Lecture 5: Oscillatory motions for the RPE3BP

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Outline

- Oscillatory motions for the RPE3BP
- Instabilities in the RPE3BP (Arnold diffusion)
- Abundance of the different types of final motions.

The RPE3BP

- The RPE3BP in polar coordinates

$$H(r, \phi, y, G; \mu) = \frac{y^2}{2} + \frac{G^2}{2r^2} - U(r, \alpha, t; \mu),$$

where

$$U(r, \alpha, t; \mu) = \frac{(1-\mu)}{\|re^{i\alpha} - q_1(t)\|} + \frac{\mu}{\|re^{i\alpha} - q_2(t)\|}.$$

and q_1, q_2 are the positions of the primaries.

- Three parameters: μ, e_0 and G_0.
- The Sitnikov problem and the RPC3BP could be reduced to a 2-dim Poincaré map.
- The RPE3BP has a phase space of dimension 5.

The RPE3BP

- The increase of dimension makes more difficult to construct symbolic dynamics as in the Moser approach.
- We will use simpler techniques to construct oscillatory motions
- We will not be able to construct orbits with any combination of past and future.
- We use techniques from Arnold diffusion: a transition chain.

Infinity and their invariant manifolds

- Now at infinity we have a cylinder

$$\Lambda = \left\{(r, y) = (+\infty, 0), G \in [G_1, G_2], (\alpha, t) \in \mathbb{T}^2\right\}$$

- In McGehee coordinates $r = 2x^{-2}$,

$$\Lambda = \left\{(x, y) = (0, 0), G \in [G_1, G_2], (\alpha, t) \in \mathbb{T}^2\right\}$$

- Dynamics in Λ is foliated by periodic orbits.

$$\dot{G} = 0$$
$$\dot{\alpha} = 0$$
$$\dot{t} = 1$$

- Consider the Poincaré map $\mathcal{P}$ to the section $\{t = 0\}$
- The cylinder $\Lambda_0 = \Lambda \cap \{t = 0\}$ is foliated by fixed points.

Infinity and their invariant manifolds

$$\Lambda_0 = \{(x, y) = (+\infty, 0), G \in [G_1, G_2], \alpha \in \mathbb{T}\}$$

- Each fixed point $z = (\alpha, G)$ has stable/unstable invariant manifolds $W^{u,s}(z)$.
- All together form $W^s(\Lambda_0)$ and $W^u(\Lambda_0)$.
- Take $G_1 \gg 1$
- For $e_0 = 0$ (RPC3BP) we have proven transversality of the invariant manifolds.
- For $e_0 \ll 1$, the invariant manifolds of the cylinder intersect transversally.
- Thus: there are heteroclinic orbits connecting different fixed points of infinity.

Goal

- Construct an infinite sequence of heteroclinic orbits $\{\gamma_i\}_{i\geq 0}$ to the cylinder Λ_0 such that the forward limit of γ_i coincides with the backward limit of γ_{i+1}.
- Such sequence is called transition chain.
- We will construct such sequence
- We will look for orbits which "shadow" such sequence getting closer to the heteroclinics and the cylinder.
- $\liminf$ of distance between the orbit and the cylinder equal to zero implies the orbit is oscillatory (in forward time).

The scattering map

- First step: construct a transition chain (a sequence of heteroclinic orbits).
- We use the scattering map defined by Delshams–de la Llave–Seara for normally hyperbolic invariant manifolds.
- It also can be defined in this degenerate setting.
- It was introduced to study Arnold diffusion in nearly integrable Hamiltonian systems.
- Consider the cylinder Λ_0.
- Their invariant manifolds $W^{u,s}(\Lambda)$ intersect transversally along a homoclinic manifold Γ.

The scattering map

- Scattering map associated to the homoclinic manifold Γ.

$$S : \Lambda_0 \to \Lambda_0$$

given by: $z_+ = S(z_-)$ provided $\exists \tilde{z} \in \Gamma$ such that

$$W^s(z_+) \cap W^u(z_-) \cap \Gamma \neq \emptyset$$

- $z_+ = S(z_-)$ implies that z_- and z_+ are connected by a heteroclinic orbit.

