Nearly integrable systems with orbits accumulating to KAM tori

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(joint work with Vadim Kaloshin)

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Nearly integrable Hamiltonian systems

- Hamiltonian

\[ H(\varphi, I) = H_0(I) + \varepsilon H_1(\varphi, I) \]

where

- \( \varphi \in T^n = (\mathbb{R}/\mathbb{Z})^n \) are the angles.
- \( I \in V \subset \mathbb{R}^n \) are the actions.
- \( \varepsilon \ll 1 \).

- Ergodic hypothesis (Maxwell, Boltzmann): Consider a typical \( H \) and a typical energy surface. Almost every point has a dense orbit (in the energy surface).

- (Not only for nearly integrable systems.)

- Disproved by KAM Theory in the 60’s
The quasiergodic hypothesis

- **Quasiergodic hypothesis** (Ehrenfest, Birkhoff): A typical $H$ in a typical energy surface has a **dense orbit**.
- Disproved by Herman for systems in $\mathbb{T}^{2n} \times [-\delta, \delta]^2$ (existence of codimension 1 tori).
- In systems in $\mathbb{T}^n \times \mathbb{R}^n$: Completely open, believed to be true and out of reach.
- Results for a weak form of such conjecture: orbits accumulating densely in a set of (large) positive measure.
- We deal with the nonautonomous case

$$H(\varphi, l, t) = H_0(l) + \varepsilon H_1(\varphi, l, t), \quad 2\pi \text{ – periodic in } t$$
Integrable Hamiltonian systems: $\varepsilon = 0$

- **Equations**
  \[
  \dot{\varphi} = \partial_l H_0(l) \\
  \dot{l} = 0
  \]

- Orbits confined in tori $\mathbb{T}^{n+1} = \{l = l_0\}$.

- Dynamics in these tori is a rigid rotation with frequency $(\omega(l_0), 1) = (\partial_l H_0(l_0), 1)$.

- QEH not possible for these systems.

- What happens for $\varepsilon > 0$ small?

- Stability: KAM
Is QEH typically true?

Actions do not move when $\varepsilon = 0$ and we want them to cover the action space densely.

Easier question (not easy!): Are there orbits whose action component $I(t)$ makes a change independent of $\varepsilon$?

$$\|I(T) - I(0)\| \geq 1$$

This is called Arnold Diffusion (Arnold 1964).

Hypotheses:

- $H_0$ is strictly convex: $D^{-1}\text{Id} \leq \partial^2 H_0(I) \leq D\text{Id}$ for some $D > 1$.
- $H_1$ is $C^r$ for some $r$ large enough.
- $H_0$ is $C^{r+3}$.

Convexity implies that $\omega(I) = \partial_I H_0(I)$ is a global diffeo.
Stability: KAM Theory

- \((\omega, 1) \in U \subset \mathbb{R}^{n+1}\) is \((\eta, \tau)\)-Diophantine if
  \[ |(\omega, 1) \cdot k| \geq \frac{\eta}{|k|^{n+\tau}}, \quad \text{for all } k \in \mathbb{Z}^{n+1} \setminus \{0\} \]

- Call \(\mathcal{D}_{\eta, \tau}\) to the set of such frequencies.
- For any \(\tau > 0\), \(\text{Meas}(\mathcal{D}_{\eta, \tau}) = 1 - O(\eta)\).

Theorem (Kolmogorov-Arnold-Moser, Pöschel)

Fix \(\eta > 0, \tau > 0\). There exists \(\varepsilon_0(\text{KAM}) > 0\), such that for all \(\varepsilon \in (0, \varepsilon_0(\text{KAM}))\), all tori with frequency in \(\mathcal{D}_{\eta, \tau}\) persist for \(H_0 + \varepsilon H_1\).

- Call \(\text{KAM}_{\eta, \tau}\) to the union of these tori: \(\text{Meas}(\text{KAM}_{\eta, \tau}) = 1 - O(\eta)\).
- One can take \(\eta \sim \sqrt{\varepsilon}\).
- Here we stick to fixed \(\eta > 0\).
Existence of instabilities

- It depends on the dimension.
- 2 degrees of freedom
  - Energy surface has dimension 3 and KAM tori dimension 2.
  - KAM tori act as a barrier: actions almost constant for all orbits and all time.
  - QEH and Arnold diffusion are not possible
- More than 2 degrees of freedom: QEH and Arnold diffusion expected to be typically true.
- From now on we focus on $2^{\frac{1}{2}}$ degrees of freedom (2 dof plus periodic time dependence).
Arnold Diffusion

- Orbits with a change in actions independent of $\varepsilon$ in the complement of $\text{KAM}_{\eta, \tau}$.
- Recent results by P. Bernard, V. Kaloshin and K. Zhang.

