Growth of Sobolev norms for the analytic non-linear Schrödinger equation

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The analytic gauge invariant NLS

- Consider the equation

$$-iu_t + \Delta u = \pm |u|^{2(d-1)} u + G'(|u|^2)u, \quad d \in \mathbb{N},\ d \geq 2$$

where $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$.

- $G(y)$ is an analytic function with a zero of degree at least $d+1$.

- In this talk we consider the defocusing setting $(+)$.

- The result presented is also valid for the focusing NLS $(-)$.

Transfer of energy

- L^2 norm (mass) and energy are preserved. Thus,

$$\|u(t)\|_{H^1(\mathbb{T}^2)} \leq C\|u(0)\|_{H^1(\mathbb{T}^2)} \text{ for all } t \geq 0.$$

- Fourier series of u,

$$u(x,t) = \sum_{n\in\mathbb{Z}^2} a_n(t)e^{inx}.$$

- Can we have transfer of energy to higher and higher modes as $t \to +\infty$?
- We measure it with the growth of s-Sobolev norms ($s > 1$)

$$\|u(t)\|_{H^s(\mathbb{T}^2)} := \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} := \left(\sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2s}|a_n(t)|^2\right)^{1/2},$$

where $\langle n\rangle = (1+|n|^2)^{1/2}$.

How fast the energy transfer can be?

- Dimension 1, $d = 2$, $G = 0$ (cubic case), a priori bounds for all H^s.
- Dimension $D \geq 2$ or power $d > 2$: growth of H^s expected to happen.
- Bourgain: Polynomial upper bounds for the growth of H^s, $s > 1$:

$$\|u(t)\|_{H^s} \leq t^A \|u(0)\|_{H^s} \quad \text{for} \quad t \to +\infty.$$

  for some $A > 0$.

- Question by Bourgain (2000): Are there solutions u such that for $s > 1$,

$$\|u(t)\|_{H^s} \to +\infty \qquad \text{as } t \to +\infty?$$

The cubic case in $\mathbb{T}^2$

- Cubic case: $-iu_t + \Delta u = |u|^2 u$, $x \in \mathbb{T}^2$.
- Kuksin (1997): growth of Sobolev norms starting from an already large initial data.

Theorem (Colliander, Keel, Staffilani, Takaoka, Tao (2010))

Fix $s > 1$, $\mathcal{C} \gg 1$ and $\mu \ll 1$. Then there exists a global solution u of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(0)\|_{H^s} \leq \mu, \qquad \|u(T)\|_{H^s} \geq \mathcal{C}.$$

- Valid on any $\mathbb{T}^D$, $D \geq 2$.

The cubic case

- M. G. and V. Kaloshin: $T \sim e^{\left(\frac{\mathcal{C}}{\mu}\right)^A}$ for some $A > 0$.
- M. G. and V. Kaloshin also in the cubic case: Fix $\mathcal{K} \gg 1$, there exists a solution u of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(T)\|_{H^s} \geq \mathcal{K}\|u(0)\|_{H^s}, \qquad T \sim \mathcal{K}^B, \quad \text{for some } B > 0$$

  and

$$\|u(t)\|_{L^2} \leq \mathcal{K}^{-\sigma} \quad \text{for some } \sigma > 0.$$

- E. Haus and M. Procesi generalized the I-team result to the quintic NLS ($d = 3$ and $D \geq 2$).

$$-iu_t + \Delta u = |u|^4 u$$

- Z. Hani, B. Pausader, N. Tzvetkov, N. Visciglia proved unbounded growth for the cubic NLS in $\mathbb{R} \times \mathbb{T}^2$.

Main result

$$-iu_t + \Delta u = |u|^{2(d-1)}u + G'(|u|^2)u, \qquad x \in \mathbb{T}^2$$

Theorem (M. G. – E. Haus – M. Procesi)

Let $d \geq 2$ and $s > 1$. There exists $A > 0$ such that for any large $\mathcal{C} \gg 1$ and small $\mu \ll 1$, there exists a global solution $u(t) = u(t, \cdot)$ of NLS and a time T satisfying

$$T \leq e^{\left(\frac{\mathcal{C}}{\mu}\right)^A}$$

such that

$$\|u(0)\|_{H^s} \leq \mu \ \text{ and } \ \|u(T)\|_{H^s} \geq \mathcal{C}.$$

- Valid on any $\mathbb{T}^D$, $D \geq 2$.
- If we do not assume small initial Sobolev norm, we do not get better time estimates.

The I-team approach for the cubic case

- Cubic NLS as an ode (of infinite dimension) for the Fourier coefficients of u:

$$-i\dot{a}_n = |n|^2 a_n + \sum_{\substack{n_1, n_2, n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1} \overline{a_{n_2}} a_{n_3}, \qquad n \in \mathbb{Z}^2.$$

- Drift through resonances.
- Resonant monomial

$$n_1 - n_2 + n_3 - n = 0 \quad \text{and} \quad |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0$$

- Non-degenerate resonances form a rectangle in $\mathbb{Z}^2$.
- Slider solutions supported in a rectangle (heteroclinics) push energy from two modes to the other two.
- For well chosen rectangles: growth of Sobolev norms by a constant factor.

