# Exponentially small splitting of separatrices for $1\frac{1}{2}$ degrees of freedom Hamiltonian Systems close to a resonance

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## $1\frac{1}{2}$ degrees of freedom Hamiltonian systems

We consider close to completely integrable Hamiltonian

$$h(I,x,\tau) = h_0(I) + \delta h_1(I,x,\tau)$$

where

- $\delta \ll 1$ is a small parameter.
- $(x,\tau) \in \mathbb{T}^2$ and $I \in U \subset \mathbb{R}$.
- $h$ is analytic.

Equations of motion:

$$\begin{cases} \dot{x} = \partial_I h_0(I) + \delta \partial_I h_1(I,x,\tau) \\ \dot{I} = -\delta \partial_x h_1(I,x,\tau) \\ \dot{\tau} = 1 \end{cases}$$

## Unperturbed system

When $\delta = 0$, the equations of motion are

$$\begin{cases} \dot{x} = \partial_I h_0(I) \\ \dot{I} = 0 \\ \dot{\tau} = 1 \end{cases}$$

Then,

- $I$ is an integral of motion and therefore the phase space is foliated by 2-dimensional tori.
- The dynamics in the invariant tori is a rotation of frequency $\omega(I) = (\partial_I h_0(I), 1)$.

## The perturbed system

What happens when $0 < \delta \ll 1$:

- If $\partial_I h_0(I)$ is irrational enough (Diophantine) the unperturbed torus with frequency $\omega(I) = (\partial_I h_0(I), 1)$ is preserved provided $\delta$ is small enough (Kolmogorov-Arnol'd-Moser Theory).
- If $\partial_I h_0(I)$ is rational we are in a resonance.
- The unperturbed torus with frequency $\omega(I) = (\partial_I h_0(I), 1)$ breaks down for $\delta > 0$.

We want to study the new invariant objects that appear in the resonances.

- We study what happens close to the resonance $\omega = (0, 1)$
- Doing a change of variables, one can deduce similar results to the ones presented in this talk for any other resonance.

## Close to a resonance

- By translation, we can locate the resonance $\omega = (0, 1)$ at $I = 0$.
- Since $\omega(I) = (\partial_I h_0(I), 1)$, this implies that $h_0(I) = \dfrac{I^2}{2} + G(I)$ with $G(I) = \mathcal{O}\left(I^3\right)$.
- The size of the resonant zone is of order $\mathcal{O}\left(\sqrt{\delta}\right)$.
- To study it, we make a rescaling to magnify it.
- Namely, we perform the change

$$I = \sqrt{\delta} y, \quad \tau = t/\sqrt{\delta}$$

and we take $\varepsilon = \sqrt{\delta}$.

## Rescaled Hamiltonian (I)

New Hamiltonian

$$H\left(y, x, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + \frac{1}{\varepsilon^2} G(\varepsilon y) + h_1\left(\varepsilon y, x, \frac{t}{\varepsilon}\right)$$

where

$$h_0(I) = \frac{I^2}{2} + G(I) \text{ with } G(I) = \mathcal{O}\left(I^3\right)$$

- The term $\varepsilon^{-2} G(\varepsilon y)$ is of order $\mathcal{O}(\varepsilon)$.
- Now the perturbation term $h_1$ has the same order as the integrable system but is fast and periodic in time.
- The fast oscillating terms are expected to *average out* (at first order) and then to have a small influence.
- So, we first study the average of $h_1$ with respect to $t$.

## Rescaled Hamiltonian (II)

- We split the Hamiltonian

$$H\left(y, x, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + \frac{1}{\varepsilon^2} G(\varepsilon y) + h_1\left(\varepsilon y, x, \frac{t}{\varepsilon}\right)$$

as

$$H\left(y, x, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + \frac{1}{\varepsilon^2} G(\varepsilon y) + V(x) + F\left(x, \frac{t}{\varepsilon}\right) + R\left(\varepsilon y, x, \frac{t}{\varepsilon}\right)$$

where

$$V(x) = \langle h_1(0, x, \tau)\rangle = \frac{1}{2\pi}\int_0^{2\pi} h_1(0, x, \tau) d\tau$$

$$F(x, \tau) = h_1(0, x, \tau) - \langle h_1(0, x, \tau)\rangle$$

$$R(I, x, \tau) = h_1(I, x, \tau) - h_1(0, x, \tau)$$

- The term $R\left(\varepsilon y, x, \frac{t}{\varepsilon}\right)$ is of order $\mathcal{O}(\varepsilon)$.

