Lecture 2: The invariant manifolds of infinity and their intersection

Marcel Guardia

Universitat Politècnica de Catalunya

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Moser approach

- McGehee coordinates send “infinity” to the origin. Then, infinity becomes a parabolic fixed point (linearization equal to the identity).

- This parabolic point has invariant manifolds (Parabolic motions).

- Prove that they intersect transversally.

- Establish symbolic dynamics close to these invariant manifolds.
The McGehee transformation and the local invariant manifolds of infinity.

The transversality between the invariant manifolds: the Poincaré–Melnikov method

Simó and Llibre result: the transversality of invariant manifolds provided $\mu \ll e^{-\frac{G_0^3}{3}}$ and $G_0 \gg 1$. 
The RPC3BP

- The RPC3BP in rotating polar coordinates

\[ H(r, \phi, y, G; \mu) = \frac{y^2}{2} + \frac{G^2}{2r^2} - G - U(r, \phi; \mu), \]

where \( U(r, \phi; \mu) \) is the Newtonian potential

\[ U(r, \phi) = \frac{(1 - \mu)}{||re^{i\phi} + \mu||} + \frac{\mu}{||re^{i\phi} - (1 - \mu)||}. \]

- Two parameters: \( \mu \) and \( G_0 \).

- Now \( H \) corresponds to the Jacobi constant and we can identify it with \( G_0 \).

- Infinity: \( (r, y) = (+\infty, 0) \)
The McGehee coordinates

- McGehee change of coordinates \( r = \frac{2}{x^2} \) sends infinity to \( x = 0 \).
- Only the region \( x > 0 \) is meaningful.
- Integrable case \( \mu \to 0 \),
  \[
  \dot{x} = -\frac{1}{4}x^3 y \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \dot{\phi} = -1 + \frac{1}{4}x^4 G \\
  \dot{y} = \frac{1}{8}G^2 x^6 - \frac{1}{4}x^4 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \dot{G} = 0
  \]
- \( G \) and
  \[
  K_0(x, y, G) = H_0 \left( \frac{2}{x^2}, y, G \right) = \frac{y^2}{2} + \frac{G^2 x^4}{8} - \frac{x^2}{2}.
  \]
  are first integrals.
\[ \begin{align*}
\dot{x} &= -\frac{1}{4} x^3 y \\
\dot{\phi} &= -1 + \frac{1}{4} x^4 G \\
\dot{y} &= \frac{1}{8} G^2 x^6 - \frac{1}{4} x^4 \\
\dot{G} &= 0
\end{align*} \]

- The cylinder at infinity is \((x, y) = (0, 0)\), which is invariant.
- For any value of \(G_0\):
  \[ \Lambda_{G_0} = \{ (x, y, \phi, G) = (0, 0, \phi, G_0), \phi \in \mathbb{T} \} \]
  is a periodic solution.

- The cylinder of infinity \(\Lambda = \bigcup_{G_0} \Lambda_{G_0}\) is foliated by periodic orbits (one at each level of energy).
Invariant manifolds of infinity when $\mu = 0$

$$K_0(x, y, G) = \frac{y^2}{2} + \frac{G^2 x^4}{8} - \frac{x^2}{2}.$$

- For fixed $G = G_0$, dynamics in the $(x, y)$ plane
  $$\dot{x} = -\frac{1}{4} x^3 y = -\frac{1}{4} x^3 \partial_y K_0$$
  $$\dot{y} = \frac{1}{8} G_0^2 x^6 - \frac{1}{4} x^4 = \frac{1}{4} x^3 \partial_x K_0$$

  The vector field has no linear part at $(x, y) = (0, 0)$.
  It is a parabolic point (linear part equal to zero).
  Parabolic points do not always have invariant manifolds.
  In this case it is enough to use the first integral

$$K_0(x, y, G_0) = \frac{y^2}{2} + \frac{G_0^2 x^4}{8} - \frac{x^2}{2}.$$
Invariant manifolds of infinity when $\mu = 0$

- Stable/ unstable manifolds defined by
  \[ \frac{y^2}{2} + \frac{G_0^2 x^4}{8} - \frac{x^2}{2} = 0 \]

- We have a separatrix as for the Duffing equation.

