Lecture 1: Oscillatory motions in the restricted three body problem

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Outline of the course

- Lecture 1: The restricted 3 body problem (RPC3BP and RPE3BP) and oscillatory motions
- Lecture 2: The invariant manifolds of infinity and their intersection
- Lecture 3: Exponentially small transversality between invariant manifolds
- Lecture 4: Symbolic dynamics and oscillatory motions
- Lecture 5: Oscillatory motions for the RPE3BP

Lecture 1

- The two body problem
- The restricted three body problem: RPC3BP and the RPE3BP
- Final motions and oscillatory motions
- Moser approach to construct oscillatory motions

The two body problem

- Motion of two bodies $q_1, q_2 \in \mathbb{R}^2$ with masses $m_1, m_2 > 0$, under the effects of the Newtonian gravitational force.
- Equations:

$$\ddot{q}_1 = m_2 \frac{q_2 - q_1}{\|q_2 - q_1\|^3}$$
$$\ddot{q}_2 = m_1 \frac{q_1 - q_2}{\|q_1 - q_2\|^3}$$

- Taking $p_i = m_i \dot{q}_i$, Hamiltonian

$$H(q, p) = \frac{\|p_1\|^2}{2m_1} + \frac{\|p_2\|^2}{2m_2} + \frac{m_1 m_2}{\|q_1 - q_2\|}.$$

of four degrees of freedom

The Kepler problem

- Center of mass $m_1 q_1 + m_2 q_2$ moves with uniform rectilinear motion.
- Assume it does not move and $m_1 q_1 + m_2 q_2 = 0$.
- Define $q_0 = q_2 - q_1$,

$$\ddot{q}_0 = (m_1 + m_2)\frac{q_0}{\|q_0\|^3}$$

- This is called the Kepler Problem.
- Hamiltonian $H(q_0, p_0) = \dfrac{\|p_0\|^2}{2} + \dfrac{m_1 + m_2}{\|q_0\|}$ has two degrees of freedom.

The Kepler Laws

- The Kepler problem is integrable.
- Its motion is governed by the Kepler laws
- Orbits are conic sections in the q_0-plane.
- Non degenerate motions: ellipses, parabolas and hyperbolas.
  - Ellipses are the only bounded motions
  - Parabolas: $q_0(t) \to +\infty$ and $\dot{q}_0(t) \to 0$ as $t \to \pm\infty$.
  - Hyperbolas: $q_0(t) \to +\infty$ and $|\dot{q}_0(t)| \to c > 0$ as $t \to \pm\infty$.
- Back to the two body problem:

$$q_1(t) = -\frac{m_2}{m_1 + m_2} q_0(t) \qquad q_2(t) = \frac{m_1}{m_1 + m_2} q_0(t)$$

Equations for the motion over ellipses

- Assume $m_1 + m_2 = 1$.
- Primaries over ellipses of eccentricity $e_0 \in (0,1)$:

$$q_1(t) = -\frac{m_2}{m_1+m_2} q_0(t) \qquad q_2(t) = \frac{m_1}{m_1+m_2} q_0(t)$$

where

$$q_0(t) = (r_0(t)\cos f(t), r_0(t)\sin f(t))$$

is the solution of the Kepler problem.

- The radius $r_0(t)$ satisfies $r_0 = r_0(t; e_0) = \dfrac{1-e_0^2}{1+e_0\cos f(t)}$.
- The true anomaly $f(t) = f(t; e_0)$ satisfies $\dfrac{df}{dt} = \dfrac{(1+e_0\cos f)^2}{(1-e_0^2)^{3/2}}$.
- Circular orbit $e_0 = 0$:

$$q_0(t) = (\cos t, \sin t).$$

The restricted planar elliptic three body problem (RPE3BP)

- Motion of three bodies q_1, q_2 and q under the effects of the Newtonian gravitational force.
- Asteroid q has mass 0.
- Assume total mass is equal to one.
- Primaries: q_1 and q_2 have positive mass μ, $1-\mu$ $(0 < \mu \leq 1/2)$.
- The primaries are not influenced by the Asteroid: q_1 and q_2 form a two body problem.
- Assume they move on ellipses (elliptic case): RPE3BP
- Particular case when they move in circles (circular case): RPC3BP.

The equations of the RP3BP

- The motion of the Asteroid $q = (q^1, q^2) \in \mathbb{R}^2$ (planar problem) is described by

$$\frac{d^2q}{dt^2} = \frac{(1-\mu)(q_1(t)-q)}{\|q_1(t)-q\|^3} + \frac{\mu(q_2(t)-q)}{\|q_2(t)-q\|^3},$$

- As a first order system,

$$\begin{aligned} \frac{dq}{dt} &= p \\ \frac{dp}{dt} &= \frac{(1-\mu)(q_1(t)-q)}{\|q_1(t)-q\|^3} + \frac{\mu(q_2(t)-q)}{\|q_2(t)-q\|^3}, \end{aligned}$$

- Phase space has dimension 5 (one has to add time).