New Hamiltonian:

$$H\left(y, x, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + \frac{1}{\varepsilon^2} G(\varepsilon y) + V(x) + F\left(x, \frac{t}{\varepsilon}\right) + R\left(\varepsilon y, x, \frac{t}{\varepsilon}\right)$$

Since

- $F(x, t/\varepsilon)$ is fast oscillating in time
- $\varepsilon^{-2} G(\varepsilon y)$ and $R(\varepsilon y, x, t/\varepsilon)$ are of order $\mathcal{O}(\varepsilon)$

we can study our system as a perturbation of

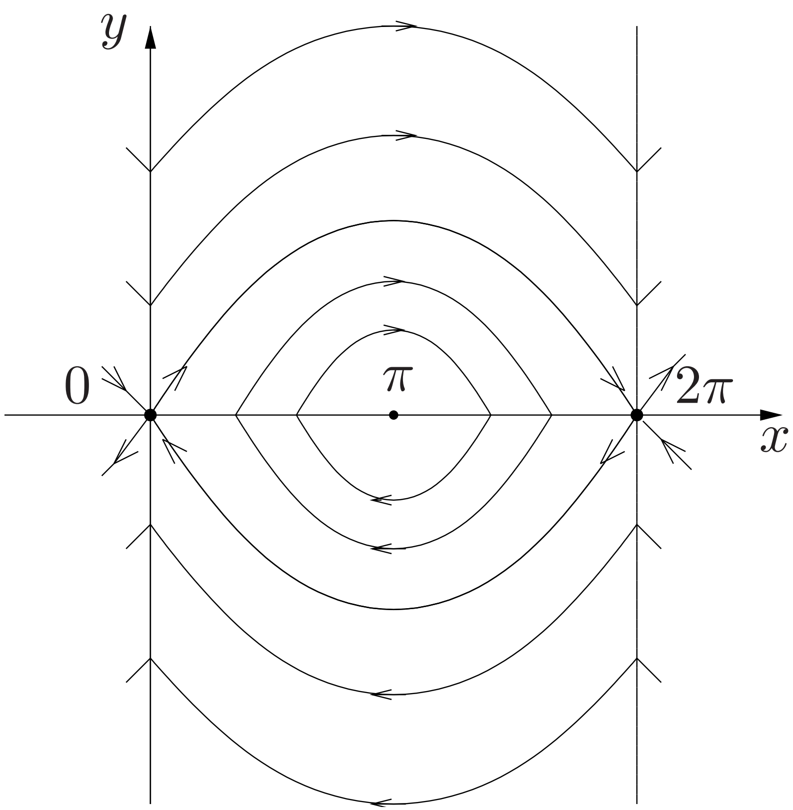
$$H_0(x, y) = \frac{y^2}{2} + V(x)$$

## The new unperturbed system

- Given by the Hamiltonian

$$H_0(x, y) = \frac{y^2}{2} + V(x).$$

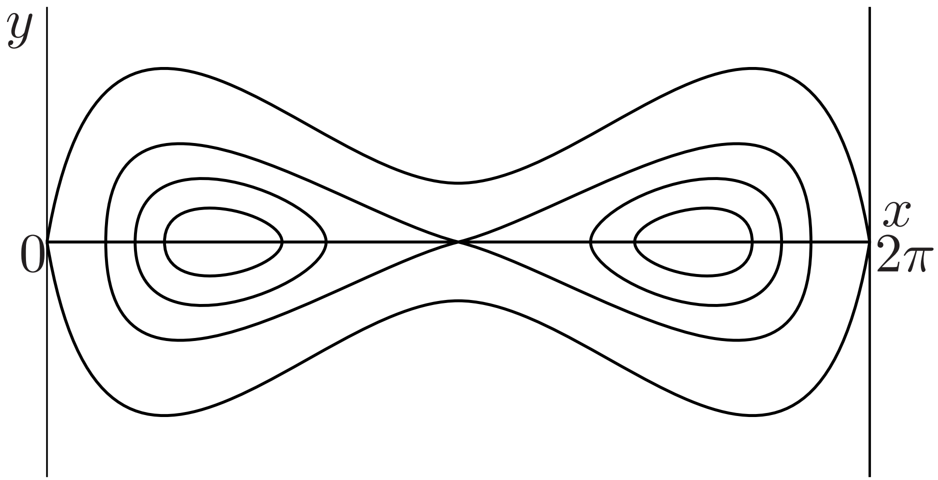
- $H_0$ is a first integral and therefore the orbits are the level curves of $H_0$.




