Exponentially small splitting of separatrices of the pendulum: two different examples

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A fast periodic perturbation of the pendulum

We consider a non-autonomous periodic perturbation of the pendulum

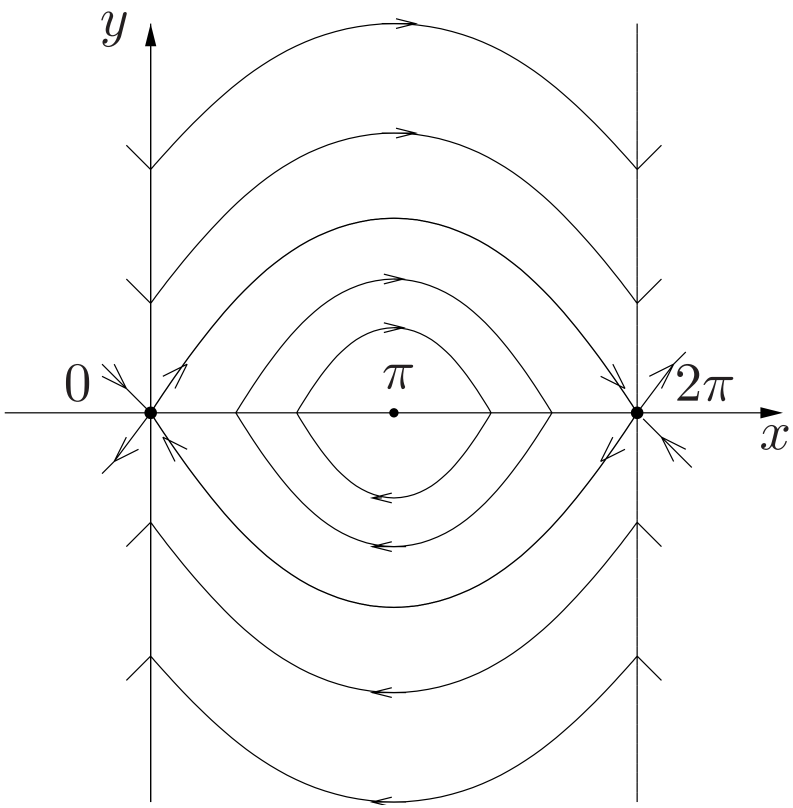
$$\ddot{x} = \sin x + \mu \sin \frac{t}{\varepsilon}$$

where $\varepsilon \ll 1$ and μ is not necessarily small.

Hamiltonian:

$$H(x, y, t) = \frac{y^2}{2} + (\cos x - 1) - \mu x \sin \frac{t}{\varepsilon}$$

Non perturbed system $\mu = 0$



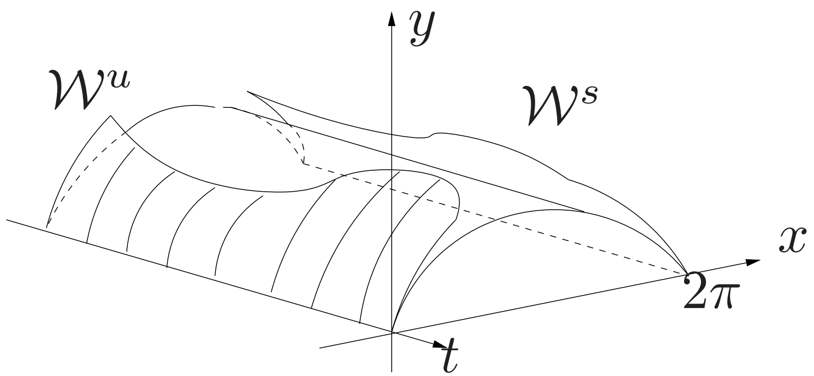
Non-perturbed system: The classical pendulum

$$\ddot{x} = \sin x$$

Moreover:

- Phase space: $(x, y) \in \mathbb{T}^1 \times \mathbb{R}$.
- It has an hyperbolic fixed point at $(0, 0)$ with two separatrices.