- Resonance in frequency space: Fix $k \in \mathbb{Z}^3$

  \[ \Gamma_k = \{(\omega, 1) \cdot k = 0\}. \]

- Main idea: drift along resonances.
Bernard-Kaloshin-Zhang results

- $2^{\frac{1}{2}}$ dof Hamiltonian

$$H(\varphi, I, t) = H_0(I) + \varepsilon H_1(\varphi, I, t), \quad 2\pi - \text{periodic in } t$$

- Fix two open sets $V_1, V_2$ in action space and a finite sequence of resonances $\{ \Gamma_{k_j} \}_{j=1}^{N}$ such that

$$V_1 \cap \Gamma_{k_1} \neq \emptyset, \quad \Gamma_{k_j} \cap \Gamma_{k_{j+1}} \neq \emptyset, \quad j = 1, \ldots, N - 1, \quad V_2 \cap \Gamma_{k_N} \neq \emptyset$$

- We have a resonant path from $V_1$ and $V_2$
Bernard-Kaloshin-Zhang results

**Theorem**

Consider $H_0$, two sets $V_1$, $V_2$ and a resonant path from $V_1$ to $V_2$. Then, for a “$C^r$-typical” $H_1$ and $\varepsilon \ll 1$, there exists an orbit $(\varphi(t), I(t), t)$ of $H = H_0 + \varepsilon H_1$ and a time $T > 0$ such that

$$I(0) \in V_1 \quad \text{and} \quad I(T) \in V_2.$$

- The orbit drifts along the resonant path: $\text{dist} \left( I(t), \bigcup_{j=1}^{N} \Gamma_{k_j} \right) \lesssim \sqrt{\varepsilon}$ for all $t \in [0, T]$.
- Similar results announced by J. Mather, J. P. Marco, C. Cheng.
Alternative version of the theorem: Fix $\rho > 0$. Then for $\varepsilon \ll 1$, $H_0 + \varepsilon H_1$ has a $\rho$-dense orbit (its $\rho$-neighborhood covers the phase space).

For QEH we want true density instead of $\rho$-density.

Nevertheless if we take $\rho \to 0$, we need $\varepsilon \to 0$.

Weak form of QEH: obtain an orbit accumulating densely to a set of positive (large) measure.

Kaloshin-Zhang-Zheng: An example of nearly integrable system with an orbit covering densely a set of positive measure.
Weak form of QEH

- Recall $\text{KAM}_{\eta, \tau}$ is union of tori with Diophantine frequency in $D_{\eta, \tau}$

**Theorem (G.-Kaloshin)**

There exists $r_0$ such that for any $\eta > 0$, $\tau > 0$ small and any $r \geq r_0$ there is a dense set

$$A \subset S' = \{\|H_1\|_{C^r} = 1\}$$

such that for any $H_1 \in A$ there exists $\varepsilon$ small enough such that there exists an orbit $(\varphi(t), l(t), t)$ of $H_0 + \varepsilon H_1$ satisfying

$$\text{KAM}_{\eta, \tau} \subset \bigcup_{t \in \mathbb{R}} (\varphi(t), l(t), t)$$

Therefore $\text{Meas} \left( \bigcup_{t \in \mathbb{R}} (\varphi(t), l(t), t) \right) \geq 1 - c\eta$ for some constant $c > 0$ independent of $\eta$ and $\varepsilon$. 
Some remarks

- \( \varepsilon < \varepsilon_0(\text{KAM}) \).
- Corollary: KAM tori are unstable (also recent results by J. Zhang and Cheng).
- Recall that these tori are Lagrangian (they do not have invariant manifolds).
- With our methods we can probably get \( \text{Meas} \geq 1 - c\varepsilon^\alpha \) for some \( \alpha < 1/2 \).
- Also true for 3 dof in the region \( \partial_{l_3} H_0 \geq \nu > 0 \): orbits accumulating in a set of positive measure in the energy level.
- We only get a \( C^r \)-density result.
Our result is very related to instability of elliptic points.

R. Douady: instability of elliptic points is a flat phenomenon.

Take $f_0 \in C^\infty$ a symplectic mapping (non degenerate) with an elliptic point at the origin.

