Growth of Sobolev norms for the cubic defocusing NLS

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The cubic defocusing nonlinear Schrödinger equation

- Consider the equation

$$\begin{cases} -i\partial_t u + \Delta u = |u|^2 u \\ u(0,x) = u_0(x) \end{cases}$$

  where $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$.
- Defocusing implies well posed globally in time in Sobolev spaces $H^s(\mathbb{T}^2)$, $s \geq 1$.
- Solutions of NLS conserve two quantities:
  - The Hamiltonian

$$E[u](t) = \int_{\mathbb{T}^2} \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{4}|u|^4 \right) dx(t)$$

  - The mass

$$\mathcal{M}[u](t) = \int_{\mathbb{T}^2} |u|^2 dx(t) = \int_{\mathbb{T}^2} |u|^2 dx(0),$$

    the square of the L^2-norm.

Transfer of energy

- Fourier series of u,

$$u(x,t) = \sum_{n\in\mathbb{Z}^2} a_n(t)e^{inx}$$

- Can we have transfer of energy to higher and higher modes as $t \to +\infty$?
- It is possible that a solution u starts oscillating only on scales comparable to the spatial period and eventually oscillates on arbitrarily small scale?

Sobolev norms

- Sobolev norms

$$\|u(t)\|_{H^s(\mathbb{T}^2)} := \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} := \left(\sum_{n\in\mathbb{Z}^2} \langle n\rangle^{2s} |a_n(t)|^2\right)^{1/2},$$

  where $\langle n\rangle = (1+|n|^2)^{1/2}$.
- Thanks to mass and energy conservation,

$$\|u(t)\|_{H^1(\mathbb{T}^2)} \leq C\|u(0)\|_{H^1(\mathbb{T}^2)} \text{ for all } t \geq 0.$$

- The L^2 norm is conserved.
- The energy transfer can be measured with the growth of the Sobolev norms with $s > 1$.
- Only possibility for H^s to grow indefinitely: the energy of u moves to higher and higher Fourier modes.

Growth of Sobolev norms

- Zakharov-Shabat equation: cubic defocusing NLS for $x \in \mathbb{T}$:

$$-i\partial_t u + \Delta u = |u|^2 u$$

- There are a priori bounds for all s-Sobolev norms and therefore there cannot be transfer of energy.
- In dimension $d \geq 2$, there are no a priori bounds and growth of s-Sobolev norms may happen.

How fast the energy transfer can be?

Polynomial upper bounds for the growth of Sobolev norms were first obtained by Bourgain (1996),

Theorem

Let us consider a solution u of the cubic defocusing NLS on $\mathbb{T}^2$, then for any $\delta > 0$,

$$\|u(t)\|_{H^s} \leq t^{2(s-1)+\delta} \|u(0)\|_{H^s} \quad \textit{for} \quad t \to +\infty.$$

- Question by Bourgain (2000): Are there solutions u such that for $s > 1$,
$$\|u(t)\|_{H^s} \to +\infty \qquad \text{as } t \to +\infty?$$
- Moreover, he conjectured that if such solutions exist, the growth should be subpolynomial in time. That is,
$$\|u(t)\|_{H^s} \ll t^{\varepsilon} \|u(0)\|_{H^s} \qquad \text{for } t \to +\infty, \text{ for all } \varepsilon > 0.$$

Previous results

- Kuksin (1997) obtained orbits with arbitrarily big fixed growth of Sobolev norms for NLS taken large enough initial data.
- For large initial data, dispersion is weaker than the nonlinearity.
- We are interested in small initial data.
- Stability of $u = 0$?
  - From the point of view of dynamical systems, $u = 0$ is an elliptic critical point.
  - For the linear equation, the solutions have constant s-Sobolev norms, for any s, as time evolves.
  - Thus, for the linear equation, $u = 0$ is stable in any H^s topology.
  - It is stable for the nonlinear equation?
  - Growth of Sobolev norms for small solutions implies its instability.
- How long does it take such instability to be noticeable?

