Lecture 4: Symbolic dynamics and oscillatory motions

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Moser approach

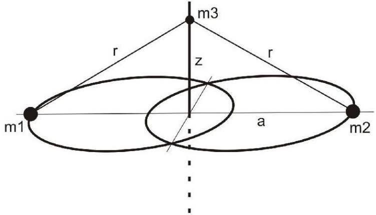
- McGehee coordinates send “infinity” to the origin. Then, infinity becomes a parabolic fixed point (linearization equal to the identity).
- This parabolic point has invariant manifolds.
- Prove that they intersect transversally.
- Establish symbolic dynamics (Smale horseshoe) close to these invariant manifolds.
- We prove how to establish dynamics for the Sitnikov problem.
- Instead of conjugating dynamics (in some subset of phase space) to the shift of two symbols, we conjugate it to a shift of infinite symbols.

Outline

- The Sitnikov problem
- Construction of symbolic dynamics
- Construction of oscillatory motions and orbits with different future and past final motions.

The Sitnikov problem

- The primaries have mass $\mu = 1/2$ and move on ellipses of small enough eccentricity ε.
- The Asteroid moves on the (invariant) vertical axis.



- Equations:

$$\ddot{z} = -\frac{z}{\left(z^2 + r^2(t)\right)^{3/2}}$$

  where $r(t) = r(t + 2\pi) > 0$.
- If we call ε to the eccentricity of the primaries

$$r(t) = \frac{1}{2}(1 - \varepsilon \cos t) + \mathcal{O}(\varepsilon^2).$$

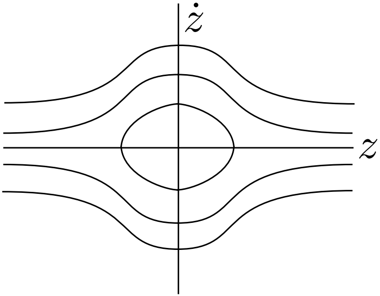
The Sitnikov problem for $\varepsilon = 0$

- Hamiltonian $H = \dfrac{\dot z^2}{2} - \dfrac{1}{\left(z^2 + \frac{1}{4}\right)^{1/2}}$

- It is integrable.

- Parabolic motion given by

$$H = \frac{\dot z^2}{2} - \frac{1}{\left(z^2 + \frac{1}{4}\right)^{1/2}} = 0$$



- Parabolic motions cross $z = 0$ at $|\dot z| = 2$.
- Orbits with $|\dot z| < 2$ cross $z = 0$ infinitely many times.
- Orbits with $|\dot z| > 2$ cross $z = 0$ only once.

The Sitnikov problem for $\varepsilon > 0$

- Hamiltonian $H = \frac{\dot{z}^2}{2} - \frac{1}{\left(z^2 + r^2(t)\right)^{1/2}}$

- Infinity is a parabolic periodic orbit as for the RPC3BP

- It has invariant manifolds.

- Applying classical Melnikov Theory, one can prove that the invariant manifolds intersect transversally.

The Sitnikov problem for $\varepsilon > 0$

$$H = \frac{\dot{z}^2}{2} - \frac{1}{\left(z^2 + r^2(t)\right)^{1/2}}$$

- We reduce it to a mapping at $\{z = 0\}$.
- We describe any solution $z(t)$ by giving t_0 such that $z(t_0) = 0$ and $\dot{z}_0 = \dot{z}(t_0)$.
- Such t_0 always exists.
- $\dot{z}_0 \neq 0$ except if $z(t) \equiv 0$.
- Since the equation is invariant by reflection $z \to -z$, we analyze $z = 0$ with the coordinates $v = |\dot{z}|$ and $t \in \mathbb{T}$.
- We treat (v, t) as polar coordinates

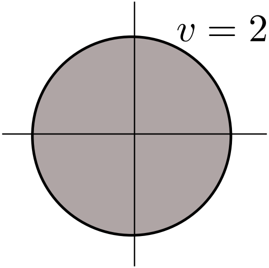
The return map ϕ

- Take a solution $z(t)$ with $z(t_0) = 0$ and $\dot{z}(t_0) = v_0$.
- Take $t_1 > t_0$ next time such that $z(t_1) = 0$ (if it exists).
- Set $v_1 = |\dot{z}(t_1)|$.
- We define $\phi(v_0, t_0) = (v_1, t_1)$.
- ϕ preserves the area form $vdvdt$.
- Where it is defined?