- Generically $H_0$ has (at least) a hyperbolic critical point whose invariant manifolds coincide along a separatrix.
- With a translation one of the critical points can be located at $(0, 0)$.
- When the hyperbolic periodic orbit is unique, we have a pendulum-like phase portrait.

## The new unperturbed system (II)

The system can have more than one hyperbolic critical points, with their invariant manifolds coinciding along separatrices.



## The perturbed system (I)

- Recall that the system is given by

$$H\left(y,x,\frac{t}{\varepsilon}\right)=\frac{y^2}{2}+V(x)+\frac{1}{\varepsilon^2}G(\varepsilon y)+F\left(x,\frac{t}{\varepsilon}\right)+R\left(\varepsilon y,x,\frac{t}{\varepsilon}\right)$$

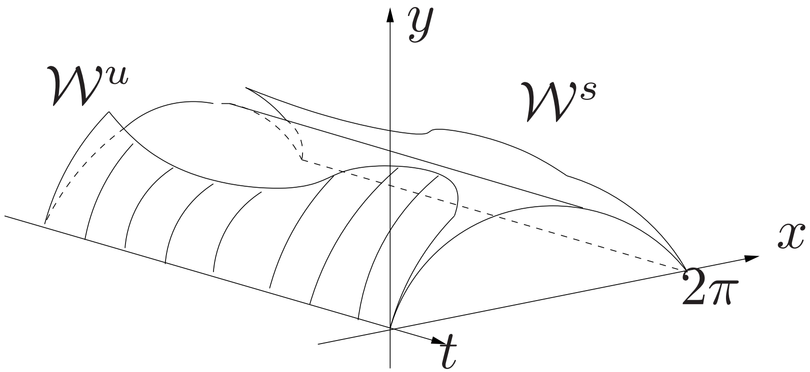
- Since the perturbative terms are either small or fast oscillating we can perform one step of averaging, which transforms the system into a system of the form

$$H\left(\tilde{y},\tilde{x},\frac{t}{\varepsilon}\right)=\frac{\tilde{y}^2}{2}+V(\tilde{x})+\varepsilon\tilde{H}_1\left(\tilde{y},\tilde{x},\frac{t}{\varepsilon},\varepsilon\right)$$

- Classical perturbation applied to this new system ensures that close to $(0,0)$, there exists a hyperbolic periodic orbit and its invariant manifolds.

- These objects are $\varepsilon$-close to the unperturbed ones.

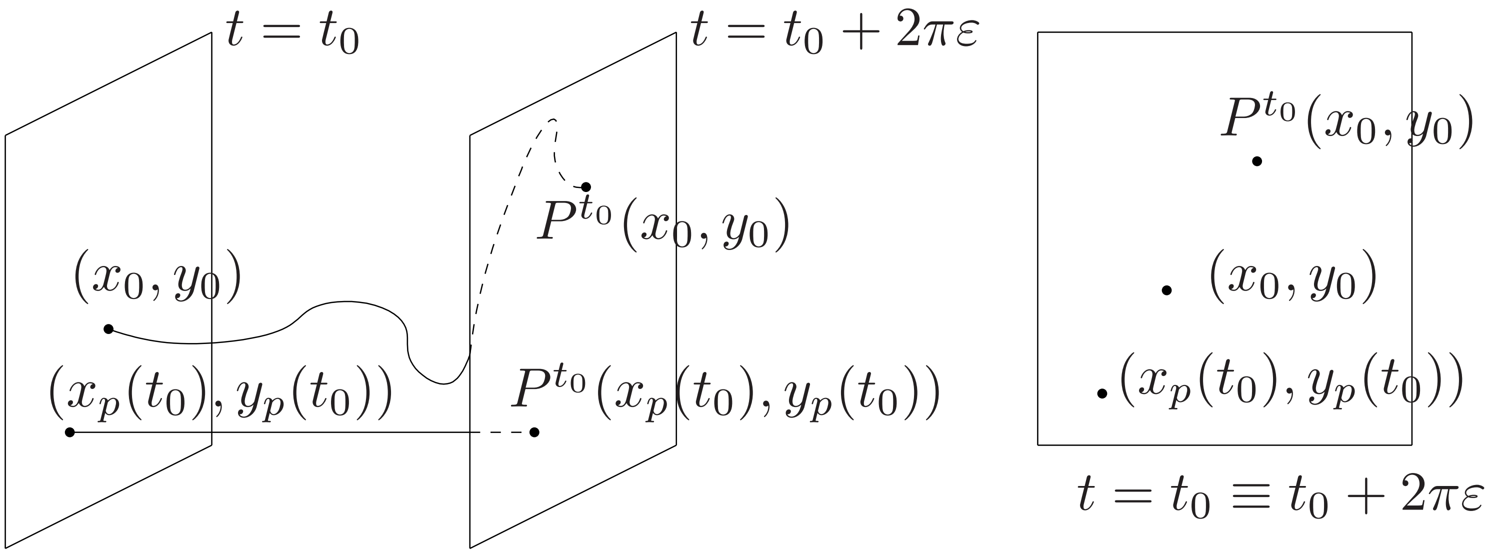
## The perturbed system (II)