Previous results

Colliander, Keel, Staffilani, Takaoka, Tao (2010) proved the following deep result:

Theorem

Fix $s > 1$, $\mathcal{C} \gg 1$ *and* $\mu \ll 1$. *Then there exists a global solution* $u(t, x)$ *of NLS on* $\mathbb{T}^2$ *and* T *satisfying that*

$$\|u(0)\|_{H^s} \leq \mu, \qquad \|u(T)\|_{H^s} \geq \mathcal{C}.$$

- The solutions they obtain have small initial mass and energy (smaller than μ).
- They remain small as time evolves whereas the s-Sobolev norm grows considerably.

Refining the methods used in that paper, we estimate the speed of the growth of Sobolev norms.

Theorem (V. Kaloshin-M. G.)

Let $s > 1$. Then, there exists $c > 0$ with the following property: for any large $\mathcal{K} \gg 1$ there exists a a global solution $u(t,x)$ of NLS on $\mathbb{T}^2$ and a time T satisfying

$$T \sim \mathcal{K}^c,$$

such that

$$\|u(T)\|_{H^s} \geq \mathcal{K} \|u(0)\|_{H^s}.$$

Moreover, this solution can be chosen to satisfy

$$\|u(0)\|_{L^2} \leq \mathcal{K}^{-\alpha}, \ \alpha > 0.$$

We can impose to start with small Sobolev norm but we get a slower growth.

Theorem (V. Kaloshin-M. G.)

Fix $s > 1$. Then, there exists $c > 0$ with the following property: for any small $\mu \ll 1$ and large $\mathcal{C} \gg 1$ there exists a global solution $u(t,x)$ of NLS on $\mathbb{T}^2$ and a time T satisfying that

$$T \sim \left(\frac{\mathcal{C}}{\mu}\right)^{c \ln(\mathcal{C}/\mu)}$$

such that

$$\|u(0)\|_{H^s} \leq \mu, \qquad \|u(T)\|_{H^s} \geq \mathcal{C}.$$

- Our result are valid in any $\mathbb{T}^d$, $d \geq 2$ taking solutions which only depend on two spatial variables.

Comparison with Bourgain conjecture

- Bourgain conjecture:

$$\|u(t)\|_{H^s} \ll t^{\varepsilon}\|u(0)\|_{H^s} \qquad \text{for } t \to +\infty, \text{ for all } \varepsilon > 0.$$

- Our result

$$\|u(T)\|_{H^s} \geq T^{\frac{1}{c}}\|u(0)\|_{H^s}. \qquad \text{for } T \gg 1.$$

- Our result does not contradict Bourgain conjecture about the subpolynomial growth:
  - The theorem deals with arbitrarily large but finite growth in the Sobolev norms.
  - Bourgain conjecture refers to unbounded growth.
- Growth of the s-Sobolev norm may slow down as time grows.

Main ideas of the proof and comparison with the I-team approach

- Source of instability are resonances.
- NLS as an ode (of infinite dimension) for the Fourier coefficients of u:

$$-i\dot{a}_n = |n|^2 a_n + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2 \\ n_1-n_2+n_3=n}} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n\in\mathbb{Z}^2.$$

- Eigenvalues of $a=0$ are

$$\lambda_n^\pm = \pm i|n|^2$$

- A term in the cubic nonlinearity is resonant provided

$$|n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0$$

- NLS is extremely resonant.

The I-team approach

- I-team chooses carefully a finite set of modes which interact through the resonances in a very particular way.
- Using the resonant interactions, they introduce a finite dimensional (toy) model

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 1,\dots N.$$

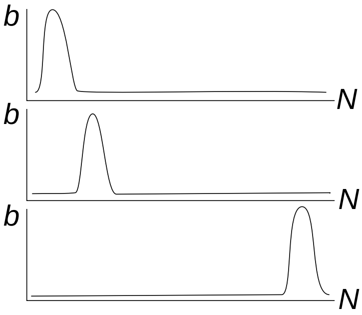
  which approximates well certain solutions of NLS.
- It can be seen as a Hamiltonian system on a lattice $\mathbb{Z}$ with nearest neighbor interactions.
- Hamiltonian:

$$h(b) := \frac{1}{4}\sum_{j=1}^{N}|b_j|^4 - \frac{1}{2}\sum_{j=1}^{N}\left(\overline{b}_j^2 b_{j-1}^2 + b_j^2\overline{b}_{j-1}^2\right)$$

The toy model

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 1,\ldots N.$$

- We want an orbit $b(t)$ of the toy model such that at $t = 0$ is localized in b_1 and at a certain $t = T \gg 1$ is localized in b_N.
- We need to analyze the dynamics of the system.