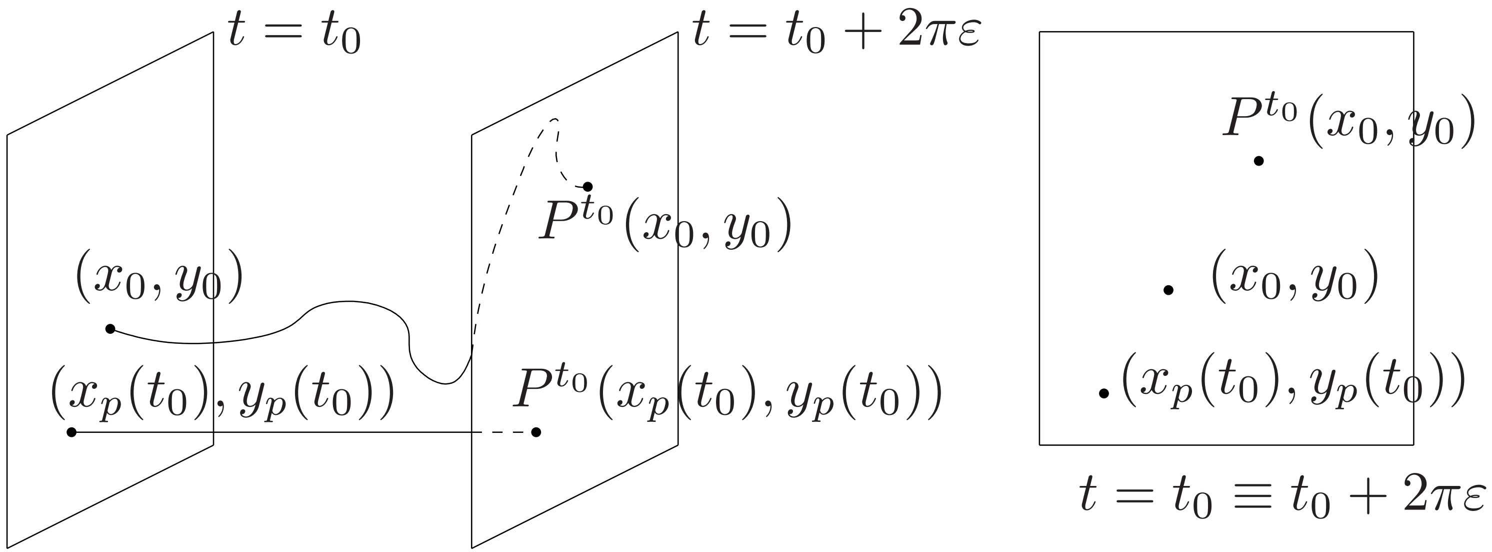
For $\varepsilon \neq 0$: The perturbed system



- 3-dimensional phase space.
- There exists a hyperbolic periodic orbit.
- The invariant manifolds are now 2-dimensional.
- They do not coincide $\longrightarrow$ the separatrix breaks down creating transversal intersections which lead to chaos in a layer of the former separatrix.

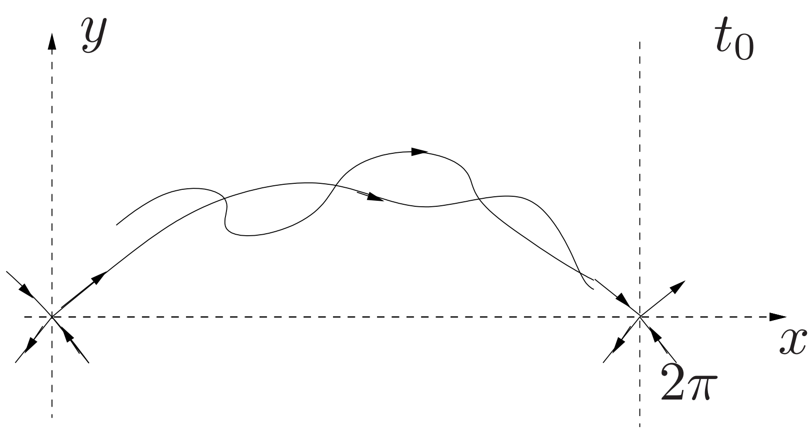
The width of this layer turns out to be exponentially small (even if the perturbation is big).