- 3-dimensional phase space.
- The invariant manifolds are now 2-dimensional.

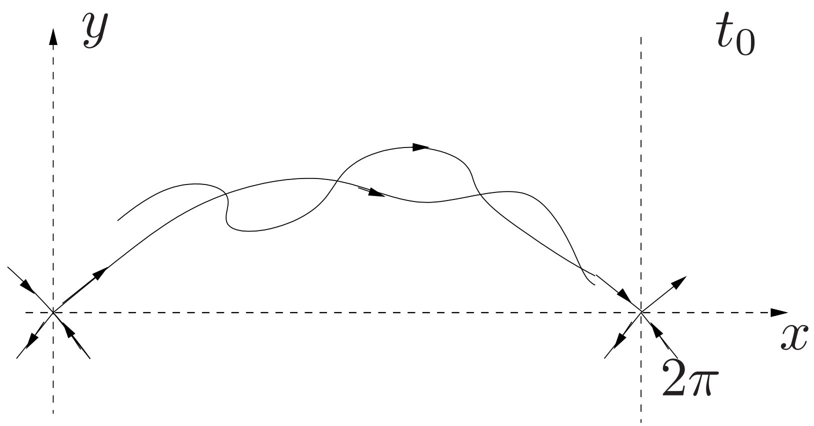
- Tipically they do not coincide anymore (the separatrix splits) but they intersect transversally.
- The transversality of the invariant manifolds implies the existence of chaotic orbits.
- We want to prove this fact and give quantitative measures of the splitting.
- These quantitative estimates give a bound for the region of the phase space where chaotic orbits are confined.

## The $2\pi\varepsilon$-time Poincaré map formulation




From the perturbed system it can be derived a discrete dynamical system considering the $2\pi\varepsilon$-time Poincaré map.

## The splitting of separatrices in the Poincaré map




- Considering the Poincaré map, we obtain this picture.
- We can measure several quantities to study the splitting.

- Since we want to prove the existence of transversal homoclinic points, a natural quantity to measure would be the angle between the invariant manifolds at the homoclinic point.
- Nevertheless, the angle depend on the homoclinic point and it is not a symplectic invariant.
- The distance between the invariant manifolds also depends on the chosen coordinates and is neither a symplectic invariant.
- Between transversal homoclinic points, the invariant manifolds create lobes.
- The area of these lobes is invariant by iteration of the Poincaré map due to the symplectic structure.
- We will measure the splitting in terms of the area of these lobes.

The first rigorous result dealing with this problem is the following.