Dynamics of the toy model

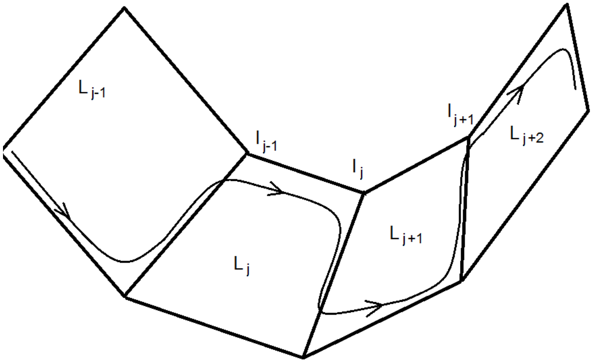
- We analyze the dynamics of the toy model

$$\dot{b}_j = -i b_j^2 \overline{b}_j + 2i \overline{b}_j \left(b_{j-1}^2 + b_{j+1}^2 \right), \ j = 0, \ldots N,$$

- Each 4-dimensional plane

$$L_j = \{ b_1 = \cdots = b_{j-1} = b_{j+2} = \cdots = b_N = 0 \}$$

is invariant.



- We construct solutions that stay close to the planes $\{L_j\}_{j=2}^{N-1}$ and go from one intersection $l_j = L_j \cap L_{j+1}$ to the next one $l_{j+1} = L_{j+1} \cap L_{j+2}$ consequently for $j = 3, \ldots, N-1$.
- In the intersections l_j only b_j is nonzero.
- The planes L_j have normal positive Lyapunov exponents.

The shadowing

- We want an orbit which shadows the concatenation of periodic and heteroclinic orbits.
- The I-team does the shadowing using Gronwall-like estimates.
- Their methods would lead to bad time estimates

$$T > C^{\mathcal{K}^\alpha}, \ \ C > 0, \alpha \geq 2.$$

- Our main contribution: analysis of the toy model model using
  - Dynamical systems tools (normal forms, Shilnikov boundary problem).
  - A careful choice of the initial conditions

Last step

- The toy model has an orbit which has growth of Sobolev norms.
- We need to prove that solutions of NLS can be approximated by the solutions of the toy model for long enough time.
- We need ensure that the energy is localized in the modes in the support of the toy model.

Robustness of the I-team mechanism

- The mentioned results deal with a concrete equation.
- Is this instability mechanism still valid if we modify the equation?
- If one adds higher order terms, one obtains the same results since we are dealing with small solutions.
- What happens if one modifies the linear part of the equation?

The cubic defocusing NLS equation with a convolution potential

- Equation
$$\begin{cases} -i\partial_t u + \Delta u + V(x) * u = |u|^2 u \\ u(0,x) = u_0(x) \end{cases}$$
where
  - $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$.
  - $V \in H^{s_0}(\mathbb{T}^2)$, $s_0 > 0$, has real Fourier coefficients.
- It still has conservation of energy and mass.
- Simplified model of NLS with a multiplicative potential.
- Bourgain and Kuksin-Eliasson: existence of (small) invariant tori with quasiperiodic behavior.

NLS with a convolution potential as an infinite ode

- Potential in Fourier series

$$V(x) = \sum_{n\in\mathbb{Z}^2} v_n e^{inx}.$$

- Equation

$$-i\dot{a}_n = \left(|n|^2 + v_n\right)\, a_n + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2\\ n_1-n_2+n_3=n}} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n\in\mathbb{Z}^2.$$

- Eigenvalues of $a = 0$ are

$$\lambda_n^{\pm} = \pm i\left(|n|^2 + v_n\right).$$

- The potential might kill the resonances.
- One would expect stronger stability properties for the critical elliptic point $a = 0$.