Traveling through rectangles

- One needs to concatenate many rectangles to attain a growth by a factor $\mathcal{C}/\mu$.
- Number of concatenations: $N \sim \log(\mathcal{C}/\mu) \gg 1$.
- At each step, the Sobolev norm is pushed to half of the modes (the ones further out)
- One needs many modes to be able to push Sobolev norm through the N "generations".
- The I-team considers a finite set of modes $\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_N \subset \mathbb{Z}^2$ of large size $|\Lambda_j| = 2^{N-1}$, $N \sim \log(\mathcal{C}/\mu)$.

Traveling through rectangles

$$\Lambda = \Lambda_1 \cup \ldots \cup \Lambda_N \subset \mathbb{Z}^2, \qquad |\Lambda_j| = 2^{N-1}, \qquad N \sim \log(\mathcal{C}/\mu)$$

- They choose carefully Λ such that the modes interact in a very particular way.
- Each rectangle has modes only in two consecutive generations.
- Each mode in generation Λ_j pumps energy from a rectangle involving modes in Λ_{j-1} to a rectangle involving modes in Λ_{j+1}.
- Symmetry condition: take each mode in Λ_j with the same initial condition. Then, they remain equal through time.
- All modes in one generation are reduced to one variable.

The I-team approach for the cubic case

- After these reductions: finite dimensional (toy) model

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 1, \dots N.$$

  which approximates well certain solutions of NLS.

- Each b_j represents the 2^{N-1} modes in Λ_j.

- They look for orbits $b(t)$ that are localized for $t = 0$ at b_1 and at a certain $t = T \gg 1$ are localized at b_N.

The I-team approach for the cubic case

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right), \ j = 1, \ldots N.$$

- It can be seen as a Hamiltonian system on a lattice $\mathbb{Z}$ with nearest neighbor interactions.
- Hamiltonian:

$$h(b) := \frac{1}{4}\sum_{j=1}^{N}|b_j|^4 - \frac{1}{2}\sum_{j=1}^{N}\left(\overline{b}_j^2 b_{j-1}^2 + b_j^2\overline{b}_{j-1}^2\right).$$

- It has the mass as a first integral $M = \sum_{i=1}^{N}|b_i|^2$.

Dynamics of the cubic toy model

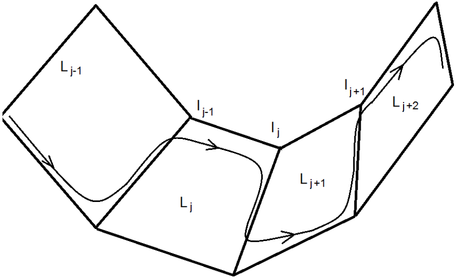
$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 0, \ldots N,$$

- Each 4-dimensional plane

$$L_j = \{b_1 = \cdots = b_{j-1} = b_{j+2} = \cdots = b_N = 0\}$$

  is invariant and corresponds to two generations interacting

- In L_j, $\mathbb{T}_j = \{b_j \neq 0, b_{j+1} = 0\}$ and $\mathbb{T}_{j+1} = \{b_j = 0, b_{j+1} \neq 0\}$ are (partially hyperbolic) periodic orbits.
- They are connected through heteroclinic orbits.
- To travel close to L_j from $\mathbb{T}_j$ to $\mathbb{T}_{j+1}$, they shadow these heteroclinics.



- Shadowing the concatenation of periodic orbits and heteroclinics we go from (close to) the first to (close to) the last plane.
- Problems:
    - The periodic orbits are partially hyperbolic and partially elliptic
    - The hyperbolic eigenvalues of these periodic orbits are resonant.
    - We do not have transversality between invariant manifolds of objects.
- The shadowing argument is delicate.

The resonant sets for $-iu_t + \Delta u = |u|^{2(d-1)}u + G'(|u|^2)u$

- Resonant monomials

$$\sum_{i=1}^{2d} (-1)^i n_i = 0 \quad \text{and} \quad \sum_{i=1}^{2d} (-1)^i |n_i|^2 = 0.$$

- Combinatorics of resonances are far more complicated.
- Consider solutions supported in a single resonant set: we want to pump energy from some modes to the others.
- Dynamics: we want two invariant objects corresponding to some modes set to zero connected by a heteroclinic orbit.

Drifting through rectangles

- Generalization of the I-team approach: consider simple resonances $\{j_1, \ldots, j_{2k}\}$ with $2 < k \leq d$

  (simple $=$ it does not factor out as a sum of lower order resonances).

- The associated Hamiltonian has periodic orbits but they are not connected by heteroclinic connections.

- Procesi and Haus for the quintic NLS (2014): one can still use rectangles as building blocks.