**Theorem (Neishtadt 84)** Let us fix $\varepsilon_0 > 0$. Then, for $\varepsilon \in (0, \varepsilon_0)$, there exists an $\varepsilon$-close to the identity canonical transformation that transforms system

$$H\left(y, x, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + V(x) + \frac{1}{\varepsilon^2} G(\varepsilon y) + F\left(x, \frac{t}{\varepsilon}\right) + R\left(\varepsilon y, x, \frac{t}{\varepsilon}\right)$$

into

$$H\left(\tilde{y}, \tilde{x}, \frac{t}{\varepsilon}\right) = \frac{\tilde{y}^2}{2} + V(\tilde{x}) + \varepsilon \tilde{H}_0(y, x, \varepsilon) + P\left(\tilde{y}, \tilde{x}, \frac{t}{\varepsilon}\right)$$

and $P$ satisfies

$$|P| \leq c_2 e^{-\frac{c_1}{\varepsilon}}$$

for certain constants $c_1, c_2 > 0$.

This theorem implies that:

- The system is exponentially small close to integrable.
- The invariant manifolds are exponentially close.
- Nevertheless, it does not give any information about the splitting of the separatrix (we do not know whether they coincide or not).
- Namely, it gives an exponentially small upper bound of the area of the lobes, but we need also lower bounds to know that they split.
- In particular, we do not know whether the system has chaotic orbits or not.

## Perturbative approach in $\varepsilon$

To see whether the separatrix splits, we can look for parameterizations of the invariant manifolds

$$x^u(r, \varepsilon), \;\; x^s(r, \varepsilon)$$

$\to$ Since $\varepsilon$ is small, we can look for formal solutions as a power series of $\varepsilon$:

$$x^\alpha(r, \varepsilon) = x_0(r) + \varepsilon x_1^\alpha(r) + \varepsilon^2 x_2^\alpha(r) + \dots \quad \text{for } \alpha = u, s$$

For these problems which are analytic and have a fast periodic perturbation:

$$x_k^u(r) = x_k^s(r) \;\; \forall k \in \mathbb{N}$$

Conclusion:

$$x^u(r, \varepsilon) - x^s(r, \varepsilon) = \mathcal{O}(\varepsilon^k) \;\; \forall k \in \mathbb{N}$$

$\to$ Proceeding formally we see that their difference is beyond all orders.

## What is happening?

Two options:

1 Both manifolds coincide also in the perturbed case (the perturbed system is also integrable) $\rightarrow$ the power series in $\varepsilon$ is convergent:

2 Both manifolds do not coincide $\rightarrow$ the power series in $\varepsilon$ is divergent and the difference between manifolds has to be flat with respect $\varepsilon$.

*Tipically* is happening the second option

- Since the splitting is a phenomenon beyond all orders, is hard to make a perturbative approach in $\varepsilon$.
- Trick: add a new parameter $\mu$.
- Namely, we rewrite the rescaled Hamiltonian

$$H\left(y, x, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + V(x) + \frac{1}{\varepsilon^2} G(\varepsilon y) + F\left(x, \frac{t}{\varepsilon}\right) + R\left(\varepsilon y, x, \frac{t}{\varepsilon}\right)$$

as

$$H\left(y, x, \frac{t}{\varepsilon}\right) = H_0(y, x) + \mu H_1\left(y, x, \frac{t}{\varepsilon}, \varepsilon\right)$$

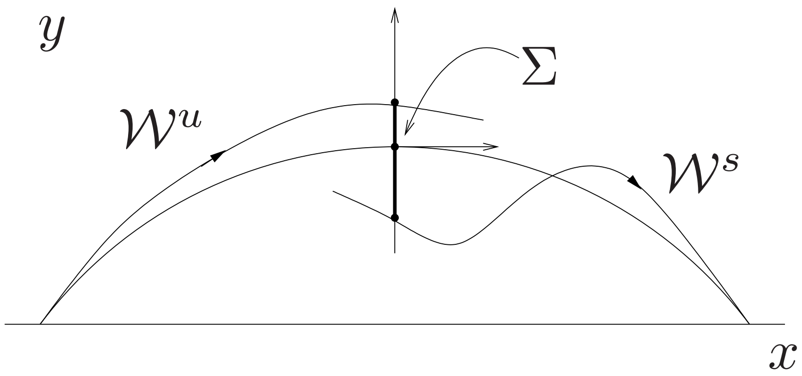
where $\mu = 1$ is a fake parameter and

$$H_0(y, x) = \frac{y^2}{2} + V(x)$$

$$H_1(y, x, \tau, \varepsilon) = F(x, \tau) + \frac{1}{\varepsilon^2} G(\varepsilon y) + R(\varepsilon y, x, \tau)$$

- Even if the general case corresponds to $\mu = 1$, the first results studying the splitting of separatrices considered $\mu$ as a small parameter.
- Namely, let's forget for a moment that $\varepsilon$ is a small parameter and let us consider $\mu$ as an arbitrarily small parameter.
- In that case, we can consider a perturbative approach in $\mu$, which is usually called Poincaré-Melnikov Method.

