Lecture 3: Exponentially small splitting of separatrices

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Exponentially small splitting of separatrices

- For the RPC3BP we want to prove transversality of the invariant manifolds for fixed $\mu \in (0, 1/2]$ and $G_0 \gg 1$.
- The distance between the invariant manifolds is exponentially small in G_0.
- Melnikov Theory does not apply unless $\mu \ll e^{-\frac{G_0^3}{3}}$.
- To prove splitting for any $\mu \in (0, 1/2]$ is a difficult problem since classical perturbative techniques do not apply.
- We have to deal with exponentially small splitting of separatrices.
- We show how to deal with this problem in a simpler model.

The perturbed pendulum

- Hamiltonian

$$H\left(y, x, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + (\cos x - 1) + \delta(\cos x - 1)\sin\frac{t}{\varepsilon}$$

- Equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \sin x + \delta \sin x \sin\frac{t}{\varepsilon} \end{aligned}$$

- This is a model of the resonances of nearly completely integrable Hamiltonian systems.
- Tipically in this problems δ is a fixed non small constant.
- In the RPC3BP one has: $\varepsilon \sim G_0^{-3}$ and $\delta \sim \mu G_0^{-2}$.
- Luckily in the RPC3BP ressembles this model with $\delta \sim \varepsilon^{2/3}$.
- This makes a big difference for analyzing the splitting.

The perturbed pendulum

- From now on:

$$H\left(y, x, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + (\cos x - 1) + \varepsilon(\cos x - 1)\sin\frac{t}{\varepsilon}$$

- Equations

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \sin x + \varepsilon \sin x \sin\frac{t}{\varepsilon} \\ \dot{t} &= 1\end{aligned}$$

- $(x, y) = (0, 0)$ is a hyperbolic periodic orbit for this system.
- We want to compute the splitting between its invariant manifolds.

First rigorous result dealing with this problem:

Theorem (Neishtadt 84)

Let us fix $\varepsilon_0 > 0$*. Then, for* $\varepsilon \in (0, \varepsilon_0)$*, there exists an* ε*-close to the identity canonical transformation that transforms system*

$$H\left(y, x, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + (\cos x - 1) + H_1\left(x, y, \frac{t}{\varepsilon}\right)$$

with H_1 *analytic into*

$$H\left(\tilde{y}, \tilde{x}, \frac{t}{\varepsilon}\right) = \frac{\tilde{y}^2}{2} + (\cos \tilde{x} - 1) + \varepsilon \tilde{H}_0(\tilde{y}, \tilde{x}, \varepsilon) + P\left(\tilde{y}, \tilde{x}, \frac{t}{\varepsilon}\right)$$

and P *satisfies*

$$|P| \leq c_2 e^{-\frac{c_1}{\varepsilon}}$$

for certain constants $c_1, c_2 > 0$*.*

This theorem implies that:

- The system is exponentially small close to integrable.
- The invariant manifolds are exponentially close.
- It does not give any information about the splitting of the separatrix
- We do not know whether they coincide or not.

We need to compute the first order of the distance between the invariant manifolds.

Plan

- Look for suitable parameterizations of the invariant manifolds.
- Extend these parameterizations to certain regions of the complex plane.
- Analyze the difference of the parameterizations in these regions.
- Deduce from this analysis the first order of the difference in the reals.
- Simplifications of our example: the unperturbed separatrix is a graph over the base

Parameterizations of the invariant manifolds

- We look for parameterizations of the invariant manifolds as graphs

$$y = \partial_x S^{u,s}(x, t/\varepsilon)$$

- The generating functions $S^{u,s}$ satisfies the Hamilton-Jacobi equation.

$$\varepsilon^{-1}\partial_\tau S(x,\tau) + H(x, \partial_x S(x,\tau), \tau) = 0$$

- Pendulum example

$$\varepsilon^{-1}\partial_\tau S + \frac{(\partial_x S)^2}{2} + \cos x - 1 + \varepsilon(\cos x - 1)\sin\tau = 0$$