Then, $\exists f, g$ such that
- $f - f_0, g - f_0$ are flat at the origin.
- The origin is Lyapunov unstable for $f$ and stable for $g$.

Lyapunov stability is not an open property.

Only open if we take perturbations $H_1$ supported away from $\text{KAM}_{\eta,\tau}$. 
Robustness of the result

- Consider the $C^r$-Whitney topology for $H_1$’s vanishing on $\text{KAM}_{\eta,\tau}$ with some (strong) decay.

- Then if $H_0 + \varepsilon H_1$ satisfies the theorem, so does any Hamiltonian $H_0 + \varepsilon H_1 + \varepsilon \Delta H_1$ with $\Delta H_1$ small with respect to this $C^r$-Whitney topology.

- Equivalently, take $\Delta H_1 = f \cdot \tilde{\Delta} H_1$ with $f|_{\text{KAM}_{\eta,\tau}} = 0$ with strong decay.

- Then, the theorem is true for $\tilde{\Delta} H_1$ in a small ball with respect to the usual $C^r$ topology.
Some ideas of the proof

- Construct a set of resonant segments which contain $\mathcal{D}_{\eta,\tau}$ in its closure.
- Adapt the result of Bernard-Kaloshin-Zhang to drift along resonances in a small neighborhood of KAM tori.
Approaching one Diophantine frequency

- Dirichlet theorem: Fix $\omega \in \mathbb{R}^2$, $R \gg 1$. There exists $k \in \mathbb{Z}^3 \setminus \{0\}$, $|k| \leq R$ such that
  
  $$|(\omega, 1) \cdot k| \leq R^{-2}$$

- We need to modify it to avoid angle between segments $\rightarrow 0$.

- We cannot apply this idea to all Diophantine frequencies at the same time (we cannot control the intersections between resonant segments).
We construct a tree of resonances approaching all frequencies in $\mathcal{D}_{\eta,\tau}$

Take sequence $\rho_{n+1} = \rho_n^{1+2\tau}$, $\rho_0 \ll 1$.

By Vitali Covering lemma: take a sequence of grids $\mathcal{D}^n_{\eta,\tau} \subset \mathcal{D}_{\eta,\tau}$, $n \geq 1$ such that

- $3\rho_n$-balls of $\omega \in \mathcal{D}^n_{\eta,\tau}$ cover $\mathcal{D}_{\eta,\tau}$.
- $\rho_n$-balls of $\omega \in \mathcal{D}^n_{\eta,\tau}$ are disjoint.

We have $\mathcal{D}_{\eta,\tau} \subset \bigcup_{n \geq 1} \mathcal{D}^n_{\eta,\tau}$.

We make approximation by generations.
The tree of resonant segments

- **Generation 0**: horizontal and vertical resonant segments which are $\rho_0$ close.
- **Generation $n \geq 1$**: we construct resonances
  - $\rho_n$-close to each $\omega \in \mathcal{D}_{\eta,\tau}^n$.
  - connected to the resonances of the previous generation.
- $\mathcal{D}_{\eta,\tau}$ belongs to the closure of the resonances tree.
Drifting along resonances: Bernard-Kaloshin-Zhang construction

Two regimes
- Single resonance: one resonant relation \( \Gamma_k = \{ \omega : (\omega, 1) \cdot k = 0 \} \).
- Double resonance: \( \omega \) such that \( \exists k_1, k_2 \) such that \( (\omega, 1) \cdot k_i = 0 \).
- Double resonances are dense in the single resonant lines.
- Only matter if \( k_1 \) and \( k_2 \) are of similar size.

Split the path between single and strong double resonances.
There exists a normally hyperbolic invariant cylinder along the resonance.

To prove this fact:
- Normal forms.
- Normally hyperbolic invariant manifolds theory.

By adding a small perturbation, the invariant manifolds of the cylinder intersect transversally.
Double resonance regime: the kissing cylinders

- The single resonance cylinders arrive at double resonances.
- Some of them cross the resonance.
- To prove existence of these cylinders
  - Several normal forms as before.
  - Variational methods: we reduce the system to a geodesic flow of a Finsler metric.

- We have a net of cylinders.
We use Mather-Fathi-Bernard variational techniques.

Single resonance: we drift along the cylinders.

Double resonance: we use the cylinders that cross it to
  - Cross double resonances: go through one cylinder.
  - Make a turn: jump from one cylinder to another one.

We analyze the Aubry-Mather sets that belong to the different cylinders.

There are orbits which shadow these Aubry-Mather sets.