The scattering map

- Different homoclinic manifolds lead to different scattering maps
- Usually, S is only defined locally since it is hard to analyze the homoclinic manifold (often we only can define them locally).
- In the case of RPC3BP and RPE3BP, the homoclinic channel is diffeomorphic to Λ_0.
- The associated scattering map is globally defined
- One can obtain formulas for the scattering map using Melnikov functions

Scattering map versus separatrix map

- The scattering map and the separatrix map are strongly related but quite different.
- Separatrix map: involves finite (long) time and its domain are points "close" to the invariant manifolds of Λ_0.
- Scattering map: involves infinite time and its domain is Λ_0
- Orbits of the Separatrix maps are true orbits of the original system.
- Orbits of the Scattering map correspond to heteroclinic orbits to Λ_0 in the original system.

The scattering map for the RPC3BP and the RPE3BP

- In the previous lecture: invariant manifolds of infinity intersect for any $\mu \in (0, 1/2]$ and $G_0 \gg 1$.
- We can define a scattering map.

$$\mathcal{S}_0 : \begin{pmatrix} \alpha \\ G \end{pmatrix} \longmapsto \begin{pmatrix} \alpha + f(\mu, G) \\ G \end{pmatrix}.$$

- Defined for all $\mu \in (0, 1/2]$ and $G \geq G_0 \gg 1$.
- It is integrable and a twist map: $\partial_G f(\mu, G) > 0$.
- RPE3BP: transversality of $W^{u,s}(\Lambda_0)$ in $[G_1, G_2]$ and e_0 small enough.
- The scattering map $\mathcal{S}_{e_0}$ is defined $S : \Lambda_0 \cap [G_1, G_2] \to \Lambda_0$.
- $\mathcal{S}_{e_0}$ is e_0–close to $\mathcal{S}_0$.

Orbits of the scattering map

- We want an infinite transition chain: a sequence of points in Λ connected by heteroclinic orbits.
- This is given by an orbit of the scattering map.
- Problem: we want the transition chain to be in a compact set $\mathbb{T} \times [G_1, G_2]$.
- If the transition chain leaves $\mathbb{T} \times [G_1, G_2]$, goes to regions were we do not know whether the invariant manifolds intersect (scattering maps are not defined).
- Nearly integrable twist maps have bounded orbits.
- A bounded orbit of the scattering map corresponds to a transition chain of periodic orbits such that all periodic orbits belong to $\mathbb{T} \times [G_1, G_2]$.

A $\mathcal{C}^0$ (reversed) Lambda lemma

- There exists a sequence of points $\{x_i\}_{i\geq 1} \subset \Lambda_0$ connected by heteroclinic orbits.
- We want orbits shadowing such sequence.
- We use a Lambda lemma.

Theorem

Let Γ be a curve which transversally intersects $W^u(\Lambda_0)$ at P such that $P \in W^u(X_0)$ for some $X_0 \in \Lambda_0$.

Then $W^s(X_0) \subset \overline{\cup_{j\geq 0} f^{-j}(\Gamma)}$.

- Compared to the Lambda lemma used for the Sitnikov problem, we have extra “central variables”.

Shadowing argument

- Let $\{P_i\}_{i=1}^{+\infty}$ be a sequence of transition periodic orbits in Λ.
- That is: $W^u(P_i) \pitchfork W^s(P_{i+1})$.
- Given $\{\varepsilon_i\}_{i=1}^{+\infty}$ a sequence of strictly positive numbers, we can find a point P and a increasing sequence of numbers T_i such that

$$\Phi_{T_i}(P) \in N_{\varepsilon_i}(P_i)$$

  where $N_{\varepsilon_i}(P_i)$ is a neighborhood of size ε_i of the periodic orbit P_i.

Proof

- Let $x \in W^s_{P_1}$. We can find a closed ball B_1, centered on x, and such that
$$\Phi_{P_1}(B_1) \subset N_{\varepsilon_1}(P_1).$$
- By the Lambda Lemma
$$W^s_{P_2} \cap B_1 \neq \emptyset.$$
- Hence, we can find a closed ball $B_2 \subset B_1$, centered in a point in $W^s_{P_2}$ such that
$$\Phi_{T_2}(B_2) \subset N_{\varepsilon_2}(P_2).$$
(and also $\Phi_{P_1}(B_1) \subset N_{\varepsilon_1}(P_1)$).
- By induction, there is a sequence of closed balls
$$B_i \subset B_{i-1} \subset \cdots \subset B_1$$
$$\Phi_{T_j}(B_i) \subset N_{\varepsilon_j}(P_j), \quad i \leq j.$$
- Since the balls are compact, $\cap B_i \neq \emptyset$. A point P in the intersection satisfies the required property.