The equations of the RP3BP

- This system is a 2π-periodic in time Hamiltonian system (2 and 1/2 degrees of freedom) with Hamiltonian

$$\mathcal{H}(q,p,t) = \frac{p^2}{2} - \frac{(1-\mu)}{\|q - q_1(t)\|} - \frac{\mu}{\|q - q_2(t)\|}.$$

- The Hamiltonian $\mathcal{H}$ is not preserved since the system is nonautonomous.

- Two parameters:

  - the mass of the smaller primary $\mu \in [0, 1/2]$ (mass ratio)
  - the eccentricity of the ellipse of the primaries $e_0 \in [0, 1)$ ($q_1(t)$ and $q_2(t)$ depend on e_0 and μ).

The circular case $e_0 = 0$

- There is a rotational symmetry.
- The RPC3BP has a first integral called Jacobi constant

$$\mathcal{J}(q,p,t) = \mathcal{H}(q,p,t) - (q_1 p_2 - q_2 p_1).$$

- $G = q_1 p_2 - q_2 p_1$ is the angular momentum.
- It can be reduced to a 2 degrees of freedom system.

- Dynamics of RPC3BP and RPE3BP is very complicated.
- Question we want to study: possible types of behavior as $t \to \pm\infty$:

Final motions

Final motions

- Chazy (1922): classification of all possible states that a three body problem can approach as time tends to infinity.
- For the restricted three body problem the possible final states are reduced to four:
  - $H^{\pm}$ (hyperbolic): $\|q(t)\| \to \infty$ and $\|\dot{q}(t)\| \to c > 0$ as $t \to \pm\infty$.
  - $P^{\pm}$ (parabolic): $\|q(t)\| \to \infty$ and $\|\dot{q}(t)\| \to 0$ as $t \to \pm\infty$.
  - $B^{\pm}$ (bounded): $\limsup_{t\to\pm\infty} \|q\| < +\infty$.
  - $OS^{\pm}$ (oscillatory):

$$\limsup_{t\to\pm\infty} \|q\| = +\infty \text{ and } \liminf_{t\to\pm\infty} \|q\| < +\infty.$$

- The two body problem only has $H^{\pm}$, $P^{\pm}$, $B^{\pm}$ and

$$H^{+} = H^{-}, \; P^{+} = P^{-} \text{ and } B^{+} = B^{-}.$$

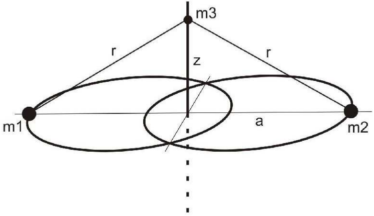
Questions by Chazy

For the R3BP

- Do oscillatory motions exist?
- Does the past predict the future?

The Sitnikov problem

- Restricted spatial elliptic three body problem.
- The primaries have mass $\mu = 1/2$ and move on ellipses of small enough eccentricity ε.
- The Asteroid moves on the (invariant) vertical axis.



- Equations

$$\ddot{z} = -\frac{z}{\left(z^2 + r^2(t)\right)^{3/2}}$$

where $r(t) = r(t + 2\pi) > 0$.

- If we call ε to the eccentricity of the primaries

$$r(t) = \frac{1}{2}(1 - \varepsilon \cos t) + \mathcal{O}(\varepsilon^2).$$

Oscillatory motions

Sitnikov (1960) proved that for $\varepsilon > 0$ small enough:

- $OS^-, OS^+ \neq \emptyset$.
- Future and past are independent.
- All combinations of past and future is possible.

Moser (1973) gave a new proof of Sitnikov results in his book Stable and random motions.

Oscillatory motions

- Sitnikov: Oscillatory motions for a symmetric model and reduced range of parameters.
- Goal after Sitnikov and Moser: extend the proof to other settings (RPC3BP and RPE3BP).

Theorem (Llibre-Simó (1980))

Fix $\mu > 0$ small enough. Then, there exists an orbit $(q(t), p(t))$ of RPC3BP which is oscillatory. Namely, it satisfies

$$\limsup_{t\to\pm\infty} \|q\| = +\infty \quad \text{and} \quad \liminf_{t\to\pm\infty} \|q\| < +\infty.$$

- Concretely: oscillatory motions exist in each $\mathcal{J}(q, p, t; \mu) = J_0$ for large enough J_0 and μ exponentially small with respect to J_0:

$$\mu \ll e^{-\frac{J_0^3}{3}}$$

Oscillatory motions in the RCP3BP

- J. Galante and V. Kaloshin (2011) use Aubry-Mather theory to prove the existence of orbits which initially are in the range of our Solar System and become oscillatory as $t \to +\infty$ with $\mu = 10^{-3}$ (realistic for the Jupiter-Sun).