The $2\pi\varepsilon$-time Poincaré map formulation



From the perturbed system it can be derived a discrete dynamical system considering the 2π-time Poincaré map.

The splitting of separatrices in the Poincaré map



- Considering the Poincaré map, we obtain this picture.
- The manifolds intersect forming lobes between them.
- The area of these lobes is invariant by iteration of the Poincaré map due to the symplectic structure.

Perturbative approach in μ: Classical Melnikov Theory

If we take μ as a small parameter and consider a perturbative approach in μ, we can apply classical Melnikov Theory:

- The distance between manifolds is given by:

$$d(\varepsilon) = 2\pi e^{-\frac{\pi}{2\varepsilon}}\mu + \mathcal{O}(\mu^2)$$

- The area of the lobes is given by:

$$A(\varepsilon) = 8\pi\varepsilon^{-1}e^{-\frac{\pi}{2\varepsilon}}\mu + \mathcal{O}(\mu^2)$$

Consider the area formula:

$$A(\varepsilon) = 8\pi\varepsilon^{-1}e^{-\frac{\pi}{2\varepsilon}}\mu + \mathcal{O}(\mu^2)$$

If we take $\mu = \varepsilon^p$ for $p > 0$ (which is the natural relation):

$$A(\varepsilon) = 8\pi e^{-\frac{\pi}{2\varepsilon}}\varepsilon^{p-1} + \mathcal{O}(\varepsilon^{2p})$$

Therefore:

- If $\mu = \varepsilon^p$, the remainder is bigger than the Melnikov prediction.
- In order to be valid the Melnikov prediction we must have $\mu = \mathcal{O}\left(e^{-\frac{\pi}{2\varepsilon}}\right)$: μ has to be exponentially small with respect to ε

Another classical perturbative approach

To understand what is happening, we can look for parameterizations of the manifolds

$$x^u(r,\varepsilon),\ \ x^s(r,\varepsilon)$$

→ Since ε is small, we can look for formal solutions as a power series of ε:

$$x^\alpha(r,\varepsilon) = x_0(r) + \varepsilon x_1^\alpha(r) + \varepsilon^2 x_2^\alpha(r) + \dots \quad \text{for } \alpha = u, s$$

where we have omitted the dependency on μ.

For these problems of fast perturbation:

$$x_k^u(r) = x_k^s(r) \ \ \forall k \in \mathbb{N}$$

Conclusion:

$$x^u(r,\varepsilon) - x^s(r,\varepsilon) = \mathcal{O}(\varepsilon^k) \ \ \forall k \in \mathbb{N}$$

→ Proceeding formally we see that their difference is beyond all orders.

What is happening?

Two options:

1 Both manifolds coincide also in the perturbed case (the perturbed system is also integrable) $\longrightarrow$ the power series in ε is convergent:

2 Both manifolds do not coincide $\longrightarrow$ the power series in ε is divergent and the difference between manifolds has to be flat with respect ε.

In the perturbed pendulum equation is happening the second option $\longrightarrow$ In fact, we will see that their difference is exponentially small with respect ε.

Questions

- When does the Melnikov Theory predict correctly the asymptotic formula of the splitting?
- When it does not, how is the asymptotic formula?

- We will show that for

$$\ddot{x} = \sin x + \mu\varepsilon^p \sin\frac{t}{\varepsilon}$$

provided $p \geq -6$:

1. There exists a hyperbolic periodic orbit.
2. The difference between its invariant manifolds is exponentially small

- Attention: if $p \in (-6, 0)$ the perturbation is bigger than the original system but the splitting is still exponentially small
- For $p \geq -2$ we will give an asymptotic formula for the splitting.
- For $p \in (-6, -2)$ we will give exponentially small bounds.