- The vertical axis is infinity.

- **Attention**: Orbits on the separatrix do not tend to $(x, y) = (0, 0)$ exponentially but polynomially.
Local behavior of the RPC3BP

- Local behavior in McGehee Coordinates

\[ \dot{x} = -\frac{1}{4}x^3y \]
\[ \dot{y} = \frac{1}{8}G^2x^6 + \partial_r U \left( 2x^{-2}, \phi \right) = -\frac{1}{4}x^4 + x^6O_0 \]
\[ \dot{\phi} = -1 + \frac{1}{4}Gx^4 \]
\[ \dot{G} = \partial_\alpha U \left( 2x^{-2}, \phi \right) = x^6O_0 \]

where \( O_k = O(\| (x, y) \| ^k) \).

- Reducing by the energy, we can eliminate \( G \).
- Reparameterizing time we can identify it with \( \phi \).
- So, we can consider a non-autonomous system in the plane (without modifying the first order).
The local invariant manifolds

- Local behavior

\[
x' = -\frac{1}{4}x^3y + x^6f_1(x, y, \phi)
\]

\[
y' = -\frac{1}{4}x^4 + x^6f_2(x, y, \phi)
\]

- The change of coordinates

\[
q = \frac{1}{2}(x - y)
\]

\[
p = \frac{1}{2}(x + y)
\]

leads to

\[
q' = \frac{1}{4}(q + p)^3(q + (q + p)^3\mathcal{O}_0)
\]

\[
p' = -\frac{1}{4}(q + p)^3(q + (q + p)^3\mathcal{O}_0)
\]
Local invariant manifolds

- New system, defined for $q + p > 0$,

\[
q' = \frac{1}{4}(q + p)^3(q + (q + p)^3O_0)
\]
\[
p' = -\frac{1}{4}(q + p)^3(q + (q + p)^3O_0)
\]

- General case of invariant manifolds of parabolic objects is quite open.

- Here just look for invariant manifolds for periodic orbits (fixed points for the Poincaré map).
McGehee proved the existence of the local invariant manifolds as graphs for all $\mu \in [0, 1/2]$ and $G_0 > 0$.

**Theorem**

Fix $G_0 > 0$. The periodic orbit $\Lambda_{G_0}$ at $(p, q) = (0, 0)$ possesses invariant stable and unstable manifolds, $W^u_{G_0}$ and $W^s_{G_0}$.

More concretely,

$$W^u_{G_0} = \{(q, p, \phi, G) \mid p = \gamma^u(q, \phi_0, G_0), \ q \in [0, q_0)\}$$

where

1. $\gamma^u$ is $C^\infty$ with respect to $q$ and analytic with respect to $(\phi_0, G_0)$,
2. $\gamma^u(q, \phi_0, G_0) = O(q^2)$.

The analogous statement holds for $W^s$, as a graph over $p$. 
The invariant manifolds are analytic away from $q = 0$.

We need regularity up to the origin to perform a change of coordinates which straightens the invariant manifolds.

Hyperbolic case: stable/unstable invariant manifolds of periodic orbits of analytic systems are analytic.

Parabolic case: they are only $C^\infty$ at $q = 0$.

Baldomá-Fontich-Martin: the invariant manifolds are $1/3$-Gevrey at $q = 0$: the Taylor coefficients of the parameterization grow as

$$|P_k| \leq C_1 C_2^k (k!)^{1/3}$$
After a change of coordinates

\[ \tilde{q}' = \frac{1}{4}(\tilde{q} + \tilde{p})^3 \tilde{q}(1 + O_2) \]

\[ \tilde{p}' = -\frac{1}{4}(\tilde{q} + \tilde{p})^3 \tilde{p}(1 + O_2) \]

The invariant manifolds are the axes.

We will use this system on Lecture 4 to study the dynamics close to infinity.
The global invariant manifolds

- We want to study the global invariant manifolds and prove that they intersect transversally.

- Poincaré–Melnikov Theory allows to prove the transversality of the invariant manifolds (in some settings).

- We apply Melnikov Theory to the RPC3BP.

- Tomorrow, we will explain how to prove transversality when Melnikov Theory cannot be applied.