Drifting through rectangles

- Take a rectangle

$$n_1 - n_2 + n_3 - n_4 = 0 \quad \text{and} \quad |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 0$$

- Rectangles induce infinitely many resonances for the quintic NLS

$$n_1 - n_2 + n_3 - n_4 + m - m = 0, \ |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |m|^2 - |m|^2 = 0$$

for any $m \in \mathbb{Z}^2$.

- Analogously for any other power.
- We want to build up the concatenation of generations building upon rectangles.
- Problem: the combinatorial analysis of resonances becomes more involved as d grows.

The resonant sets in the general case

- We need to impose much more conditions to avoid non-desired resonances than in the cubic case.
- Some resonant interactions are unavoidable: take two rectangles with a common vertex n_4,

$$n_1 - n_2 + n_3 - n_4 = 0 \qquad |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 0$$
$$n_4 - n_5 + n_6 - n_7 = 0 \qquad |n_4|^2 - |n_5|^2 + |n_6|^2 - |n_7|^2 = 0$$

  They create the resonant sextuple with six different modes

$$n_1 - n_2 + n_3 - n_5 + n_6 - n_7 = 0$$
$$|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_5|^2 + |n_6|^2 - |n_7|^2 = 0.$$

- Each mode receives and pumps energy not only through two rectangles but through much more resonant interactions.

The toy model in the general case

- Impose the I-team intra-generational symmetry:

$$h(b) = \left(\sum_{i=1}^{N} |b_i|^2\right)^{d-2} \left[\frac{1}{4}\sum_{i=1}^{N} |b_i|^4 - \sum_{i=1}^{N-1} \mathrm{Re}\left(b_i^2 \bar{b}_{i+1}^2\right)\right] + \frac{1}{2^N}\mathcal{P}\left(b, \bar{b}, \frac{1}{2^N}\right).$$

- As noticed by Procesi and Haus for the quintic case: unavoidable "non-rectangular resonances" only appear at higher order in 2^{-N}.
- I-team symmetry condition for modes of the same generation implies that the first order is just the cubic toy model with a power of the mass factored out.
- $M = \sum_{i=1}^{N} |b_i|^2$ is a first integral.

The toy model in the general case

$$h(b) = \left(\sum_{i=1}^{N} |b_i|^2\right)^{d-2} \left[\frac{1}{4}\sum_{i=1}^{N} |b_i|^4 - \sum_{i=1}^{N-1} \mathrm{Re}\left(b_i^2 \bar{b}_{i+1}^2\right)\right] + \frac{1}{2^N}\mathcal{P}\left(b, \bar{b}, \frac{1}{2^N}\right).$$

- In the quintic case $\mathcal{P}$ is explicit.
- Monomials involve at most three consecutive generations.
- $\mathcal{P}$ is not explicit for higher powers: some monomials involve more generations and/or more separated generations.

The toy model in the general case

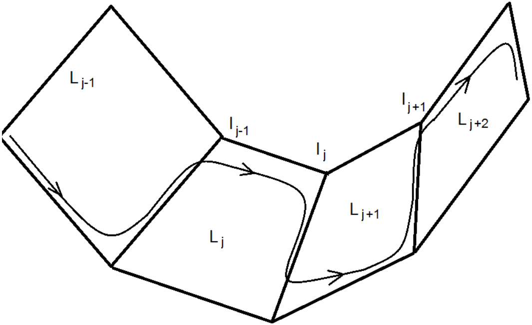
$$h(b) = \left(\sum_{i=1}^{N} |b_i|^2\right)^{d-2} \left[\frac{1}{4}\sum_{i=1}^{N} |b_i|^4 - \sum_{i=1}^{N-1} \mathrm{Re}\left(b_i^2 \bar{b}_{i+1}^2\right)\right] + \frac{1}{2^N}\mathcal{P}\left(b, \bar{b}, \frac{1}{2^N}\right).$$

- We want orbits $b(t)$ with transfer of mass as in the cubic case.
- The drift obtained for the cubic toy model takes long time.
- We have to choose carefully the modes in Λ so that $\mathcal{P}$ preserves the "dynamics" of the cubic case.
- For instance, one can see that:
  - All monomials in $\mathcal{P}$ are of even degree in $(b_j, \bar{b}_j)$.
  - The subspaces $\{b_j = 0\}$ are invariant (same invariant plane structure).

Properties of the toy model

- The shadowing argument by the I-team (also G.-Kaloshin) relies on the particular form of the toy model.
- Restricted to an invariant plane, the Hamiltonian depending on $(b_j, \overline{b}_j)$, $(b_{j+1}, \overline{b}_{j+1})$ is j independent and symmetric with respect to the exchange $j \longleftrightarrow j+1$.
- This implies that we have periodic orbits with resonant hyperbolic eigenvalues.
- Now we do not have nearest neighbor interaction.
- The strongest non-nearest neighbor interaction is integrable: a monomial depending on two modes i, j, $|i-j| \neq 1$, is of the form $|b_i||b_j|^{d-2}$.

Shadowing the invariant planes



- We proceed as in M. G.– V. Kaloshin.
- We construct solutions that drift through the planes
- These shadowing orbits are a good first order of orbits of NLS undergoing growth of Sobolev norms