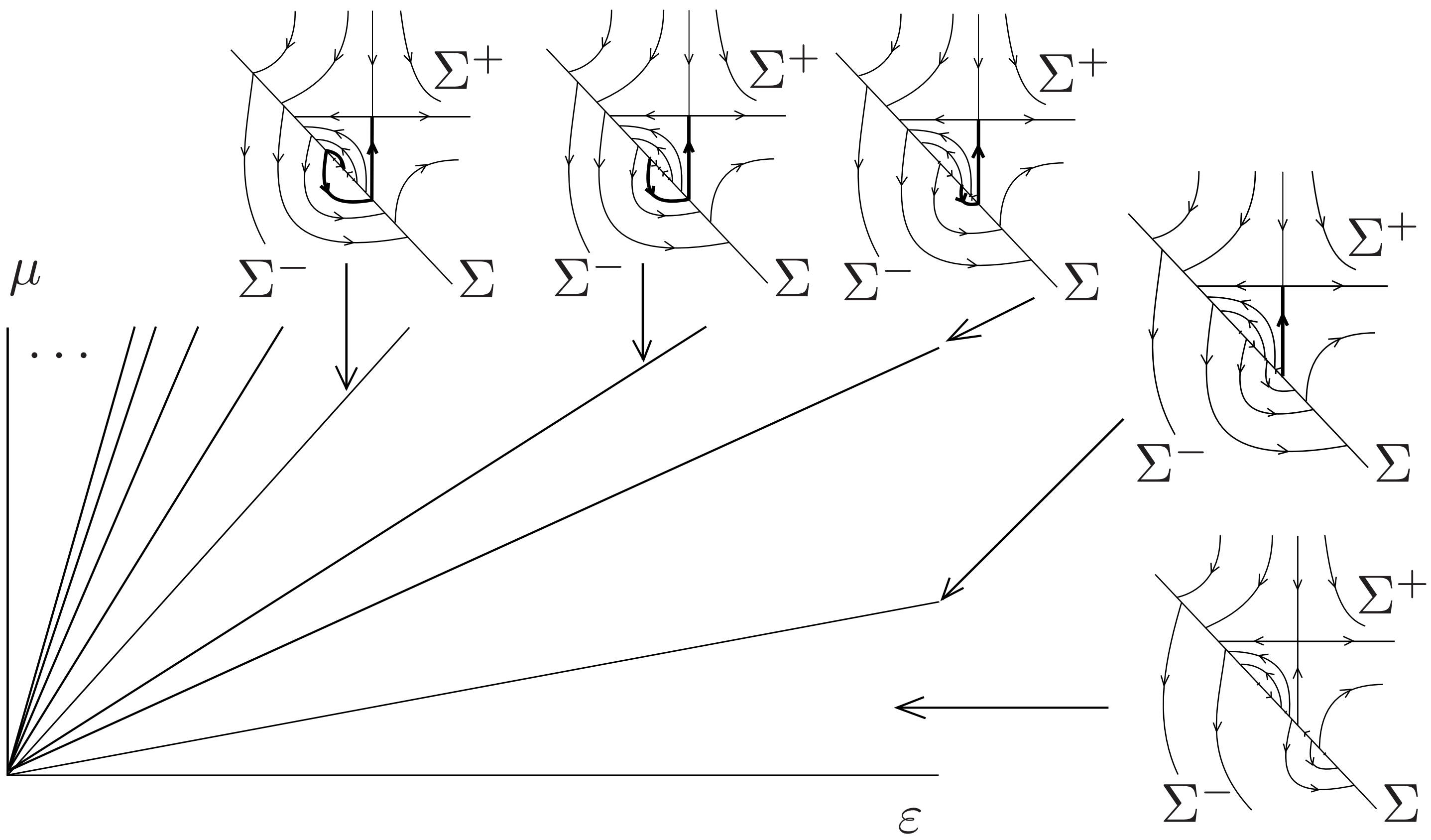
The unfolding for $\mu < 0$



The unfolding for $\mu > 0$



Global bifurcations in R_1



Global bifurcations in R_2