## Poincaré-Melnikov Theory (I)




1. We fix $\Sigma$, a transversal section to the unperturbed separatrix in order to measure in it the splitting.
2. We consider a parameterization $\gamma$ of the unperturbed separatrix such that $\gamma(0)$ belongs to this section.

## Poincaré-Melnikov Theory (II)

3. We define the Melnikov function as:

$$M(s,\varepsilon) = \int_{-\infty}^{+\infty} \{H_0, H_1\} \left( \gamma(t-s), \frac{t}{\varepsilon} \right) dt$$

where

- $s$ corresponds to the time evolution through the separatrix.
- $\{H_0, H_1\}$ is the Poisson bracket:

$$\{H_0, H_1\} = \frac{\partial H_0}{\partial x}\frac{\partial H_1}{\partial y} - \frac{\partial H_1}{\partial x}\frac{\partial H_0}{\partial y}$$

$\rightarrow$ $M$ can be computed since $H_0$, $H_1$ and $\gamma$ are known.

## Poincaré-Melnikov Theory (III)

Then:

- The distance between both invariant manifolds for $\mu > 0$ is given by:

$$d(s,\mu,\varepsilon) = \mu \frac{M(s,\varepsilon)}{\|DH_0(\gamma(-s))\|} + \mathcal{O}\left(\mu^2\right)$$

- If there exists $s_0$ such that

$$(i)\; M(s_0,\varepsilon) = 0 \quad (ii)\; \left.\frac{\partial M}{\partial s}\right|_{s=s_0} \neq 0$$

  Then the invariant manifolds intersect transversally in a point which is close to $\gamma(s_0)$.

- If $s_0$ y $s_1$ are two consecutive simple zeros of the Melnikov function, the area of the corresponding lobe is given by:

$$\mathcal{A} = \mu \int_{s_0}^{s_1} M(s,\varepsilon) ds + \mathcal{O}\left(\mu^2\right)$$

Conclusion: Poincaré-Melnikov theory allows to

- Prove the existence of transversal homoclinic orbits
- Compute the distance between manifolds, and therefore to compute asymptotically for $\mu \to 0$ the region of the phase space where chaos is confined.
- Nevertheless, these results are for $\mu$ arbitrarily small and $\varepsilon$ fixed.

- To see whether Poincaré-Melnikov Theory is valid also for small $\varepsilon$ we have to study the dependence on $\varepsilon$ of the Melnikov function.
- The dependence on $\varepsilon$ of the Melnikov functions is extremely sensitive on the analyticity properties of the Hamiltonian.
- So, we need to impose conditions on the Hamiltonian to compute this dependence.

## Hypothesis on the Hamiltonian

- Recall that the original system was given by

$$h(I, x, \tau) = h_0(I) + \delta h_1(I, x, \tau)$$

- Then, we have to assume that the perturbation $h_1(I, x, \tau)$ is a trigonometric polynomial in $x$.
- The potential $V$ has been defined as

$$V(x) = \frac{1}{2\pi} \int_0^{2\pi} h_1(0, x, \tau) d\tau$$

- We have also to assume the generic hypothesis that $V(x)$ has the same degree as $h_1$.

## Hypotheses on the separatrix

- Let us consider the parameterization of the separatrix of $H_0(y,x) = \frac{y^2}{2} + V(x)$:

$$\gamma(u) = (x_0(u), y_0(u))$$

- $y_0(u)$ has always singularities in the complex plane.
- Then, there exists $a > 0$ such that $y_0(u)$ is analytic in the complex strip $\{|\mathrm{Im}\, u| < a\}$ and cannot be extended in any wider strip.
- We assume that $y_0(u)$ has only one singularity in each of the boundary lines $\{\mathrm{Im}\, u = \pm a\}$.
- Then, they are order one poles.
- Moreover, the parameterization of $\gamma$ can be chosen such that they are located at $u = \pm ia$.