- Asymptotic conditions:

$$\begin{cases} \lim\limits_{x\to 0} \partial_x S^u(x, t/\varepsilon) = 0 \\ \lim\limits_{x\to 2\pi} \partial_x S^s(x, t/\varepsilon) = 0 \end{cases}$$

Reparameterization using the time over the separatrix

- Parameterization of the pendulum separatrix:

$$x_0(v) = 4\arctan(e^v), \qquad y_0(v) = \frac{2}{\cosh v}.$$

- Reparametrize $x = x_0(v)$, $T^{u,s}(v,\tau) = S^{u,s}(x_0(v),\tau)$
- Equation

$$\varepsilon^{-1}\partial_\tau S + \frac{(\partial_x S)^2}{2} + \cos x - 1 + \varepsilon(\cos x - 1)\sin\tau = 0$$

becomes

$$\varepsilon^{-1}\partial_\tau T + \frac{\cosh^2 v}{8}(\partial_v T)^2 - \frac{2}{\cosh^2 v} - \varepsilon\frac{2}{\cosh^2 v}\sin\tau = 0$$

- Asymptotic conditions

$$\begin{cases} \lim_{\mathrm{Re}\, v\to-\infty} \cosh v\,\partial_v T^u(v,\tau) = 0 \\ \lim_{\mathrm{Re}\, v\to+\infty} \cosh v\,\partial_v T^s(v,\tau) = 0. \end{cases}$$

New parameterizations

$$\varepsilon^{-1}\partial_\tau T + \frac{\cosh^2 v}{8}(\partial_v T)^2 - \frac{2}{\cosh^2 v} - \varepsilon\frac{2}{\cosh^2 v}\sin\tau = 0$$

- Parameterizations of the form

$$x = 4\arctan(e^v), \quad y = \frac{\cosh v}{2}\partial_v T^{u,s}(v,\tau).$$

- For the unperturbed pendulum: $\partial_v T_0(v) = \dfrac{4}{\cosh^2 v}$.
- We can consider a formal power series expansion

$$T^{u,s}(v,\tau) = T_0(v) + \sum_{k\geq 1}\varepsilon^k T_k^{u,s}(v,\tau)$$

Formal power series

- We have: $T_k^u(v,\tau) = T_k^s(v,\tau)\ \forall k \in \mathbb{N}$
- Namely: $T^u(v,\tau) - T^s(v,\tau) = \mathcal{O}(\varepsilon^k)\ \forall k \in \mathbb{N}$
- Proceeding formally, their difference is beyond all orders.
- What is happening?

  1. The power series is convergent: both manifolds coincide in the perturbed case
  2. The power series in ε is divergent: the manifolds do not coincide and their difference is flat with respect ε.

- We will see that is happening the second option

New parameterizations

$$\varepsilon^{-1}\partial_\tau T + \frac{\cosh^2 v}{8}(\partial_v T)^2 - \frac{2}{\cosh^2 v} - \varepsilon\frac{2}{\cosh^2 v}\sin\tau = 0$$

- Unperturbed pendulum: $\partial_v T_0(v) = \dfrac{4}{\cosh^2 v}$.
- Perturbed pendulum: $T^{u,s} = T_0 + \mathcal{T}^{u,s}$.
- Parameterizations of the invariant manifolds:

$$\begin{cases} x = 4\arctan(e^v) \\ y = \dfrac{2}{\cosh v} + \dfrac{\cosh v}{2}\partial_v \mathcal{T}^{u,s}(v,\tau) \end{cases}$$

- To study the difference between manifolds: analyze $\mathcal{T}^u - \mathcal{T}^s$.

$$\varepsilon^{-1}\partial_\tau T + \frac{\cosh^2 v}{8}(\partial_v T)^2 - \frac{2}{\cosh^2 v} - \varepsilon\frac{2}{\cosh^2 v}\sin\tau = 0$$

- Plug $T^{u,s} = T_0 + \mathcal{T}^{u,s}$ into the Hamilton-Jacobi equation.
- Recall $\partial_v T_0(v) = \dfrac{4}{\cosh^2 v}$.
- New equation:

$$\varepsilon^{-1}\partial_\tau \mathcal{T} + \partial_v \mathcal{T} + \frac{\cosh^2 v}{8}(\partial_v \mathcal{T})^2 - \varepsilon\frac{2}{\cosh^2 v}\sin\tau = 0$$

- $\mathcal{T}^{u,s}$ are expected to be small.
- We can solve this equation by inverting the linear differential operator.