Shadowing argument

- Given $\{\varepsilon_i\}_{i=1}^{+\infty}$ a sequence of strictly positive numbers, we can find a point P and a increasing sequence of numbers T_i such that

$$\Phi_{T_i}(P) \in N_{\varepsilon_i}(P_i)$$

where $N_{\varepsilon_i}(P_i)$ is a neighborhood of size ε_i of the periodic orbit P_i.

- This is true for any sequence $\{\varepsilon_i\}_{i=1}^{+\infty}$.
- Take any sequence $\varepsilon_i \to 0$ as $i \to \infty$.
- The $\liminf$ of the distance between the orbit and the cylinder Λ_0 is 0.
- Undoing McGehee change of coordinates $r = \dfrac{2}{x^2}$, $\limsup$ of the orbit is infinity.

Oscillatory motions for the RPE3BP

Theorem (Guardia-Martin-Seara-Sabbagh)

Fix any $\mu \in (0, 1/2]$ *and* e_0 *small enough. There exists an orbit* $(q(t), p(t))$ *of the RPE3BP which satisfies*

$$\limsup_{t\to+\infty} \|q\| = +\infty \quad \textit{and} \quad \liminf_{t\to+\infty} \|q\| < +\infty.$$

Drift in angular momentum

$$H(r, \phi, y, G; \mu) = \frac{y^2}{2} + \frac{G^2}{2r^2} - U(r, \alpha, t; \mu),$$

- Besides oscillatory motions, several other interesting phenomena take place close to parabolic motion.
- There exists Arnold diffusion: drift of actions
- The angular momentum G is a first integral when $\mu = 0$ (two body problem).
- Traveling close to the invariant manifolds of infinity, G can make big excursions.

Arnold diffusion in the RPE3BP

Theorem (de la Rosa - Delshams - Kaloshin - Seara)

Fix $G_2 > G_1 \gg 1$. Then for e_0 and μ small enough, there exists a trajectory of the RPE3BP and some $T > 0$ such that

$$G(0) < G_1, \qquad G(T) > G_2.$$

- Main idea: use (several) scattering maps to construct transition chains with a big increase in angular momentum.
- Shadow this transition chain.
- It is only proven for $\mu \ll 1$ and $e_0 \ll 1$.
- It should happen for any $\mu \in (0, 1/2]$ and $e_0 \in (0, 1)$.
- Even more: orbits such that $G(t) \to +\infty$ as $t \to +\infty$.

Measure of final motions

- For the RPC3BP, we have construct any possible combinations $X^- \cap Y^+$ with $X, Y = B, P, H, OS$.

- How abundant these combinations are?

- Do they have zero or positive measure?

Measure of final motions

- Any combination involving $P^{\pm}$ must have measure zero since $P^{\pm}$ are the invariant manifolds of infinity.
- For the other combinations

	B^+	H^+	OS^+
B^-	Measure > 0	Measure $= 0$	Measure $= 0$
H^-	Measure $= 0$	Measure > 0	Measure $= 0$
OS^-	Measure $= 0$	Measure $= 0$	?

- The only one still not known is $OS^- \cap OS^+$.

Measure conjecture

- Question by Arnold in the Conference in honor of the 70th anniversary of Alexeev:

  Is the Lebesgue measure of the set of oscillatory motions positive?

- Arnold considered this problem the central problem of celestial mechanics.
- Conjecture (Alexeev, Kolmogorov): The Lebesgue measure is zero.

Hausdorff dimension

- Kaloshin/Gorodetski consider the set of oscillatory motions for the RPC3BP (also Sitnikov)

- The Hausdorff dimension of the set of oscillatory motions is maximal for a Baire generic subset of an open set of parameters (the mass ratio and the Jacobi constant in the RPC3BP).

Existence of oscillatory motions

- Proved for all μ in the circular case.
- Proved for all μ and $e_0 \ll 1$ for the elliptic problem.
- Problem to extend the results $e_0 \in (0, 1)$: prove the transversality between the invariant manifolds.
- Non restricted 3BP:
  - Oscillatory motions only proved for a Sitnikov like configuration (Alexeev) with a third body with small mass.
  - Arnold diffusion should imply exchange of angular momentum between the parabola of body 3 and the ellipse of bodies 1 and 2.
- For more bodies no results at all.