- This will be related to exponentially small phenomena.
Consider a Hamiltonian with $1 + \frac{1}{2}$ degrees of freedom with $2\pi$–periodic time dependence:

$$H(q, p, t; \delta) = H_0(q, p) + \delta H_1(q, p, t; \delta),$$

Assume that the origin $(q, p) = (0, 0)$ is an equilibrium of saddle type at $H_0 = 0$.

It has associated separatrices included in $H_0^{-1}(0)$.

Consider one of the separatrices (for instance the one with $p > 0$) and write it as

$$\{(q_0(t), p_0(t)), t \in \mathbb{R}\} =: \Gamma.$$
The easiest possible example: perturbations of the pendulum

- Example: the pendulum
  \[ H_0(q, p) = \frac{p^2}{2} + (\cos q - 1). \]

- Parameterization of the upper separatrix
  \[
  q_0(t) = 4 \arctan(e^t),
  
  p_0(t) = \frac{2}{\cosh t}
  
  \text{so that}
  \[
  (q_0(t), p_0(t)) \to (0 \mod 2\pi, 0)
  \]
  \text{for } t \to \pm\infty.

- So \[ |q_0(t)| + |p_0(t)| \leq C e^{-|t|}. \]
Extended phase space: add time as variable ($\dot{s} = 1$).

$\Lambda = \{(q, p) = (0, 0), s \in \mathbb{T}\}$ is a saddle periodic orbit.

$\Lambda$ has coincident stable and unstable surfaces.

Motion on the upper 2-dimensional separatrix

$W^0(\tilde{\Lambda}) = \{(q_0(\tau), p_0(\tau), s), \tau \in \mathbb{R}, s \in \mathbb{T}\} = \Gamma \times \mathbb{T}$

is

$\phi(t; q_0(\tau), p_0(\tau), s_0) = (q_0(\tau + t), p_0(\tau + t), s_0 + t)$
The perturbed case: $\delta \neq 0$ small

$$H(q, p, t; \delta) = H_0(q, p) + \delta H_1(q, p, t; \delta),$$

- For $\delta \ll 1$, there exists a periodic orbit $\Lambda_\delta$, with stable/unstable manifolds $W_{loc}^{s,u}(\Lambda_\delta)$ and
  $$\Lambda_\delta = \Lambda + O(\delta), \quad W_{loc}^{s,u}(\Lambda_\delta) = W_{loc}^{s,u}(\Lambda) + O(\delta)$$

- We want conditions that imply $W^s(\Lambda_\delta) \neq W^u(\Lambda_\delta)$

- Since $W^s(\Lambda_\delta)$ and $W^u(\Lambda_\delta)$ are close to $W^0(\Lambda)$, they can be also parameterized in terms of $\tau$ and $s$
We fix $\Sigma$, a transversal section to the unperturbed separatrix in order to measure in it the splitting.

We consider a parameterization $\gamma$ of the unperturbed separatrix such that $\gamma(0)$ belongs to this section.

Poincaré–Melnikov Theory: expand the parameterizations of the invariant manifolds in power series in $\delta$ and compute the first order of their difference at $\Sigma$.

The distance will be a function periodic in $s$. 
We define the Melnikov function as:

\[ M(s) = \int_{-\infty}^{+\infty} \{H_0, H_1\} (q_0(t), p_0(t), t + s) \, dt \]

We have that \(|q_0(t)| + |p_0(t)| \leq Ce^{-\lambda |t|}\) where \(\lambda > 0\) is the eigenvalue of the saddle.

This implies that the integral is convergent.

\(M\) can be computed since \(H_0, H_1\) and \(\gamma\) are known.

Often is not so easy to compute analytically this integral.

