Oscillatory motions for the restricted planar three body problem

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The restricted three body problem

- Three bodies of masses $1-\mu$, μ and 0 under the effect of the Newtonian gravitational force.
- The bodies with mass (primaries) are not influenced by the zero mass one.
- They form a two body problem.
- Assume they move on ellipses of eccentricity $e_0 \in (0,1)$ (RPE3BP) or circles $e_0 = 0$ (RPC3BP).

The Restricted Planar 3BP

- Hamiltonian

$$H(q,p,t) = \frac{\|p\|^2}{2} - \frac{1-\mu}{\|q+\mu q_0(t)\|} - \frac{\mu}{\|q-(1-\mu)q_0(t)\|}$$

  where $q, p \in \mathbb{R}^2$.
- The two primaries are located at $-\mu q_0(t)$ and $(1-\mu)q_0(t)$.
- Wlog assume that $q_0(t)$ has period 2π,

$$q_0(t) = (\rho(t)\cos v(t), \rho(t)\sin v(t))$$

  where

$$\rho(t) = \frac{1-e_0^2}{1+e_0\cos v(t)}$$

  and $v(t)$ is called the *true anomaly* and satisfies

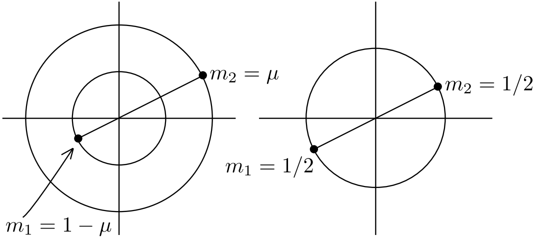
$$\frac{dv}{dt} = \frac{(1+e_0\cos v)^2}{(1-e_0^2)^{3/2}}, \qquad v(0) = 0.$$

Circular case: $e_0 = 0$, $q_0(t) = (\cos t, \sin t)$

- The system has a rotational symmetry
- First integral: the Jacobi constant

$$\mathcal{J}(q, p, t; \mu) = \mathcal{H}(q, p, t; \mu) - (q_1 p_2 - q_2 p_1).$$

- It can be reduced to a two degree of freedom system.
- When $\mu = 1/2$, the two bodies move in the same circle at diametrally opposed points.
- The Hamiltonian is π-periodic in time.



Asymptotic motions

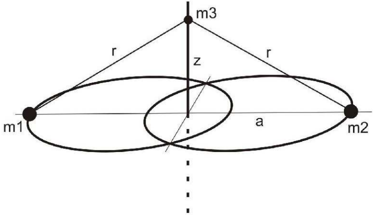
- Chazy (1922): Classification of all possible states that a 3BP can approach as $t \to \pm\infty$.
- Restricted 3BP:
  - $H^{\pm}$ (hyperbolic): $\|q(t)\| \to +\infty$ and $\|\dot{q}(t)\| \to c > 0$ as $t \to \pm\infty$.
  - $P^{\pm}$ (parabolic): $\|q(t)\| \to +\infty$ and $\|\dot{q}(t)\| \to 0$ as $t \to \pm\infty$.
  - $B^{\pm}$ (bounded): $\limsup_{t\to\pm\infty} \|q\| < +\infty$.
  - $OS^{\pm}$ (oscillatory): $\limsup_{t\to\pm\infty} \|q\| = +\infty$ and $\liminf_{t\to\pm\infty} \|q\| < +\infty$.
- Examples of all types except oscillatory already known by Chazy.
- When $\mu \to 0$ we have a 2 body problem: only has $H^{\pm}$, $P^{\pm}$, $B^{\pm}$ and

$$H^+ = H^-, \ P^+ = P^- \text{ and } B^+ = B^-.$$

Existence of oscillatory motions

Oscillatory motions were first proved by

- Sitnikov (1960) considered the restricted spatial elliptic three body problem.
- Existence of oscillatory motions when
    - Primaries have mass $\mu = 1/2$ and move on ellipses of small enough eccentricity.
    - Third body moves on the (invariant) vertical axis.
- Moser (1973) gave a new proof of Sitnikov results.