The integral operators

- Inverses of $\mathcal{L}_0 = \varepsilon^{-1}\partial_\tau + \partial_v$.
- For functions which decay to zero as $\mathrm{Re}\, v \longrightarrow -\infty$,

$$\mathcal{G}^u(h)(v,\tau) = \int_{-\infty}^{0} h(v+t, \tau+\varepsilon^{-1}t)\, dt.$$

- For functions which decay to zero as $\mathrm{Re}\, v \longrightarrow +\infty$,

$$\mathcal{G}^s(h)(v,\tau) = \int_{+\infty}^{0} h(v+t, \tau+\varepsilon^{-1}t)\, dt.$$

The fixed point equation

- $\mathcal{T}^u$ is a solution of

$$\mathcal{T}^u = \mathcal{G}^u\left(-\frac{\cosh^2 v}{8}(\partial_v \mathcal{T}^u)^2 + \varepsilon\frac{2}{\cosh^2 v}\sin\tau\right)$$

with

$$\mathcal{G}^u(h)(v,\tau) = \int_{-\infty}^{0} h(v+t, \tau+\varepsilon^{-1}t)\, dt.$$

- Look for fixed points of

$$\mathcal{F}^u(\mathcal{T}) = \mathcal{G}^u\left(-\frac{\cosh^2 v}{8}(\partial_v \mathcal{T})^2 + \varepsilon\frac{2}{\cosh^2 v}\sin\tau\right)$$

- $\mathcal{T}^s$ is a fixed point of

$$\mathcal{F}^s(\mathcal{T}) = \mathcal{G}^s\left(-\frac{\cosh^2 v}{8}(\partial_v \mathcal{T})^2 + \varepsilon\frac{2}{\cosh^2 v}\sin\tau\right)$$

The fixed point argument in the reals

- $\mathcal{T}^{u,s} = \mathcal{F}^{u,s}(\mathcal{T}^{u,s}) = \mathcal{F}^{u,s}(0) + \mathcal{O}(\varepsilon^2)$.
- $\mathcal{F}^u(0) = \varepsilon \int_{-\infty}^{0} \frac{2}{\cosh^2(v+t)} \sin\left(\tau + \varepsilon^{-1} t\right)\, dt$
- $\mathcal{F}^s(0) = \varepsilon \int_{+\infty}^{0} \frac{2}{\cosh^2(v+t)} \sin\left(\tau + \varepsilon^{-1} t\right)\, dt$
- The difference: $\mathcal{T}^u - \mathcal{T}^s = \mathcal{F}^u(0) - \mathcal{F}^s(0) + \mathcal{O}(\varepsilon^2)$.
- Difference between first iteration is the Melnikov potential

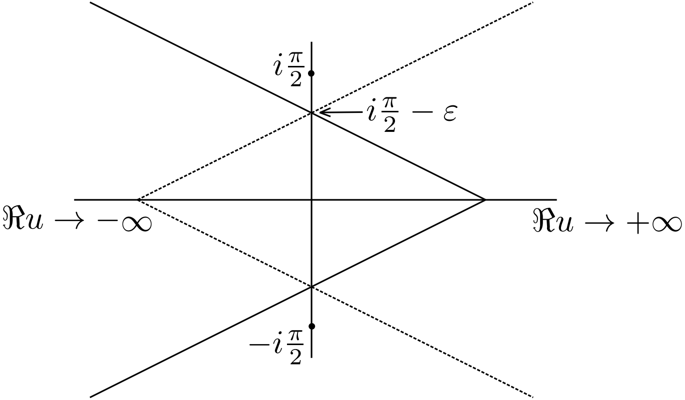
$$\mathcal{F}^u(0) - \mathcal{F}^s(0) = \varepsilon \int_{-\infty}^{+\infty} \frac{2}{\cosh^2(v+t)} \sin\left(\tau + \varepsilon^{-1} t\right)\, dt.$$

- We have recovered Melnikov theory through the Hamilton-Jacobi equation.