- With these hypotheses we can study the dependence on $\varepsilon$ of the Melnikov function .
- Namely, we can use Residuums Theorem to compute

$$M(s,\varepsilon) = \int_{-\infty}^{+\infty} \{H_0, H_1\} \left(\gamma(t-s), \frac{t}{\varepsilon}\right) dt.$$

- Then Melnikov Theory formula for the area of the lobes is

$$\mathcal{A}(\varepsilon,\mu) = \mu\mathcal{A}_0(\varepsilon) + \mathcal{O}(\mu^2)$$

- $\mathcal{A}_0(\varepsilon) = C\dfrac{1}{\varepsilon}e^{-\frac{a}{\varepsilon}}(1+\mathcal{O}(\varepsilon^\nu))$ where
  - $a$ is the imaginary part of the singularity of the separatrix.
  - $C$ is a real constant given by Melnikov Theory.
  - $\nu > 0$.
- Melnikov theory only applies if $\mu$ is exponentially small with respect to $\varepsilon$ but we want $\mu = 1$.

- Even if Melnikov Theory only applies if $\mu$ is exponentially small with respect to $\varepsilon$, one can ask whether the Melnikov function gives the true first order for a wider range in $\mu$.
- Using complex perturbation techniques, these results were improved.
- V. Gelfreich (1997), proved that Melnikov predicts correctly the area of the lobes provided $\mu = \varepsilon^\eta$ with $\eta$ big enough.
- A. Delshams and T. Seara (1997) improved this result to the case $\mu = \varepsilon^\eta$ with $\eta > 2$.

- The only results dealing with the case $\mu = 1$ (usually called singular case) are due to V. Gelfreich, D. Treschev, and T. Seara, C. Olivé and G., but only deal with the pendulum with certain perturbations which, in particular, do not depend on $y$.
- In these cases, one can see that Melnikov does not predict the area of the lobes correctly.
- In the present work we deal with the general case (assuming the explained hypotheses).

# Main theorem

There exists a constant $b$ and an analytic function $f(\mu)$ such that for any fixed $\mu$ such that $f(\mu) \neq 0$ and $\varepsilon$ small enough:

- The invariant manifolds split creating transversal intersections.
- The area of the lobes is given by the asymptotic formula:

$$\mathcal{A} = \frac{\mu}{\varepsilon} e^{-\frac{a}{\varepsilon} + \mu^2 b \ln \frac{1}{\varepsilon}} \left( f(\mu) + \mathcal{O}\left( \frac{1}{|\ln \varepsilon|} \right) \right)$$

- In particular, if $\mu = 1$,

$$\mathcal{A} = \frac{1}{\varepsilon^{1-b}} e^{-\frac{a}{\varepsilon}} \left( f(1) + \mathcal{O}\left( \frac{1}{|\ln \varepsilon|} \right) \right)$$

## The function $f(\mu)$

- Recall the expression of the Hamiltonian

$$H\left(y,x,t\right)=\frac{y^2}{2}+V(x)+\mu\left(F\left(x,\frac{t}{\varepsilon}\right)+\frac{1}{\varepsilon^2}G(\varepsilon y)+R\left(\varepsilon y,x,\frac{t}{\varepsilon}\right)\right)$$

- The term $\frac{1}{\varepsilon^2}G(\varepsilon y)+R\left(\varepsilon y,x,\frac{t}{\varepsilon}\right)$ is of order $\varepsilon$.
- Nevertheless, the function $f(\mu)$ depends on the full jet in $y$ of $G$ and $R$.
- Namely, any finite order truncation in $\varepsilon$ of the Hamiltonian does not predict correctly the area of the lobes.

- Since for $\mu = 1$ the area is given by

$$\mathcal{A} = \frac{1}{\varepsilon^{1-b}} e^{-\frac{a}{\varepsilon}} \left( f(1) + \mathcal{O}\left( \frac{1}{|\ln \varepsilon|} \right) \right),$$

  to know whether there is splitting or not in the original example it is enough to check if $f(1) \neq 0$.

- It is difficult to know the function $f$ analytically.

- It can be studied numerically using a different problem independent of $\varepsilon$, which is usually called inner equation.

## The constant $b$

- The constant $b$ changes the algebraic order in front of the exponentially small term.
- It can be computed explicitly and generically satisfies $b \neq 0$.
- Nevertheless, satisfies $b = 0$ if $G = 0$ and $R = 0$, that is, if $H_1$ does not depend on $y$.
- Therefore, it had not been detected before since all the studied examples did not depend on $y$.