In order to simplify the notation:

$$\ddot{x} = \sin x + \frac{\mu}{\varepsilon^2} \sin \frac{t}{\varepsilon} \quad \text{with} \quad \mu = \mathcal{O}(\varepsilon^s), s \geq 0$$

We reparameterize the time $\tau = \varepsilon^{-1} t$.

New Dynamical System:

$$\begin{cases} x' = \varepsilon y \\ y' = \varepsilon \sin x + \mu \varepsilon^{-1} \sin \tau \end{cases}$$

$\longrightarrow$ For systems of the form $x' = \varepsilon f(x, t)$ periodic in time: Averaging Theory allows to focus on the dominant part of the equation.

Averaging Theory (I)

In our system, two steps of averaging correspond to the change of variables:

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x + \mu \sin \tau \\ y + \mu \varepsilon^{-1} \sin \tau \end{pmatrix}$$

New system:

$$\begin{cases} x' = \varepsilon y \\ y' = \varepsilon \sin (x - \mu \sin \tau) \end{cases}$$

Now the perturbation has the same (or smaller size) than the non perturbed system.

Main idea: In these cases of fast perturbation, we are in a perturbative frame even when the perturbation and the non-perturbed system have the same size in some complex domain

Averaging Theory (II)

Performing this change of variables, we have changed the perturbation, but also the non-perturbed integrable system.

- Since our system is of the form $x' = \varepsilon f(x,t)$ and is non-autonomous and 2π-periodic,the integrable system is given by the averaged systems $x' = \varepsilon \bar{f}(x)$ where

$$\bar{f}(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x,t)dt$$

- In our case:

$$\begin{cases} x' = \varepsilon y \\ y' = \varepsilon J_0(\mu)\sin x \end{cases}$$

where $J_0(\mu)$ is the Bessel function of first order.

The Bessel function $J_0(\mu)$

It is defined by:

$$J_0(\mu) = \frac{1}{2\pi}\int_0^{2\pi} \cos(\mu \sin\tau)d\tau$$

Moreover,

- For μ small ($\mu = \mathcal{O}(\varepsilon^s)$ with $s > 0$): $J_0(\mu) = 1 + \mathcal{O}(\mu^2)$

  $\longrightarrow$ If $s > 1/2$ the averaged system is ε-close to the classical pendulum (and we are in a classical perturbative setting).

  $\longrightarrow$ For $s \in [0, 1/2]$, the averaged system is not close enough to the original one and we will obtain different results.

- If we consider μ_0 the first zero of $J_0(\mu)$ $\rightarrow$ We have to impose that μ belongs to $[0, \mu_0)$

Main theorem

Recall the system:

$$\begin{cases} x' = \varepsilon y \\ y' = \varepsilon \sin x + \mu \varepsilon^{-1} \sin \tau \end{cases}$$

Then, for ε sufficiently small and $\mu < \mu_0$, the area of the lobes is given by the asymptotic formula:

$$\mathcal{A} = \varepsilon^{-1} e^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} \left(4|f(\mu)| + \mu\, \mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)} \right) \right)$$

where $f(\mu) = 2\pi\mu + \mathcal{O}(\mu^3)$ is an analytic function.

Attention: This asymptotic formula still holds when $\mu = \mathcal{O}(1)$ with respect to ε (that is when the size of the perturbation is the same of the unperturbed system).

Validity of the Melnikov function prediction (I)

Following the notation:

$$\ddot{x} = \sin x + \frac{\mu}{\varepsilon^2} \sin \frac{t}{\varepsilon}$$

and using

$$J_0(\mu) = 1 + \mathcal{O}(\mu^2) \text{ and } f(\mu) = 2\pi\mu + \mathcal{O}(\mu^3)$$

when μ is small, we can check the validity of the Melnikov function.

First case: $\mu \leq \mathcal{O}(\varepsilon^s)$ for $s > 1/2$.

$$\mathcal{A} = \varepsilon^{-1} e^{-\frac{\pi}{2\varepsilon}} \left(8\pi\mu + \mu\,\mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)} \right) \right)$$

Conclusion: the asymptotic formula coincide with the Melnikov formula prediction which was not *a priori* true due to the exponentially smallness.