The return map for $\varepsilon = 0$

$$H = \frac{\dot{z}^2}{2} - \frac{1}{\left(z^2 + \frac{1}{4}\right)^{1/2}}$$

- The stable/unstable invariant manifolds of infinity coincide (parabolic motions) and cross $z = 0$ at $|\dot{z}| = 2$.
- Motions out of the circle are hyperbolic: never come back.

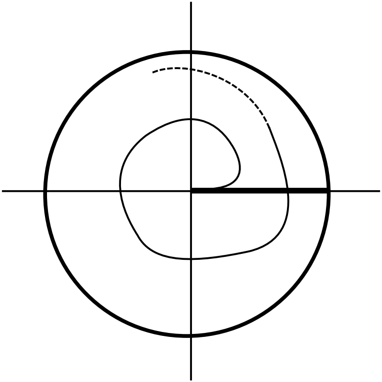


- Motion inside the circle are periodic: cross infinitely many times.
- Return map: $\phi_0(v_0, t_0) = (v_0, t_0 + T(v_0))$.

The return map for $\varepsilon = 0$

$$\phi_0(v_0, t_0) = (v_0, t_0 + T(v_0))$$

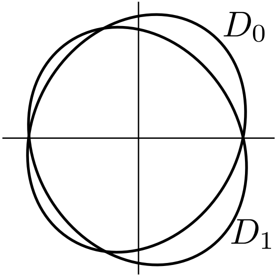
- $T(v_0) \longrightarrow \infty$ as $v_0 \to 2$.
- Take a radius from v_0 to the boundary $v_0 = 2$: The image is a curve spiralling infinitely about the origin and tending to the boundary $v_0 = 2$.



The return map for $\varepsilon > 0$

$$H = \frac{\dot{z}^2}{2} - \frac{1}{\left(z^2 + r^2(t)\right)^{1/2}}$$

- The boundary of the domain of definition D_0 of ϕ is $W^s(\text{Infinity}) \cap \{z = 0\}$.
- The boundary of its image D_1 is $W^u(\text{Infinity}) \cap \{z = 0\}$.
- So $\phi : D_0 \longrightarrow D_1$.
- ϕ is invariant under $\rho : (v, t) \to (v, -t)$. Thus, $D_1 = \rho(D_0)$.



- Both curves ∂D_i are closed and ε-close to the circle $v_0 = 2$.
- ∂D_0 and ∂D_1 intersect transversally at $t_0 = 0$.

The return map for $\varepsilon > 0$

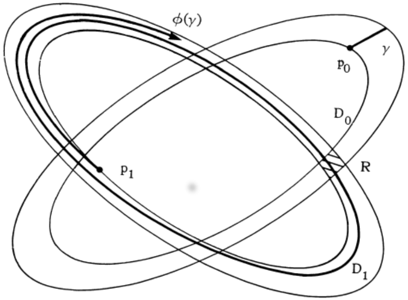
- Now the iteration of ϕ is more delicate.
- There are four options:
  - An orbit cross infinitely many times forward and backward.
  - An orbit cross infinitely many times forward and finitely backward.
  - An orbit cross infinitely many times backward and finitely forward.
  - An orbit cross finitely many times forward and backward.
- Orbits with forward/backward finite crossings are orbits which go to infinity as $t \to \pm\infty$.