- First order (Melnikov potential):
$\mathcal{F}^u(0) - \mathcal{F}^s(0) = 4\pi e^{-\frac{\pi}{2\varepsilon}} \sin(\tau - \varepsilon^{-1} v)$
- Melnikov is exponentially small and the error is $\mathcal{O}(\varepsilon^2)$.
- We need better estimates for the error.
- We consider the fixed point equation for complex values of $v \in \mathbb{C}$.
- To compute the Melnikov function, we have analyzed the singularities of the Melnikov integrand at $v = \pm i\frac{\pi}{2}$.
- We extend the parameterizations of the invariant manifolds up to a distance $\mathcal{O}(\varepsilon)$ of these singularities.

Complex domains:

- In τ: $\mathbb{T}_\sigma = \{\tau \in \mathbb{C}/(2\pi\mathbb{Z}) : |\mathrm{Im}\,\tau| \leq \sigma\}$.
- In *v*:



Close to the singularities of the unperturbed separatrix

- By the fixed point argument $\mathcal{T}^{u,s}$ are defined in these complex domains and satisfy:

$$|\mathcal{T}^{u,s}(v,\tau)| \lesssim \frac{\varepsilon^2}{(v \pm i\pi/2)^2}.$$

- Unperturbed separatrix

$$T_0(v) \sim \frac{1}{v \pm i\pi/2}$$

- Both become larger close to the singularity.
- T_0 is always larger than $\mathcal{T}^{u,s}$ (otherwise the fixed point argument would not work!).

Difference between manifolds

- Parameterizations of the invariant manifolds:

$$\begin{cases} x = 4\arctan(e^v) \\ y = \dfrac{2}{\cosh v} + \dfrac{\cosh^2 v}{2} \partial_v \mathcal{T}^u(v,\tau) \end{cases}$$

- We study $\Delta = \mathcal{T}^u - \mathcal{T}^s$.
- Subtract the equation

$$\varepsilon^{-1}\partial_\tau \mathcal{T} + \partial_v \mathcal{T} + \frac{\cosh^2 v}{8}(\partial_v \mathcal{T})^2 - \varepsilon \frac{2}{\cosh^2 v} \sin\tau = 0$$

for $\mathcal{T}^{u,s}$.
- Δ satisfies $\mathcal{L}\Delta = 0$ for the linear operator

$$\mathcal{L} = \varepsilon^{-1}\partial_\tau + \partial_v + \frac{\cosh^2 v}{8}\left(\partial_v \mathcal{T}^u(v,\tau) + \partial_v \mathcal{T}^s(v,\tau)\right)\partial_v$$

- $\mathcal{L}$ is close to $\mathcal{L}_0 = \varepsilon^{-1}\partial_\tau + \partial_v$.

Difference between manifolds

- Assume that $\Delta \in \mathrm{Ker}\mathcal{L}_0$, $\mathcal{L}_0 = \varepsilon^{-1}\partial_\tau + \partial_v$.
- Main idea: functions in $\mathrm{Ker}\mathcal{L}_0$ bounded in a complex strip are exponentially small in the reals.

Proposition

Let $\psi(v,\tau) \in \mathrm{Ker}\mathcal{L}_0$ be an analytic in τ function in $[-ir, ir] \times \mathbb{T}_\sigma$. Then,

- *ψ can be extended analytically to $\{|\mathrm{Im}\, v| \leq r\} \times \mathbb{T}_\sigma$*
- *Define $M = \max_{(v,\tau)\in[-ir,ir]\times\overline{\mathbb{T}}_\sigma} |\partial_v \psi(v,\tau)|$*
- *Then, for $\varepsilon \ll 1$,*

$$\forall (v,\tau) \in \mathbb{R} \times \mathbb{T}, |\partial_v \psi(v,\tau)| \leq 2Me^{-\varepsilon^{-1} r}$$

Why?