Validity of the Melnikov function prediction (II)

Second case: $\mu \le \mathcal{O}(\varepsilon^s)$ for $s \in (0, 1/2)$.

$$\mathcal{A} = \varepsilon^{-1} e^{-\frac{\pi}{2\varepsilon\sqrt{J_0(\mu)}}} \left(8\pi\mu + \mu\,\mathcal{O}\left(\frac{1}{\ln(1/\varepsilon)}\right)\right)$$

This is the Melnikov formula if we consider as unperturbed system de averaged system

$$\begin{cases} \dot{x} = y \\ \dot{y} = J_0(\mu)\sin x \end{cases}$$

Conclusion: In some sense, in this case the Melnikov function still works.

Validity of the Melnikov function prediction (III)

Third case: $\mu = \mathcal{O}(1)$ with respect to ε

$\longrightarrow$ in this case μ is a fixed constant independent of ε.

Then:

- Even when the perturbation and the integrable system have the same size, the splitting of separatrices is exponentially small.
- The asymptotic formula depends on the full jet of $f(\mu)$.
- The Melnikov function fails to predict the splitting of separatrices.

How do we study the difference between manifolds?

- Since the separatrix has singularities at $\pm i\pi/2$, it is expected that the perturbed manifolds have singularities close to them.
- Close to these singularities (and therefore close to $\pm i\pi/2$) these manifolds are very big $\rightarrow$ It is easier to study there their difference.
- We look for a good approximation of the manifolds close to the singularities (at a distance of order ε of $\pm i\pi/2$ they are not well aproximated by the separatrix).
- Their difference is studied through a Borel resumation process which gives $f(\mu)$ (Resurgence Theory by Jean Ecalle).
- From the (algebraically small) difference between manifolds close to the singularities we will derive the (exponentially small) difference between manifolds for real values of the variables.

Main ideas of our proof

- Following P. Lochak, J. P. Marco and D. Sauzin, we do not use parameterizations of the manifolds: we write them as a graph and we study them as solutions of the Hamilton-Jacobi equation.
- Following V. F. Lazutkin, we study the manifolds for complex time up to a distance $\mathcal{O}\left(\varepsilon \ln(1/\varepsilon)\right)$ of the singularities of the separatrix.
- Close to the singularities, the first order of the manifolds are solution of a new Hamilton-Jacobi equation called inner equation (analogous to the Reference System of V. Gelfreich).
- To compute the difference between manifolds, we use that it is a solution of a linear PDE.
- We do not use flow-box coordinates.
- This allows us to compute the splitting of separatrices in the original variables.

Splitting of separatrices for a bigger perturbation (I)

- Recall our model

$$H(x,y,t) = \frac{y^2}{2} + J_0(\mu)(\cos x - 1) + \sin x \cos\left(\mu \sin\frac{t}{\varepsilon}\right) + (\cos x - 1)\left(\cos\left(\mu\sin\frac{t}{\varepsilon}\right) - J_0(\mu)\right)$$

- Take $\mu = \varepsilon^p$ with $-4 < p < 0$, then $\mu \to \infty$ as $\varepsilon \to 0$.
- Then

$$J_0(\mu) \sim \sqrt{\frac{2}{\pi\mu}} \cos\left(\mu - \frac{\pi}{4}\right) \quad \text{as } \mu \to +\infty$$

- Take ε such that $\cos\left(\mu - \frac{\pi}{4}\right)$ is of order one with respect to ε.
- The perturbation is still bigger than the unperturbed system.
- But, for $\varepsilon > 0$ small enough (holding the previous hypotheses) the perturbed system still has a hyperbolic periodic orbit and so there is still the question of the splitting of its invariant manifolds

Splitting of separatrices for a bigger perturbation (II)