Stability results

Theorem (Bambusi-Grebert 2003, also Glauckner-Lubich 2010)

Consider $M > 0$ and take a typical (in certain measure sense) potential V. Then, there exists $s_0, \mu_0 > 0$ such that for any $s \geq s_0$ and any solution $u(t)$ of NLS with potential V with initial condition $u(0) = u_0 \in H^s$ satisfying $\mu := \|u_0\|_{H^s} < \mu_0$, one has that

$$\|u(t)\|_{H^s} \leq 2\mu \quad \text{for} \quad |t| \lesssim \frac{1}{\mu^M}.$$

Instability results

Theorem (M. G.)

Fix $s > 1$, $s_0 > 70s/17$ and and take $V \in H^{s_0}(\mathbb{T}^2)$ with real Fourier coefficients. Then, there exists $c > 0$ with the following property: for any small $\mu \ll 1$ and large $\mathcal{C} \gg 1$ there exists a global solution $u(t,x)$ of NLS with convolution potential V and a time T satisfying that

$$T \sim \left(\frac{\mathcal{C}}{\mu}\right)^{c \ln(\mathcal{C}/\mu)}$$

such that

$$\|u(0)\|_{H^s} \leq \mu, \qquad \|u(T)\|_{H^s} \geq \mathcal{C}.$$

- We obtain exactly the same result as for cubic NLS with no potential.
- We can deal with any potential V.
- The obtained orbit u only depends on $\|V\|_{H^{s_0}}$ and not on V itself.

As for cubic NLS without potential, if we do not impose initial small Sobolev norm but only large growth, we obtain a growth polynomial in time.

Theorem (M. G.)

Fix $s_0 > 0$ and $s > 1$ and take $V \in H^{s_0}(\mathbb{T}^2)$ with real Fourier coefficients. Then, there exists $c > 0$ with the following property: for any large $\mathcal{K} \gg 1$ there exists a a global solution $u(t,x)$ of NLS with convolution potential V in $\mathbb{T}^2$ and a time T satisfying

$$T \sim \mathcal{K}^c,$$

such that

$$\|u(T)\|_{H^s} \geq \mathcal{K}\,\|u(0)\|_{H^s}.$$

Moreover, this solution can be chosen to satisfy

$$\|u(0)\|_{L^2} \leq \mathcal{K}^{-\alpha}, \ \alpha > 0.$$

Main ideas of the proof

- Equation

$$-i\dot{a}_n = \left(|n|^2 + v_n\right)\ a_n + \sum_{\substack{n_1,n_2,n_3\in\mathbb{Z}^2\\ n_1-n_2+n_3=n}} a_{n_1}\overline{a_{n_2}}a_{n_3}, \qquad n\in\mathbb{Z}^2.$$

- Eigenvalues of $a = 0$ are

$$\lambda_n^{\pm} = \pm i\left(|n|^2 + v_n\right).$$

- The potential might kill the resonances,
- However, $V \in H^{s_0}(\mathbb{T}^2)$ implies

$$v_n \sim \frac{1}{|n|^{s_0}}$$

Main ideas of the proof

- Cubic resonant terms for NLS with no potential

$$|n_1|^2 - |n_2|^3 + |n_3|^2 - |n|^2 = 0$$

- Cubic resonant terms for NLS with potential,

$$|n_1|^2 - |n_2|^3 + |n_3|^2 - |n|^2 + v_{n_1} - v_{n_2} + v_{n_3} - v_n = 0$$

- The I-team mechanism for NLS already deals with high modes.
- Resonant terms for NLS with no potential involving high enough modes are almost resonant for NLS with potential:

$$|n_1|^2 - |n_2|^3 + |n_3|^2 - |n|^2 - |n|^2 + v_{n_1} - v_{n_2} + v_{n_3} - v_n \ll 1$$

- We select carefully a finite set of modes so that they interact through almost resonant terms.