Oscillatory motions for the RPC3BP

- First results in the planar case by Llibre and Simó, 1980 following Moser's approach.

Theorem (Llibre-Simó)

Fix $\mu > 0$ small enough. Then, there exists an orbit $(q(t), p(t))$ of RCP3BP which is oscillatory. Namely, it satisfies

$$\limsup_{t\to\pm\infty} \|q\| = +\infty \quad \textit{and} \quad \liminf_{t\to\pm\infty} \|q\| < +\infty.$$

- More concretely: they obtain oscillatory motions in each $\mathcal{J}(q, p, t; \mu) = J_0$ for large enough J_0 and μ exponentially small with respect to J_0:

$$\mu \ll e^{-\frac{J_0^3}{3}}$$

Other results in oscillatory motions

RPC3BP

- Xia (1992), following Llibre-Simó and using analyticity arguments: oscillatory motions for every $\mu \in (0, 1/2]$ except a finite number of values.

- J. Galante and V. Kaloshin (2011) use Aubry-Mather theory to prove the existence of orbits which initially are in the range of our Solar System and become oscillatory as $t \to +\infty$ with $\mu = 10^{-3}$ (realistic for the Jupiter-Sun).

No results for the RPE3BP

Results by Alexeev and Llibre-Simó in other models.

Abundance of the asymptotic motions

- All possible combinations $X^- \cap Y^+$ for $X, Y = H, P, B, OS$ exist.

- Measure of each $X^- \cap Y^+$?

- It is known for each of them whether they have positive or zero measure except for $OS^- \cap OS^+$.

- Conjecture (Kolmogorov, Alexeev): Lebesgue measure of $OS^- \cap OS^+$ is zero.

Abundance of oscillatory motions

- Kaloshin and Gorodetski (2011): study the Hausdorff dimension of oscillatory motions for both the Sitnikov problem and the RPC3BP.
- For the RPC3BP:
  - Fix J_0 large enough. For a a Baire generic set in an open set of mass ratio μ, oscillatory motions have maximal Hausdorff dimension in $\mathcal{J}(q, p, t; \mu) = J_0$.
  - Fix $\mu \in (0, 1/2]$. For a a Baire generic set in an open set of Jacobi constants J_0, the oscillatory motions have maximal Hausdorff dimension in $\mathcal{J}(q, p, t; \mu) = J_0$.

Oscillatory motions in the RCP3BP

- We generalize Llibre-Simó results to any value $\mu \in (0, 1/2]$.
- Recall that for $\mu = 0$ they cannot exist.

Theorem (G.–Martín – Seara)

Fix any $\mu \in (0, 1/2]$. Then, there exists an orbit $(q(t), p(t))$ of RCP3BP which is oscillatory. Namely, it satisfies

$$\limsup_{t\to\pm\infty} \|q\| = +\infty \quad \textit{and} \quad \liminf_{t\to\pm\infty} \|q\| < +\infty.$$

More precisely,

Theorem

Fix any $\mu \in (0, 1/2]$. Then, there exists $J_0 > 0$ big enough, such that for any $J > J_0$ there exists an orbit $(q_J(t), p_J(t))$ of RCP3BP in the hypersurface $\mathcal{J}(q, p, t; \mu) = J$ which is oscillatory. Namely, it satisfies

$$\limsup_{t\to\pm\infty} \|q_J\| = +\infty \quad \textit{and} \quad \liminf_{t\to\pm\infty} \|q_J\| < +\infty.$$

- These orbits satisfy $\liminf_{t\to\pm\infty} \|q_J\| \sim J^2$.
- They are far from the primaries (far from collision).

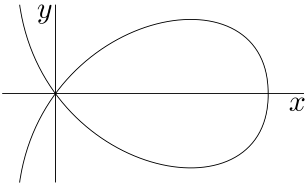
Moser and Llibre-Simó approach

- RPC3BP can be reduced to a two dimensional Poincaré map
- McGehee coordinates send "infinity" to the origin. Then, infinity becomes a parabolic critical point.
- This parabolic point has invariant manifolds.
- Prove that they intersect transversally.
- Establish symbolic dynamics close to these invariant manifolds.
- It leads to the existence of oscillatory motions.