- For these system, we can still bound the difference between the invariant manifolds.
- The maximal distance d between them is bounded by

$$d \leq C\varepsilon^{2-4\gamma} e^{-\frac{1}{\varepsilon\sqrt{J_0(\mu)}}\left(\frac{\pi}{2} - a\varepsilon^{\gamma}\right)}$$

where a and γ are any numbers holding

$$a > 0 \quad \text{and} \quad 0 < \gamma < 1 + \frac{p}{4}$$

Splitting of separatrices for a bigger perturbation (II)

Remarks

- This bound is not optimal.
- To obtain an asymptotic expression for the splitting it would be needed
  1. To perform more steps averaging and study how the singularities of the separatrix move when we add the new averaged terms.
  2. To study the perturbed invariant manifolds close to the singularity of the new averaged (integrable) system, as it has been done before.

Splitting of separatrices for μ close to zeros of the Bessel function (I)

- Recall our model

$$
\begin{aligned}
H(x,y,t) = \quad & \frac{y^2}{2} + J_0(\mu)(\cos x - 1) + \sin x \cos\left(\mu \sin \frac{t}{\varepsilon}\right) \\
& +(\cos x - 1)\left(\cos\left(\mu \sin \frac{t}{\varepsilon}\right) - J_0(\mu)\right)
\end{aligned}
$$

- Take $\mu = \mu_0 - \varepsilon^r$ with $0 < r < 2$, then

$$
J_0(\mu) \sim \mu - \mu_0 \sim \varepsilon^r
$$

- The averaged system is smaller than the perturbation.

- For $\varepsilon > 0$ small enough the perturbed system still has a hyperbolic periodic orbit and so there is still the question of the splitting of its invariant manifolds

Splitting of separatrices for μ close to zeros of the Bessel function (II)

- For these system, we can still bound the difference between the invariant manifolds.
- The maximal distance d between them is bounded by

$$d \leq C\varepsilon^{2-4\gamma} e^{-\frac{1}{\varepsilon\sqrt{J_0(\mu)}}\left(\frac{\pi}{2} - a\varepsilon^{\gamma}\right)}$$

where a and γ are any numbers holding

$$a > 0 \quad \text{and} \quad 0 < \gamma < 1 - r/2$$

Questions:

- How can we obtain an asymptotic expression of the distance between manifolds?
- What happens for $\mu = \mu_0 - \varepsilon^r$ with $r \geq 2$?

Case $\mu = \mu - \varepsilon^r$ with $r \in (0, 2)$

- To obtain an asymptotic expression for the splitting it would be needed

    1. To perform more steps averaging and study how the singularities of the separatrix move when we add the new averaged terms.
    2. To study the perturbed invariant manifolds close to the singularity of the new averaged (integrable) system, as it has been done before.

- The distance between manifolds is expected to be order

$$d \sim \varepsilon^{\beta} e^{-\alpha/\varepsilon}$$

where $\beta \in \mathbb{R}$ and $\alpha >$ is the imaginary part of the singularity of the new averaged system.

Case $\mu = \mu - \varepsilon^r$ with $r > 2$

- To understand what happens for $\mu = \mu_0 - \varepsilon^r$ with $r \geq 2$, we first study $\mu = \mu_0$.
- For $\mu = \mu_0$, the system has zero average and period $2\pi\varepsilon$.

Then,

- We perform one step more of averaging.
- We study the new averaged system.

After one step of averaging and rescaling $y = \varepsilon \bar{y}$ and $t = s/\varepsilon$ to have the averaged system of order one :

$$\begin{cases} \dfrac{d\bar{x}}{ds} = & \bar{y} + h_1\left(\dfrac{s}{\varepsilon^2}\right)\cos\bar{x} + h_2\left(\dfrac{s}{\varepsilon^2}\right)\sin\bar{x} \\ \dfrac{d\bar{y}}{ds} = & \langle m\rangle\left(\bar{x}\right) - \dfrac{1}{\varepsilon}\bar{y}\left(h_1\left(\dfrac{s}{\varepsilon^2}\right)\sin\bar{x} - h_2\left(\dfrac{s}{\varepsilon^2}\right)\cos\bar{x}\right) + \\ & + \left(m\left(\bar{x}, \dfrac{s}{\varepsilon^2}\right) - \langle m\rangle(\bar{x})\right). \end{cases}$$