If the perturbation \(h\) is \(T\)-periodic, the Melnikov potential and function are also \(T\)-periodic.
The distance between both invariant manifolds for $\delta > 0$ is given by:

$$d(s, \delta) = \delta \frac{M(s)}{\|DH_0(\gamma(0))\|} + O(\delta^2)$$

If there exists $s_0$ such that

(i) $M(s_0) = 0$  
(ii) $\frac{\partial M}{\partial s} \bigg|_{s=s_0} \neq 0$

Then the invariant manifolds intersect transversally at $\Sigma$ for some $s'$ $\delta$-close to $s_0$. 
An example: The perturbed pendulum

\[ H \left( p, q, \frac{t}{T} \right) = \frac{p^2}{2} + (\cos q - 1) + \delta(\cos q - 1) \sin \frac{t}{T} \]

- Equations

\[ \dot{q} = p \]
\[ \dot{p} = \sin q + \delta \sin q \sin \frac{t}{T} \]

- \( \Lambda_\delta = \{(0, 0)\} \) is a hyperbolic periodic orbit for this system.
- The Melnikov function is

\[ \mathcal{M}(s) = \int\limits_{-\infty}^{\infty} \{ H_0, H_1 \}(p_0(\sigma), q_0(\sigma), s + \sigma) d\sigma \]
\[ = \int\limits_{-\infty}^{\infty} p_0(\sigma) \sin q_0(\sigma) \sin \frac{s + \sigma}{T} d\sigma \]
\[ = 4 \cos \frac{s}{T} \int\limits_{-\infty}^{\infty} \frac{\sinh \sigma}{\cosh^3 \sigma} \sin \frac{\sigma}{T} d\sigma \]
An example: The perturbed pendulum

So we only need to compute the integral
\[ \int_{-\infty}^{\infty} \frac{\sinh t}{\cosh^3 t} e^{i \sigma/T} d\sigma = \frac{\pi i}{T^2} \frac{1}{2 \sinh(\pi/2T)}. \]

cosh \ t \ has \ zeros \ at \ t = \pm i\pi/2.

Applying Residues Theorem, the Melnikov function is
\[ \mathcal{M}(s) = \frac{2\pi}{T^2} \frac{1}{\sinh(\pi/2T)} \cos \frac{s}{T}. \]

The distance between manifolds is given by
\[ d(s, \delta) = \delta \frac{\mathcal{M}(s)}{\|DH_0(\gamma(0))\|} + \mathcal{O}(\delta^2) \]
so
\[ d(s, \delta) = \delta \frac{1}{\|DH_0(\gamma(0))\|} \frac{2\pi}{T^2} \frac{1}{\sinh(\pi/2T)} \cos \frac{s}{T} + \mathcal{O}(\delta^2) \]

Non-degenerate zeros of \( \mathcal{M}(s) \) give rise to transversal homoclinic orbits.
What happens if we have a fast forcing?

- Take $T = \varepsilon \ll 1$:
  
  $$H\left(p, q, \frac{t}{\varepsilon}\right) = \frac{p^2}{2} + (\cos q - 1) + \delta(\cos q - 1) \sin \frac{t}{\varepsilon}$$

- Apply Poincaré-Melnikov: fix $\varepsilon$ and expand in $\delta$.
  
  The Melnikov function is:

  $$\mathcal{M}(s, \varepsilon) = \frac{2\pi}{\varepsilon^2} \frac{1}{\sinh(\pi/2\varepsilon)} \cos \frac{s}{\varepsilon} \sim \frac{4\pi}{\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}} \cos \frac{s}{\varepsilon}.$$ 

  The distance between manifolds is given by

  $$d(s, \varepsilon, \delta) = \delta \frac{4\pi}{\|DH_0(\gamma(0))\|} e^{-\frac{\pi}{2\varepsilon}} \cos \frac{s}{\varepsilon} + \mathcal{O}(\delta^2)$$

  We need $\delta = \mathcal{O}(e^{-\frac{\pi}{2\varepsilon}})$ to make the error term smaller!

  If we want $\delta \sim \varepsilon$, Poincaré-Melnikov Theory cannot be applied.
What happens if we have a fast forcing?

- Poincaré-Melnikov Theory for periodic fast perturbations only works provided $\delta = O(e^{-\frac{\pi}{2\varepsilon}})$.

- If we want $\delta \sim \varepsilon$, Poincaré-Melinkov Theory cannot be applied.

- Exponentially small perturbation is extremely restrictive.

- Fast forcing is a very important setting: appears typically when studying invariant manifolds in the resonances of nearly integrable Hamiltonian systems.

- It appears in many Arnold diffusion problems (“a priori stable” setting).
Recall that we have two parameters $\mu$ and $G_0$.