Main difficulty in applying the approach to RPC3BP: prove the transversality of the invariant manifold of infinity.

Transversality of the invariant manifolds of infinity in Llibre-Simó

- For $\mu = 0$ the invariant manifolds coincide (parabolic orbits).
- In McGehee coordinates we have a homoclinic to a critical point.
- For $0 < \mu \ll 1$, expand in μ and compute the first order of the difference between the manifolds (Melnikov Theory).



- We know only how to compute it provided the Jacobi constant $J_0 \gg 1$.

Transversality of the invariant manifolds of infinity

- Distance between manifolds (in certain transversal section and in suitable coordinates) for $\mu \ll 1$ and $J_0 \gg 1$

$$d(\phi) = \mu C J_0^{3/2} e^{-\frac{J_0^3}{3}} \sin\phi + \mathcal{O}(\mu^2) \text{ for some } C > 0$$

- Applying Melnikov method only gives transversality provided $J_0 \gg 1$ and $\mu \leq J_0^{3/2} e^{-J_0^3/3}$.
- We prove the transversality for any $\mu \in (0, 1/2]$ and $J_0 \gg 1$.
- We want to be in a nearly integrable setting.
- For any $\mu \in (0, 1/2]$ and $J_0 \gg 1$, the RPC3BP is still close to a two body problem.

A different nearly integrable setting

- The existence of exponentially small phenomena usually arise when
  - The system possesses two different time scales.
  - The system has combined fast elliptic behavior and slow hyperbolic (or parabolic) behavior.
- Restrict to $\mathcal{J} = J_0$, $J_0 \gg 1$.
- For $J_0 \gg 1$, the zero mass body is very far away from the primaries ($\geq J_0^2$) and moves very slowly.
- Looked from the zero mass body,
  - The primaries are extremely close (at first order just one body).
  - They move much faster than the massless body.
- After a suitable rescaling:
  - The third body is at an ~ 1 position.
  - Its equation is a two body problem plus a small and fast periodic in time perturbation.

RPC3BP in rotating polar coordinates

- Hamiltonian in rotating polar coordinates:

$$H(r,\phi,y,G;\mu)=\frac{y^2}{2}+\frac{G^2}{2r^2}-G-U(r,\phi;\mu).$$

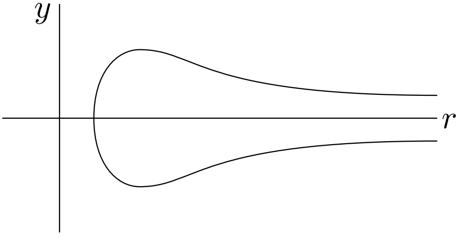
- G is the angular momentum.
- Conservation of energy corresponds to conservation of the Jacobi constant.
- Restrict to $H=-G_0$ with $G_0 \gg 1$ and consider a section $\phi=\phi_0$.
- Area preserving Poincaré map

$$\begin{array}{rcl} \mathcal{P}_{\phi_0} & : & \{\phi=\phi_0\} \longrightarrow \{\phi=\phi_0\} \\ & & (r,y) \quad \mapsto \quad \mathcal{P}_{\phi_0}(r,y) \end{array}$$

- $(r,y)=(\infty,0)$ is a fixed point with 1 dim. invariant manifolds.

The two body problem: $\mu \to 0$

- When $\mu = 0$, the massless body is only influenced by one primary, located at the origin
- The invariant manifolds of infinity coincide forming a separatrix.



- We show that the separatrix splits when we add the perturbation.
- We measure their distance in a section transversal to the unperturbed separatrix.