where

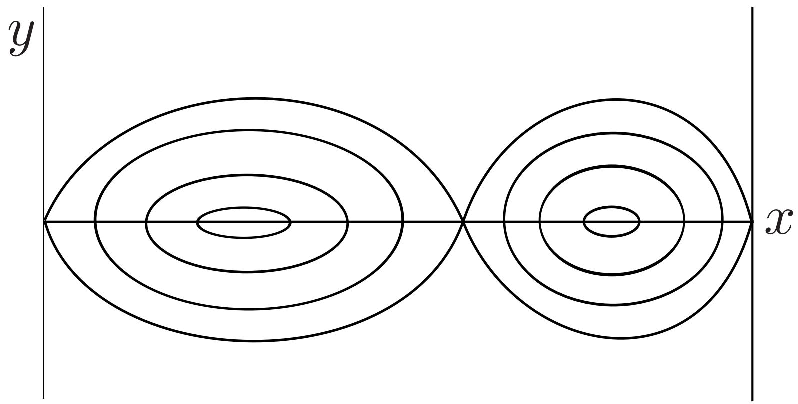
- h_i are 2π-periodic functions.
- m is 2π periodic function in the second variable with average

$$\langle m\rangle(x) = a_1 \sin(2x) + a_2 \cos(2x)$$

Now the system is a $2\pi\varepsilon^2$-periodic perturbation of the integrable averaged system

$$\left\{ \begin{array}{l} \dfrac{d\bar{x}}{ds} = \bar{y} \\ \dfrac{d\bar{y}}{ds} = \langle m \rangle (\bar{x}), \end{array} \right.$$

Averaged system: Has a double well potential with two hyperbolic critical points in the same level of energy.

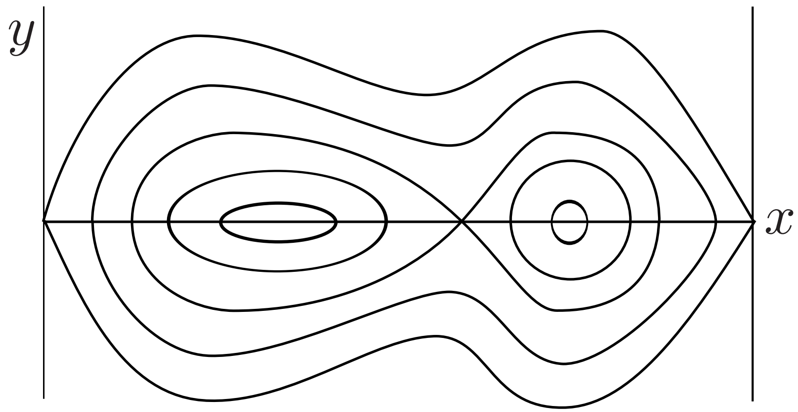


Taking $\mu = \mu_0 - \varepsilon^r$ with $r > 2$, the averaged system is

$$\begin{cases} \dfrac{d\bar{x}}{ds} = \bar{y} \\ \dfrac{d\bar{y}}{ds} = \langle m \rangle (\bar{x}) + \mathcal{O}(\varepsilon^{r-2}) \sin x. \end{cases}$$

Then,

- The two hyperbolic critical points belong to different levels of energy.
- The heteroclinic connections bifurcate into four homoclinic orbits.



To study the splitting of any of these homoclinic orbits:

- Study the singularities of the heteroclinic orbits for $\mu = \mu_0$.
- See how they change when we consider the full averaged system.
- Study the full system close to these singularities to obtain an asymptotic expression for the distance between manifolds.

Then, it is expected that this distance is of order

$$d \sim \varepsilon^{\beta} e^{-d/\varepsilon^2}$$

where $\beta \in \mathbb{R}$ and d is the imaginary part of the singularity of the averaged system.