Construction of the symbolic dynamics

- Consider the case of infinitely many forward and backward crossings.
- Consider the sequences $\{t_k\}_{k=-\infty}^{k=+\infty}$ of crossing times, $t_k < t_{k+1}$.
- We introduce the integers $s_k = \left\lfloor \dfrac{t_{k+1} - t_k}{2\pi} \right\rfloor$.
- s_k measure the complete number of revolutions of the primaries between two zeros of $z(t)$.
- We will prove the existence of orbits having crossings with any prescribed sequence of s_k with $s_k \geq m \gg 1$.
- An unbounded sequence of $\{s_k\}$ will correspond to an oscillatory orbit.

The spiralling for the return map

- Fix $\delta \ll 1$. Consider the set Δ_0 of all points inside D_0 and δ–close to D_0.
- Fix $\delta' \ll 1$. Consider the set Δ_1 of all points inside D_1 and δ'–close to D_1.
- Take an arc γ connecting the two boundaries of Δ_0



- (Part of) $\phi(\gamma)$ spirals inside Δ_1 and approaching D_1.

The spiralling for the return map

- We want to prove that $\phi(\gamma)$ spirals inside Δ_1 and approaching D_1.
- We need to analyze the behavior close to infinity.
- We need to show that the iterates of γ accumulate at the unstable invariant manifold of infinity.
- This is usual done using a Lambda lemma.
- Lambda lemmas for parabolic points are delicate.

The spiralling for the return map

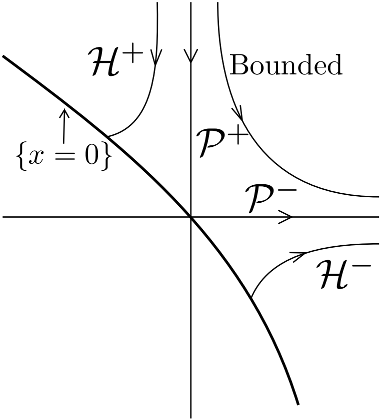
- We split the return map $\phi = \phi_- \circ \psi \circ \phi_+$
- $\phi_\pm$ are maps along the invariant manifolds away from infinity. They are regular maps (they involve finite time).
- The map ψ analyzes the behavior in a neighborhood of infinity ($(0,0)$ in McGehee coordinates).
- Time involved is unbounded.
- We treat it in McGehee coordinates

Local behavior close to infinity: el Λ lemma

- Behavior close to infinity in suitable coordinates.

- Local behavior

$$\widetilde{q}' = \frac{1}{4}(\widetilde{q}+\widetilde{p})^3\widetilde{q}(1+\mathcal{O}_2)$$
$$\widetilde{p}' = -\frac{1}{4}(\widetilde{q}+\widetilde{p})^3\widetilde{p}(1+\mathcal{O}_2)$$



- The invariant manifolds are the axes.

- Consider the Poincaré map

$$f \equiv f_{t_0} : \quad \Sigma \longrightarrow \Sigma$$
$$(\widetilde{q},\widetilde{p}) \mapsto f(\widetilde{q},\widetilde{p}),$$

A parabolic Lambda lemma

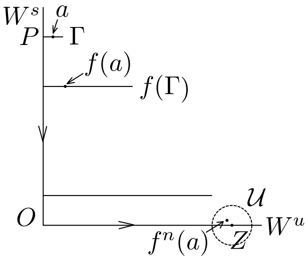
Theorem

Let Γ *be a curve intersecting transversally* $W^s(\Lambda_{t_0})$ *at* $P \in W^s(X_0)$*,* $X_0 \in \Lambda_{t_0}$*.*

Consider $Z \in W^u(X_0)$ *and a neighborhood* $\mathcal{U} \subset \mathbb{R}^4$ *of* Z*.*

Then, there exists $a \in \Gamma$ *and* $n \in \mathbb{Z}^+$ *such that* $f^n(a) \in \mathcal{U}$*. Thus,*

$$W^u(X_0) \subset \overline{\cup_{j\geq 0} f^j(\Gamma)}.$$

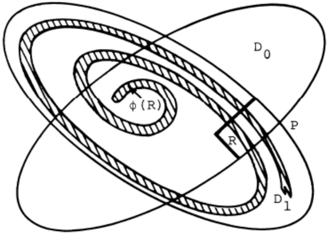


- In our picture, we had γ intersecting transversally W^s of infinity.
- The lambda lemma (essentially) gives the spiralling of the image of γ.

