Analytic properties of one and a half degrees of freedom
Hamiltonian Systems and exponentially small splitting of
separatrices

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**Integrable Hamiltonian Systems with a fast periodic perturbation**

Non-autonomous fast periodic perturbation of a one degree of freedom hamiltonian

\[ H \left( x, y, \frac{t}{\varepsilon} \right) = H_0(x, y) + \mu H_1 \left( x, y, \frac{t}{\varepsilon} \right) \]

such that

- The Hamiltonian is analytic.
- \( H_0(x, y) = \frac{y^2}{2} + V(x) \).
- \( \varepsilon > 0 \) is a small parameter.
- \( \mu \) is a parameter not necessarily small.
- The perturbation is \( 2\pi\varepsilon \) periodic in time and has zero average

\[ \int_0^{2\pi} H_1(x, y, \tau)d\tau = 0 \]
Features of the integrable Hamiltonian Systems

Assume that the integrable system

\[ H_0(x, y) = \frac{y^2}{2} + V(x) \]

- Has a hyperbolic critical point at \((0, 0)\).
- Its stable and unstable invariant manifolds coincide along a separatrix.
Examples
What happens when we add the perturbation?

- The parameter $\mu$ is not small and then the perturbation has the same size as the unperturbed system.
- *A priori* we cannot say anything about the perturbed system.
- Nevertheless, since the perturbation is fast periodic and has zero average, we can do one step of averaging.
- This change of variables is $\varepsilon$-close to the identity and transforms

$$
H \left( x, y, \frac{t}{\varepsilon} \right) = \frac{y^2}{2} + V(x) + \mu H_1 \left( x, y, \frac{t}{\varepsilon} \right)
$$

into

$$
\tilde{H} \left( \tilde{x}, \tilde{y}, \frac{t}{\varepsilon} \right) = \frac{\tilde{y}^2}{2} + V(\tilde{x}) + \mu \varepsilon \tilde{H}_1 \left( \tilde{x}, \tilde{y}, \frac{t}{\varepsilon}, \varepsilon \right)
$$
Now, the perturbation has size $\mu \varepsilon$ and therefore is small.

Classical perturbation theory ensures that:

- There exist a hyperbolic $2\pi \varepsilon$-periodic orbit $\mu \varepsilon$-close to $(0,0)$.
- Its stable and unstable invariant manifolds are $\mu \varepsilon$-close to the unperturbed separatrix.

Question: Do the invariant manifolds of the perturbed periodic orbit still coincide or not?
3-dimensional phase space

Recall

\[ H \left( x, y, \frac{t}{\varepsilon} \right) = H_0(x, y) + \mu H_1 \left( x, y, \frac{t}{\varepsilon} \right) \]
The $2\pi\varepsilon$-time Poincaré map formulation

From the perturbed system it can be derived a discrete dynamical system considering the $2\pi\varepsilon$-time Poincaré map.
The splitting of separatrices in the Poincaré map

- Considering the Poincaré map, we obtain this picture.
- We can measure several quantities to study the splitting.
• Since we want to prove the existence of transversal homoclinic points, a natural quantity to measure would be the angle between the invariant manifolds at the homoclinic point.

• Nevertheless, the angle depend on the homoclinic point and it is not a symplectic invariant.

• The distance between the invariant manifolds also depends on the chosen coordinates and is neither a symplectic invariant.

• Between transversal homoclinic points, the invariant manifolds create lobes.

• The area of these lobes is invariant by iteration of the Poincaré map due to the symplectic structure.

• We will measure the splitting in terms of the area of these lobes.
Perturbative approach in $\varepsilon$

To understand what is happening, we can look for parameterizations of the invariant manifolds

$$x^u(r, \varepsilon), \quad x^s(r, \varepsilon)$$

$\rightarrow$ Since $\varepsilon$ is small, we can look for formal solutions as a power series of $\varepsilon$:

$$x^{\alpha}(r, \varepsilon) = x_0(r) + \varepsilon x_1^{\alpha}(r) + \varepsilon^2 x_2^{\alpha}(r) + \ldots \text{ for } \alpha = u, s$$

where we have omitted the dependence on $\mu$.

For these problems which are analytic and have a fast periodic perturbation:

$$x_k^u(r) = x_k^s(r) \quad \forall k \in \mathbb{N}$$

Conclusion:

$$x^u(r, \varepsilon) - x^s(r, \varepsilon) = \mathcal{O}(\varepsilon^k) \quad \forall k \in \mathbb{N}$$

$\rightarrow$ Proceeding formally we see that their difference is beyond all orders.
What is happening?

Two options:

1. Both manifolds coincide also in the perturbed case (the perturbed system is also integrable) → the power series in $\varepsilon$ is convergent:

2. Both manifolds do not coincide → the power series in $\varepsilon$ is divergent and the difference between manifolds has to be flat with respect $\varepsilon$.