- $\varepsilon^{-1}\partial_\tau\psi + \partial_v\psi = 0$ implies $\psi(v,\tau) = \Lambda(\tau - \varepsilon^{-1}v)$ for some Λ.
- We can extend analytically ψ to $\{|\mathrm{Im}\, v| \leq r\} \times \mathbb{T}_\sigma$.
- Let us take and example $\psi(v,\tau) = A\sin(\tau - \varepsilon^{-1}v)$ for some constant $A > 0$.
- We have defined $M = \max_{(v,\tau)\in[-ir,ir]\times\overline{\mathbb{T}}_\sigma} |\partial_v\psi(v,\tau)|$.
- Take $(v,\tau) = (ir, 0)$,

$$M \geq A\varepsilon^{-1}\cosh(\varepsilon^{-1}r).$$

- So $A \leq 2M\varepsilon e^{-\frac{r}{\varepsilon}}$.
- For real values of (v,τ): $|\psi(v,\tau)| \leq 2M\varepsilon e^{-\frac{r}{\varepsilon}}$.

Better estimates for the error

- We apply this lemma to $\Delta - \mathrm{Melnikov} = \mathcal{T}^u - \mathcal{T}^s - \mathrm{Melnikov}$:
  - (Approximately) $\Delta - \mathrm{Melnikov} \in \mathrm{Ker}\mathcal{L}_0$
  - The fixed point argument gives bounds for $\Delta - \mathrm{Melnikov}$ for $(v, \tau) \in [-ir, ir] \times \mathbb{T}_\sigma$ with $r = \frac{\pi}{2} - \varepsilon$.
  - The bounds of $\Delta - \mathrm{Melnikov}$ are smaller than the size of Melnikov (in the complex).
- We deduce exponentially small bounds for real values of the variables.

Conclusion

- Parameterizations

$$\begin{cases} x = x_0(v) \\ y = y_0(v) + y_0^{-1}(v)\partial_v \mathcal{T}^u\left(v, \frac{t}{\varepsilon}\right) \end{cases}$$

- We obtain

$$\begin{aligned} \partial_v \Delta(v,\tau) &= \partial_v \mathcal{T}^u(v,\tau) - \partial_v \mathcal{T}^s(v,\tau) \\ &= -4\pi\varepsilon^{-1} e^{-\frac{\pi}{2\varepsilon}} \left(\sin(\tau - \varepsilon^{-1} v) + \mathcal{O}(\varepsilon)\right). \end{aligned}$$

- We have a first order of the difference between the invariant manifolds.

Splitting of separatrices

- We have proved splitting of separatrices for

$$H\left(y, x, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + (\cos x - 1) + \varepsilon(\cos x - 1)\sin\frac{t}{\varepsilon}$$

- Typically in resonances of nearly integrable systems

$$H\left(y, x, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + (\cos x - 1) + \delta(\cos x - 1)\sin\frac{t}{\varepsilon}$$

  for fixed $\delta \in \mathbb{R}$.
- This case is much harder: Melnikov does not give the first order.
- Analysis of the invariant manifolds close to the singularities gives a new first order.

Back to the circular problem

- One can apply the explained ideas to the circular problem
- Then, we have an asymptotic formula for the distance between the invariant manifolds for any $\mu \in (0, 1/2)$ and $G_0 \gg 1$

  $$\partial_v \mathcal{T}^u(v,\xi) - \partial_v \mathcal{T}^s(v,\xi) = G_0^{\frac{3}{2}} e^{-\frac{G_0^3}{3}} \left(C(\mu) \sin(\xi + G_0^3 v) + \mathcal{O}(G_0^{-\frac{1}{2}}) \right)$$

  where

  $$C(\mu) = \frac{1}{2}\sqrt{\frac{\pi}{2}}\mu(1-\mu)(1-2\mu).$$

- This allows us to prove their transversality in this range of parameters.