# Validity of the Melnikov function prediction

Melnikov prediction:

$$\mathcal{A} \sim C\frac{\mu}{\varepsilon}e^{-\frac{a}{\varepsilon}}$$

True first order

$$\mathcal{A} \sim f(\mu)\frac{\mu}{\varepsilon}e^{-\frac{a}{\varepsilon}+\mu^2 b \ln\frac{1}{\varepsilon}}$$

- The function $f$ satisfies $f(\mu) = C + \mathcal{O}(\mu)$.
- Then, if

  $$\mu \ll \frac{1}{\sqrt{|\ln\varepsilon|}}$$

  Melnikov theory predicts correctly the area of the lobe.
- If $b = 0$, Melnikov works provided $\mu$ is small independently of $\varepsilon$.
- In the other cases, Melnikov fails to predict correctly.

- Namely, generically ($\mu = 1$) Melnikov Theory fails to predict
  - The constant in front of the exponential.
  - The polynomial power in front of the exponential.
- That is, it only predicts correctly the exponential coefficient.

## Some ideas of the proof

- Consider the simpler case in which:
  - The periodic orbit remains at the origin after perturbation.
  - The unperturbed separatrix is a graph over the base (like in the pendulum).
- We look for parameterizations of the invariant manifolds as graphs using the Hamilton-Jacobi equation.
- Namely, we look for functions $S^{u,s}$ solutions

$$\partial_t S^{u,s}\left(x, \frac{t}{\varepsilon}\right) + H\left(x, \partial_x S^{u,s}\left(x, \frac{t}{\varepsilon}\right), \frac{t}{\varepsilon}\right) = 0$$

which satisfy

$$\begin{cases} \lim_{x\to 0} \partial_x S^u\left(x, t/\varepsilon\right) = 0 \\ \lim_{x\to 2\pi} \partial_x S^s\left(x, t/\varepsilon\right) = 0 \end{cases}$$

- We reparametrize $x = x_0(u)$ and $T^{u,s}(u, t/\varepsilon) = S^{u,s}(x_0(u), t/\varepsilon)$ and then we look for solutions of

$$\partial_t T^{u,s}\left(u, \frac{t}{\varepsilon}\right) + H\left(x_0(u), \frac{1}{y_0(u)}\partial_u T^{u,s}\left(u, \frac{t}{\varepsilon}\right), \frac{t}{\varepsilon}\right) = 0$$

which satisfy

$$\begin{cases} \mathrm{Re} \lim_{u\to-\infty} y_0^{-1}(u)\partial_u T^u\left(u, t/\varepsilon\right) = 0 \\ \mathrm{Re} \lim_{u\to+\infty} y_0^{-1}(u)\partial_u T^s\left(u, t/\varepsilon\right) = 0 \end{cases}$$

This gives parameterizations of the invariant manifolds of the form

$$\begin{cases} x = x_0(u) \\ y = y_0(u) + y_1^{u,s}\left(u, \frac{t}{\varepsilon}\right) \end{cases}$$

- For real $u$ and $t$, the invariant manifolds are well approximated by the unperturbed separatrix.
- Nevertheless, they are exponentially close to each other and therefore it is very difficult to study the difference.
- We extend the parameterization to the complex plane for $u$ up to a distance of order $\mathcal{O}(\varepsilon)$ of the singularities of the unperturbed separatrix $u = \pm ia$.
- Close to these singularities, the invariant manifolds become bigger and therefore it is easy to study its difference.

- At a distance of order $\mathcal{O}(\varepsilon)$ of $u = \pm ia$ the unperturbed system and the perturbation have the same size.
- This implies that in this case the invariant manifolds are not well approximated by the unperturbed separatrix.
- We have to look for different first approximations of the invariant manifolds close to $u = \pm ia$.
- They are solutions of a new Hamilton-Jacobi equation independent of $\varepsilon$ usually called inner equation.
- This equation was studied I. Baldomá (2006).

- Roughly speaking, the difference of these first approximations replaces the Melnikov function in the first order.
- Namely, from the first order of the difference near the singularity, one can deduce true first order of the difference between the invariant manifolds in the reals.