Theorem (G.–Martín – Seara)

Fix any $\mu \in (0, 1/2]$. Then, there exists an orbit $(q(t), p(t))$ of RCP3BP which is oscillatory. Namely, it satisfies

$$\limsup_{t\to\pm\infty} \|q\| = +\infty \quad \textit{and} \quad \liminf_{t\to\pm\infty} \|q\| < +\infty.$$

Oscillatory motions in the RCP3BP

More precisely,

Theorem

Fix any $\mu \in (0, 1/2]$. Then, there exists $J_0 > 0$ big enough, such that for any $J > J_0$ there exists an orbit $(q_J(t), p_J(t))$ of RCP3BP in the hypersurface $\mathcal{J}(q, p, t; \mu) = J$ which is oscillatory. Namely, it satisfies

$$\limsup_{t\to\pm\infty} \|q_J\| = +\infty \quad \textit{and} \quad \liminf_{t\to\pm\infty} \|q_J\| < +\infty.$$

- These orbits satisfy $\liminf_{t\to\pm\infty} \|q_J\| \sim J^2$.
- They are far from the primaries (far from collision).

Oscillatory motions for the RPE3BP

- One would like to prove their existence for any $\mu \in (0, 1/2]$ and $e_0 \in (0, 1]$.
- Only results for $e_0 \ll 1$.

Theorem (G. – Martín – Seara – Sabbagh)

Fix $\mu \in (0, 1/2]$. There exists $e_0^*(\mu) > 0$ such that for any $e_0 \in (0, e_0^*(\mu))$ there exists an orbit $(q(t), p(t))$ of RPE3BP such that

$$\limsup_{t \to +\infty} \|q\| = +\infty \quad \textit{and} \quad \liminf_{t \to +\infty} \|q\| < +\infty.$$

- $OS^+, OS^- \neq \emptyset$ but we do not know whether $OS^- \cap OS^+ \neq \emptyset$.

Oscillatory motions

- We will spend the rest of the course explaining the proofs of these two theorems.

- We need to understand

  - Moser ideas.

  - How to drop Llibre and Simó condition $\mu \ll e^{-\frac{J_0^3}{3}}$.

  - How to construct oscillatory motions for the RPE3BP, which has more dimension than Sitnikov and RPC3BP.

The nearly integrable limit $\mu \to 0$ for the RPE3BP

- When $\mu \to 0$, the second primary does not influence the Asteroid.
- Hamiltonian $\mathcal{H}(q, p, t) = \dfrac{p^2}{2} - \dfrac{1}{|q|} + \mathcal{O}(\mu)$.
- Asteroid only perceives a body of mass 1 located at the origin (Kepler problem).
- The Asteroid moves on a conic section.
- Assume it performs a parabola.
- Case $\mu > 0$: Oscillatory motions must happen close to parabolic orbits since we want the orbits to go to infinity and come back.
- We want to stay for all time close to these parabolic orbits and get closer and closer to them (close to the infinity cylinder).

Moser and Llibre-Simó approach

Consider the Sitnikov problem:

- McGehee coordinates send “infinity” to the origin. Then, infinity becomes a fixed point.
- Prove that this point has invariant manifolds.
- Attention: This point is not hyperbolic but parabolic (linearization equal to the identity).
- Prove that the invariant manifolds intersect transversally.
- Establish symbolic dynamics (Smale horseshoe) close to these invariant manifolds.
- It leads to the existence of oscillatory motions.

To understand this fact is more convenient to consider the system in polar coordinates.

The RPE3BP in polar coordinates

- Polar coordinates for the Asteroid: $q = (r\cos\alpha, r\sin\alpha)$.
- y symplectic conjugate to r (radial velocity).
- G symplectic conjugate to α (angular momentum).
- Hamiltonian:

$$H(r, \alpha, y, G, t) = \frac{y^2}{2} + \frac{G^2}{2r^2} - U(r, \alpha, t; \mu),$$

- $U(r, \alpha, t)$ is the Newtonian potential

$$U(r, \alpha, t) = \frac{(1-\mu)}{\|re^{i\alpha} - q_1(t)\|} + \frac{\mu}{\|re^{i\alpha} - q_2(t)\|}.$$

- It satisfies $U(r, \alpha, t) = \frac{1}{r} + \mathcal{O}(\mu)$.