The difference between the manifolds

Theorem

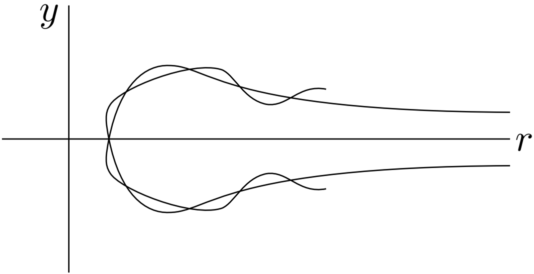
Consider the invariant manifolds of infinity of the Poincaré map $\mathcal{P}_{\phi_0}$. *Then, there exists* $G_0^* > 0$ *such that for any* $G_0 > G_0^*$ *and* $\mu \in (0, 1/2]$, *in a suitable section the distance* d *between these curves along this section is given by*

$$
\begin{aligned}
d = & C\mu(1-\mu)\left[\frac{1-2\mu}{2\sqrt{2}} G_0^{3/2} e^{-\frac{G_0^3}{3}} \sin\left(f(\phi_0)\right)\right. \\
& + 8 G_0^{7/2} e^{-\frac{2G_0^3}{3}} \sin\left(2f(\phi_0)\right) \\
& \left. + \mathcal{O}\left((1-2\mu) G_0 e^{-\frac{G_0^3}{3}} + G_0^3 e^{-\frac{2G_0^3}{3}}\right)\right],
\end{aligned}
$$

where $C > 0$ *and* $f(\phi)$ *are an explicit constant and an explicit function.*

Theorem

Fix $\mu \in (0, 1/2]$. Then, there exists $G^* > 0$ such that for any $G_0 > G^*$, the invariant manifolds of infinity of $\mathcal{P}_{\phi_0}$ intersect transversally.



This result allow us to proof the existence of oscillatory motions.

Theorem

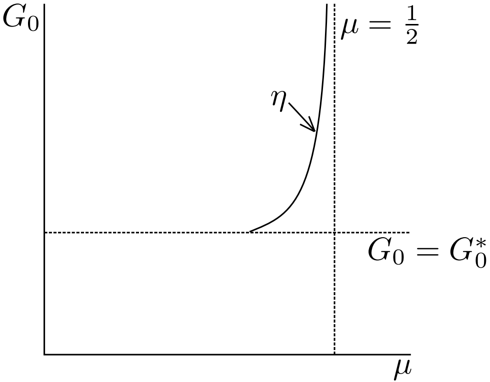
There exist G_0^* and a curve η in the parameter region

$$(\mu, G_0) \in \left(0, \frac{1}{2}\right] \times (G_0^*, +\infty),$$

of the form

$$\mu = \mu^*(G_0) = \frac{1}{2} - 16\sqrt{2} G_0^2 e^{-\frac{G_0^3}{3}} \left(1 + O\left(G_0^{-1/2}\right)\right),$$

such that, for $(\mu, G_0) \in \eta$, the invariant manifolds of infinity of $\mathcal{P}_{\phi_0}$ have a cubic homoclinic tangency and a transversal homoclinic point.



Bifurcation curve η in the parameter space where the homoclinic tangency is undergone.

Some ideas on computing the exponentially small distance between manifolds

- Study the invariant manifolds as solutions of the Hamilton – Jacobi equation as a perturbation of the separatrix of the 2BP.
- Consider the analytic continuation of the generating functions into certain regions of the complex plane "close" to the singularities of the unperturbed separatrix.
- Analysis of the difference between the generating functions in such complex regions gives the asymptotic formula for the exponentially small difference between the real invariant manifolds.

Oscillatory motions for the RPE3BP

- No previous results.
- One would like to prove their existence for any $\mu \in (0, 1/2]$ and $e_0 \in (0, 1]$.
- Only results for $e_0 \ll 1$.

Theorem (G. – Martín – Seara – Sabbagh)

Fix $\mu \in (0, 1/2]$. There exists $e_0^*(\mu) > 0$ such that for any $e_0 \in (0, e_0^*(\mu))$ there exists an orbit $(q(t), p(t))$ of RPE3BP such that

$$\limsup_{t \to +\infty} \|q\| = +\infty \quad \textit{and} \quad \liminf_{t \to +\infty} \|q\| < +\infty.$$

- $OS^+, OS^- \neq \emptyset$ but we do not know whether $OS^- \cap OS^+ \neq \emptyset$.