Unperturbed separatrix (parabolic motion) was parameterized as

$$r = \frac{1}{2} G_0^2 (1 + \tau^2)$$

$$\alpha = \alpha_0 + \pi + 2 \arctan \tau$$

$$y = \frac{2\tau}{G_0(1 + \tau^2)}$$

$$G = G_0$$

where $t = \frac{G_0^3}{2} \left( \tau + \frac{\tau^3}{3} \right)$.

Recall that $(r, y) = (0, 0)$ is parabolic.

As long as the local invariant manifolds are defined, we can (try to) apply Melnikov Theory.

To have an unperturbed problem and a separatrix independent of the parameters, we apply a scaling.
The scaled system

- Fix $G_0 > 0$.
- Rescaling: $r = G_0^2 \tilde{r}$, $y = G_0^{-1} \tilde{y}$, $\alpha = \tilde{\alpha}$, $G = G_0 \tilde{G}$.
- Time rescaling: $t = G_0^3 s$.
- New Hamiltonian

\[
\tilde{H}(\tilde{r}, \tilde{\alpha}, \tilde{y}, \tilde{G}, s) = \frac{\tilde{y}^2}{2} + \frac{\tilde{G}^2}{2\tilde{r}^2} - \frac{(1 - \mu)}{\| r e^{i(\tilde{\alpha} - G_0^3 s + \mu G_0^{-2})} \|} - \frac{\mu}{\| r e^{i\tilde{\alpha} - G_0^3 s} - (1 - \mu) G_0^{-2} \|}.
\]

- Expanding,

\[
\tilde{H}(\tilde{r}, \tilde{\alpha}, \tilde{y}, \tilde{G}, s) = \frac{\tilde{y}^2}{2} + \frac{\tilde{G}^2}{2\tilde{r}^2} - \frac{1}{\tilde{r}} + H_1(\tilde{r}, \tilde{\alpha}, \tilde{y}, \tilde{G}, G_0^3 s)
\]

with $H_1 = O\left(\mu G_0^{-2}\right)$. 
The scaled separatrix

- Scaled separatrix:

\[
\begin{align*}
    r &= \frac{1}{2} (1 + \tau^2) \\
    \alpha &= \alpha_0 + \pi + 2 \arctan \tau \\
    y &= \frac{2\tau}{(1 + \tau^2)} \\
    G &= 1
\end{align*}
\]

where \( t = \frac{1}{2} \left( \tau + \frac{\tau^3}{3} \right) \).

- In McGehee coordinates: \( x = 2 \left( 1 + \tau^2 \right)^{-1/2} \).
- We apply Poincaré-Melnikov Theory using \( \mu \) as a small parameter.
- The perturbation depends on \( G_0 \) and so \( \mathcal{M}(s, G_0) \) too.
- The computation of this Melnikov function is not so easy.
- We only know how to compute it if we assume \( G_0 \gg 1 \).
Melnikov Theory applied to the RPC3BP

- Compared with

\[ H\left(p, q, \frac{t}{\varepsilon}\right) = \frac{p^2}{2} + (\cos q - 1) + \delta (\cos q - 1) \sin \frac{t}{\varepsilon} \]

we have \( \varepsilon = G_0^{-3} \) and \( \delta = \mu \).

- Melnikov function

\[ M(s) = CG_0^{3/2} e^{-\frac{G_0^{3}}{3}} \left( \sin(G_0^3 s) + O\left(G_0^{-1}\right) \right) \]

for some computable constant \( C > 0 \).

- Distance between manifolds

\[ d(s, \mu, G_0) \sim \mu CG_0^{3/2} e^{-\frac{G_0^{3}}{3}} \left( \sin(G_0^3 s) + O\left(G_0^{-1}\right) \right) + O\left(\mu^2 G_0^{-4}\right). \]
So, Melnikov Theory only applied provided $\mu \ll G_0^{-3/2} e^{-G_0^3/3}$

Simó and Llibre used this condition to prove the transversality of the invariant manifolds.

Only proved existence of oscillatory motions under this condition.

Next day,
- We will explain how to prove the transversality for fast periodic perturbations.
- I will apply it to the RPC3BP for any $\mu \in (0, 1/2]$ and $G_0 \gg 1$.
- Construct the symbolic dynamics that leads to oscillatory motions for the RPC3BP.