- We want to construct a hyperbolic set for the return map ϕ.
- We need $\mathcal{C}^1$ estimates to have expansion/contraction rates as when building the classical horseshoe.
- This is a little bit more delicate for this parabolic point.
- With this, we are ready to prove the existence of symbolic dynamics conjugating the dynamics to a shift of infinite symbols.

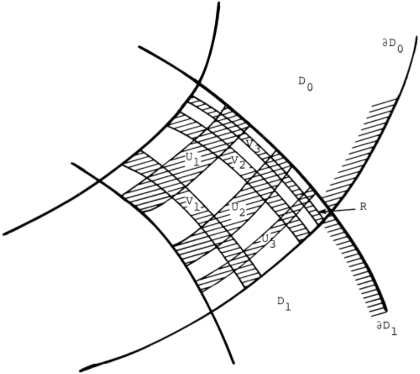
Symbolic dynamics

- Consider a R and $\phi(R)$
- $\phi(R)$ intersects R at infinitely many components.
- Except for finitely many, these components connect opposite sides of R.



Symbolic dynamics

- Denote the components connecting opposite sites by $V_1, V_2, V_3, \ldots$.
- Define $V_k = \phi(U_k)$.
- The V_k (resp. U_k) are disjoint.



- We can proceed as for the construction of the horseshoe.
- We prove that ϕ posesses the shift of infinite symbols as a subsystem.

From symbolic dynamics to the construction of final motions

- We wanted to consider the sequences $\{t_k\}_{k=-\infty}^{k=+\infty}$ of $z(t_k) = 0$, $t_k < t_{k+1}$.
- We have introduced the integers $s_k = \left\lfloor \frac{t_{k+1} - t_k}{2\pi} \right\rfloor$.
- s_k measure the complete number of revolutions of the primaries between two zeros of $z(t)$.
- Consider an initial condition in U_k. It satisfies that the return time to $\{z = 0\}$ is of the form

$$t_1 - t_0 = 2\pi(k + c + \nu)$$

  for some fixed large constant $c > 0$ which only depends on R and $\nu \in (0, 1)$.
- So we have an arbitrary sequence $\{s_k\}$ with $s_k = \lfloor c \rfloor + s'_k$, $s'_k \in \mathbb{N}$.

Different types of orbits

$$s_k = \left\lfloor \frac{t_{k+1} - t_k}{2\pi} \right\rfloor$$

- Take $\{s_k\}$ such that $s_k \to +\infty$: the orbit belongs to OS^+
- Take $\{s_k\}$ with $\limsup_{k\to+\infty} s_k$ is bounded: the orbit belongs to B^+
- Take a sequence $\{s_k\}$ such that it is finite in forward time $(\ldots, s_1, 0, s_1, s_2, \ldots s_{k^*}, \infty)$.
- This corresponds to an orbit which goes to infinity as $t \to \infty$: belongs to $H^+ \cup P^+$.

Different behavior in the past and the future

- One can do the same as $t \to -\infty$.
- We can impose any condition to the sequence $\{s_k\}$ for $k < 0$ and $k > 0$.
- We can construct orbits with any combination of past and future.
- $B^- \cap H^+$, $B^- \cap P^+$ are escape orbits.
- $H^- \cap B^+$, $P^- \cap B^+$ are capture orbits.

Conclusion

- We have constructed oscillatory motions and motions with any past/future combination for the Sitnikov problem.
- One can construct analogous symbolic dynamics for the RPC3BP
- Question: How abundant are these past/future combinations? Do they have positive or zero measure?