Oscillatory motions for the elliptic case

- RPE3BP has $2\frac{1}{2}$ degrees of freedom (4 dimensional Poincaré map $\mathcal{P}_{t_0}$).
- For $\mathcal{P}_{t_0}$, infinity is a two dimensional "normally parabolic" cylinder Λ filled with fixed points.
- Moser approach:
  1. Prove that Λ has invariant manifolds.
  2. Prove that they intersect transversally.
  3. Prove existence of symbolic dynamics close to the invariant manifolds

Oscillatory motions for the elliptic case

- Moser approach: construct a horseshoe.
- A horseshoe gives more precise information.
- Construct orbits with different final motion in the past and in the future.
- Instead, we rely on "simpler ideas" from Arnold diffusion.
- Infinite transition chain: an infinite sequence of fixed points in the cylinder Λ connected by transversal heteroclinic orbits.
- Use a Lambda lemma to prove that there exists an orbit shadowing such chain.
- Such orbit would be oscillatory.

The invariant manifolds

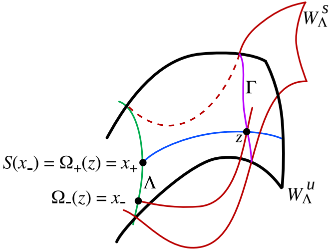
- For any $\mu \in [0, 1/2]$ and $e_0 \in [0, 1)$, any fixed point in Λ has stable/unstable invariant manifolds.
- They are analytic away from the fixed point and $\mathcal{C}^\infty$ at the fixed point.
- Fix $\mu \in (0, 1/2]$. For the RPC3BP we have transversality of the invariant manifolds of
$$\Lambda \cap [G_*, +\infty)$$
- Fix an interval $[G_1, G_2] \subset [G_*, +\infty)$. For $e_0 \ll 1$ we still have transversality.

The scattering map (after Delshams – de la Llave – Seara)

- Λ has stable and unstable manifolds intersecting along a homoclinic manifold Γ.
- Scattering map associated to the homoclinic manifold Γ

$$S : \tilde{\Lambda} \to \tilde{\Lambda}, \quad x_+ = S(x_-)$$

when
$\emptyset \neq W^s(x_+) \cap W^u(x_-) \in \Gamma$



- That is: $\exists z \in \Gamma$ such that

$$\mathrm{dist}(\mathcal{P}^n x_\pm), \mathcal{P}^n z) \to 0, \quad \text{as} \quad n \to \pm\infty$$

- Delshams– de la Llave – Seara: Scattering map is area preserving.

The scattering map for the RPE3BP

- Tipically scattering maps are not globally defined (are multivalued).
- In this problem is globally defined.
- Scattering map of the elliptic problem

$$\mathcal{S} : \mathbb{T} \times [G_1, G_2] \to \mathbb{T} \times \mathbb{R}$$

is a nearly integrable twist map

$$\mathcal{S} : \begin{pmatrix} \alpha \\ G \end{pmatrix} \longmapsto \begin{pmatrix} \alpha + f(\mu, G) + \mathcal{O}(e_0) \\ G + \mathcal{O}(e_0) \end{pmatrix}.$$

- An infinite transition chain of fixed points is equivalent to consider an orbit for the scattering map

Shadowing the transition chain

- We want orbits which stay $\mathbb{T} \times [G_1, G_2]$ for all time.
- The scattering map is a nearly integrable twist maps.
- Such orbits give infinite transition chains.
- We prove a $\mathcal{C}^0$ Lambda lemma (Shilnikov lemma) for parabolic points.
- Applying the Lambda lemma, orbits shadowing such transition chains are oscillatory.

Conclusion

- RPC3BP: proved for all $\mu \in (0, 1/2]$.
- REC3BP: proved for all $\mu \in (0, 1/2]$ and $e_0 \ll 1$ for the elliptic problem.
- For all $e_0 \in (0, 1)$ only problem is to prove transversality of the invariant manifolds.
- Non restricted 3BP: only results for a Sitnikov like configuration (Alexeev) for very small parameters range.
- The presented methods should apply: only remains to prove transversality of the invariant manifolds.
- For more bodies no results at all.