Bifurcations in the averaged system

In a curve given at first order by $\mu = \mu_0 - c\varepsilon^2$ for certain $c > 0$ occurs a saddle-center bifurcation:

- One hyperbolic and one elliptic fixed points merge.
- One of the homoclinic orbits of the figure eight disappears.
- The other one becomes a periodic orbit.

Splitting of separatrices for a meromorphic perturbation

- Consider the model

$$\ddot{x} = \sin x + \varepsilon^p \frac{\sin x}{(1-\alpha \sin x)^2} \sin \frac{t}{\varepsilon}$$

  where $\alpha \in (0,1)$.

- It has Hamiltonian

$$H(x,y,t) = \frac{y^2}{2} + \cos x - 1 + \varepsilon^p m(x) \sin \frac{t}{\varepsilon}$$

  where m is the primitive of $\dfrac{\sin x}{(1-\alpha \sin x)^2}$.

- $(0,0)$ is a hyperbolic periodic orbit even for the perturbed system.

Computation of the Melnikov function

- Melnikov function:

$$
\begin{aligned}
M(t_0) &= \varepsilon^p \int_{-\infty}^{+\infty} y(u) \frac{\sin x(u)}{(1-\alpha \sin x(u))^2} \sin\left(\frac{u+t_0}{\varepsilon}\right) du \\
&= 4\varepsilon^p \int_{-\infty}^{+\infty} \frac{\sinh u \cosh u}{\left(\cosh^2 u - 2\alpha \sinh u\right)^2} \sin\left(\frac{u+t_0}{\varepsilon}\right) du
\end{aligned}
$$

- The first order of this integral can be computed using residues theorem.

Computation of the Melnikov function (II)

- If $\alpha = \mathcal{O}(\varepsilon^q)$ with $q > 2$: Since the integral is uniformly convergent in the reals, we can expand $M(t_0)$ in power series of α and split the integral

$$M(t_0) = 4\varepsilon^p \sum_{k=0}^{\infty} (k+1) 2^k \alpha^k \int_{-\infty}^{+\infty} \frac{\sinh^{k+1} u}{\cosh^{2k+3} u} \sin\left(\frac{u+t_0}{\varepsilon}\right) du$$

- Its first term gives the bigger contribution to the splitting

$$M(t_0) \sim 4\pi\varepsilon^{p-2} e^{-\frac{\pi}{2\varepsilon}}$$

- In that case, the exponential coefficient is given by the complex singularity of the separatrix.

- Conclusion: If the analyticity strip of the perturbation is big enough, the size of the splitting is given as in the entire perturbation case.

Computation of the Melnikov function (III)

- If $\alpha = \mathcal{O}(\varepsilon^q)$ with $q \in [0,2]$, the integral of the summands is bigger as k increases.
- We look for the singularities of the integrand.
- Consider $u^* = \sigma \pm i\rho$ singularities of the integrand closest to the reals.
- If α is small $\rho = \pm\left(\frac{\pi}{2} - \sqrt{\alpha} + \mathcal{O}(\alpha)\right)$
- Then, Melnikov is given by

$$M(t_0) \sim e^{-\frac{\rho}{\varepsilon}} \left(\frac{\varepsilon^{p-1}}{\sqrt{\alpha}} + \text{smaller terms} \right)$$

Validity of the Melnikov prediction

- If $\alpha = \mathcal{O}(\varepsilon^q)$ with $q > 2$ and ε is small enough: Melnikov function predicts correctly the splitting provided $p > 0$.
- The limit case $p = 0$ (integrable system and perturbation of the same order) can be studied as in the entire case.
- If $\alpha = \mathcal{O}(\varepsilon^q)$ with $q \in [0, 2]$, ε is small enough and $\alpha < 1$: Melnikov function predicts correctly the splitting provided $p + \dfrac{q}{2} - 1 > 0$.
- The limit case $p + \dfrac{q}{2} - 1 = 0$ seems that has to be studied considering an inner equation close to the singularities of the perturbation.