Circular case

- Circular case $e_0 = 0$:

$$q_1(t) = -\mu q_0(t),\ \ q_2(t) = (1-\mu)q_0(t) \ \text{ where } \ q_0(t) = e^{it}.$$

- Potential: $\quad U(r,\alpha,t) = \dfrac{(1-\mu)}{\|re^{i\alpha} + \mu e^{it}\|} + \dfrac{\mu}{\|re^{i\alpha} - (1-\mu)e^{it}\|}.$
- U only depends on $\alpha - t$.
- Defining new angle $\phi = \alpha - t$ (rotating coordinates): we have the 2dof system

$$\mathcal{H}(r,\phi,y,G) = \frac{y^2}{2} + \frac{G^2}{2r^2} - G - \frac{(1-\mu)}{\|re^{i\phi} + \mu\|} + \frac{\mu}{\|re^{i\phi} - (1-\mu)\|},$$

- Energy $\mathcal{H} = H - G$ is equal to the Jacobi constant and is preserved.

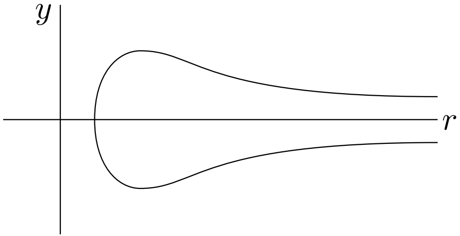
Case $\mu = 0$ in polar coordinates

- Hamiltonian: $H(r, \alpha, y, G, t; 0) = \frac{y^2}{2} + \frac{G^2}{2r^2} - \frac{1}{r}$.
- Equations

$$\dot{r} = y \qquad \dot{\alpha} = \frac{G}{r^2}$$
$$\dot{y} = \frac{G^2}{r^3} - \frac{1}{r^2} \qquad \dot{G} = 0$$

- G is a first integral.
- For fixed G, we have a 1 degree of freedom system for (r, y).
- Orbits in parabolas satisfy $r(t) \to +\infty$ and $y(t) \to 0$ as $t \to \pm\infty$.
- They belong to level of energy $H = 0$.
- $H < 0$ are elliptic orbits and $H > 0$ are hyperbolic orbits.

Parabolic motion in the (r, y) plane



- The point $(r, y) = (+\infty, 0)$ can be seen as a critical point at infinity.
- The parabola is at the same time the stable and unstable invariant manifold of infinity.

Parabolic motion in the phase space

- Consider the full phase space of dimension 5.
- For the full system

$$\Lambda = \left\{ (r, y) = (+\infty, 0), G \in [G_1, G_2], (\alpha, t) \in \mathbb{T}^2 \right\}$$

  is the infinity cylinder.
- The invariant manifolds of this infinity cylinder coincide.
- In the circular case, we have one angle less: the cylinder is foliated by periodic orbits.
- Until Lecture 5, we only work with the RPC3BP.

The parameters

- We want to prove existence of oscillatory motions for all $\mu \in (0, 1/2]$
- So, even if useful to understand what happens, the limit $\mu \to 0$ is not meaningful.
- Our results only apply for Jacobi constant $J_0 \gg 1$.
- The Jacobi constant J_0 is an extra parameter.
- To understand its role, we consider the time parameterization of the parabola of the 2BP.

Parameterization of the parabolas

- Equations

$$\dot r = y$$
$$\dot y = \frac{G^2}{r^3} - \frac{1}{r^2}$$
$$\dot\alpha = \frac{G}{r^2}$$
$$\dot G = 0$$

- Parameterization

$$r = \frac{1}{2}G_0^2(1+\tau^2)$$
$$\alpha = \alpha_0 + \pi + 2\arctan\tau$$
$$y = \frac{2\tau}{G_0(1+\tau^2)}$$
$$G = G_0$$

where $t = \frac{G_0^3}{2}\left(\tau + \frac{\tau^3}{3}\right)$.

Another parameter

- For each $G = G_0$ we have,
- Consider $G_0 \gg 1$:
  - $r = \frac{1}{2} G_0^2 (1 + \tau^2)$: all points in the parabola are of size bigger than $G_0^2/2$.
  - $t = \frac{G_0^3}{2}(\tau + \frac{\tau^3}{3})$ means that the trajectories in the parabola evolve very slowly.
  - At parabolic motion, $H = 0$. So

$$\mathcal{J} = H - G = G.$$

- For every $\mu \in (0, 1/2]$, $G_0 \longrightarrow +\infty$ gives a different nearly integrable limit.

Limit with large angular momentum

- How affects the system the size of angular momentum?
- From the point of view of the Asteroid:
  - The primaries are extremely close.
  - They move much faster than the massless body.
- The Asteroid at first order only perceive one body of mass one at the origin.
- The perturbative terms have size equal to the ratio between distances: $\mathcal{O}(G_0^{-2})$.
- So for all $\mu \in (0, 1/2]$ and as $G_0 \to +\infty$, we have a 2BP plus a $\mathcal{O}(G_0^{-2})$ perturbation whose time frequency is G_0^3.