Generically is happening the second option
→ In fact, we will see that generically the difference is exponentially small with respect $\varepsilon$. 
Perturbative approach in $\mu$: Classical Poincaré-Melnikov Theory

- For a moment, let’s forget that $\varepsilon$ is a small parameter and let us consider $\mu$ as an arbitrarily small parameter.
- In that case, we can consider a perturbative approach in $\mu$.
- This approach was initially considered by Poincaré and is usually called Poincaré-Melnikov Method.
Poincaré-Melnikov Theory (I)

1. We fix $\Sigma$, a transversal section to the unperturbed separatrix in order to measure in it the splitting.

2. We consider a parameterization $\gamma$ of the unperturbed separatrix such that $\gamma(0)$ belongs to this section.
Poincaré-Melnikov Theory (II)

3. We define the Melnikov function as:

\[ M(s, \varepsilon) = \int_{-\infty}^{+\infty} \{H_0, H_1\} \left( \gamma(t - s), \frac{t}{\varepsilon} \right) \, dt \]

where

- \( s \) corresponds to the time evolution through the separatrix.
- \( \{H_0, H_1\} \) is the Poisson bracket:

\[ \{H_0, H_1\} = \frac{\partial H_0}{\partial x} \frac{\partial H_1}{\partial y} - \frac{\partial H_1}{\partial x} \frac{\partial H_0}{\partial y} \]

\( M \) can be computed since \( H_0, H_1 \) and \( \gamma \) are known.
Poincaré-Melnikov Theory (III)

Then:

- The distance between both invariant manifolds for \( \mu > 0 \) is given by:

\[
d(s, \mu, \varepsilon) = \mu \frac{M(s, \varepsilon)}{\|DH_0(\gamma(-s))\|} + \mathcal{O}(\mu^2)
\]

- If there exists \( s_0 \) such that

\[
(i) \ M(\mathbf{s}, \varepsilon) = 0 \quad (ii) \left. \frac{\partial M}{\partial s} \right|_{s=s_0} \neq 0
\]

Then the invariant manifolds intersect transversally in a point which is close to \( \gamma(s_0) \).

- If \( s_0 \) and \( s_1 \) are two consecutive simple zeros of the Melnikov function, the area of the corresponding lobe is given by:

\[
\mathcal{A} = \mu \int_{s_0}^{s_1} M(s, \varepsilon)ds + \mathcal{O}(\mu^2)
\]
Conclusion: Poincaré-Melnikov theory allows to

- Prove the existence of transversal homoclinic orbits
- Compute the distance between manifolds, and therefore to compute asymptotically for $\mu \to 0$ the region of the phase space where chaos is confined.
- Nevertheless, these results are for $\mu$ arbitrarily small and $\varepsilon$ fixed.
To see whether Poincaré-Melnikov Theory is valid also for small $\varepsilon$, we have to study the dependence on $\varepsilon$ of the Melnikov function.

The dependence on $\varepsilon$ of the Melnikov functions is extremely sensitive on the analyticity properties of the Hamiltonian.

So, we need to impose strong conditions on the Hamiltonian to compute this dependence.
Hypotheses on the Hamiltonian

Assume

- $V(x)$ is either a polynomial or a trigonometric polynomial.
- $H_1(x, y, t/\varepsilon)$ is
  - A polynomial in $y$.
  - Either a polynomial or a trigonometric polynomial on $x$.
  - Any dependence on $t$. 
Hypotheses on the separatrix

- Parameterization of the separatrix \( \gamma(u) = (x_0(u), y_0(u)) \).

- \((x_0(u), y_0(u))\) must have singularities in the complex plane. Why?
  - \((x_0(u), y_0(u))\) are analytic functions bounded at infinity.
  - If they do not have singularities, they are entire, and then by Liouville Theorem they must be constant.
  - Therefore, they must have some singularities.
Let us assume that

- $y_0(u)$ is analytic in a complex strip \{\(\text{Im } u < a\)\}.
- $y_0(u)$ has only one singularity in the lines \{\(\text{Im } u = \pm a\)\}. We choose $\gamma$ such that they are at $u = \pm ia$.

Then, $y_0(u)$ satisfies

$$y_0(u) \sim \frac{1}{(u \mp ia)^r}$$

for certain $r \geq 1$. 
Example satisfying these hypotheses

The perturbed Duffing equation:

\[ H(x, y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} \]

The unperturbed separatrix is

\[ \gamma(u) = \left( \frac{\sqrt{2}}{\cosh t}, -\sqrt{2} \frac{\sinh t}{\cosh^2 t} \right) \]

has singularities at \( u = i\frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}. \) Moreover

\[ -\sqrt{2} \frac{\sinh t}{\cosh^2 t} \sim \frac{1}{(u \pm i\pi/2)^2} \]

That is, \( r = 2. \)
The order of the perturbation

There exists $\ell > 0$ such that

$$H_1 \left( x_0(u), y_0(u), \frac{t}{\varepsilon} \right) \sim \frac{1}{(u \mp ia)^\ell}$$
**Poincaré-Melnikov prediction**

- With these (strong) hypotheses we can compute the Melnikov function using Residuums Theory or other complex analytic techniques.

- The area of the lobes is given by,

\[
A = \mu \frac{|f_0|}{\varepsilon^\ell - 1} e^{-\frac{a}{\varepsilon}} + O\left(\mu^2\right)
\]

where

- \(a\) is the imaginary part of the singularity of the unperturbed separatrix.

- \(\ell\) is the order of the perturbation.

- \(f_0 \in \mathbb{C}\) is a constant independent of \(\mu\) and \(\varepsilon\) given by the Melnikov integral, which generically satisfies \(f_0 \neq 0\).

- The first order in \(\mu\) is exponentially small with respect to \(\varepsilon\).
• If we take $\mu = \varepsilon^p$ for $p \geq 0$ (which is the natural relation):

$$A(\varepsilon) = |f_0|e^{-\frac{a}{\varepsilon}}\varepsilon^p - \ell + 1 + O(\varepsilon^{2p}) .$$

Therefore, if $\mu = \varepsilon^p$, the remainder is bigger than the Melnikov function prediction.

• $\mu$ has to be exponentially small with respect to $\varepsilon$ to apply Poincar’e-Melinkov Theory.
**Previous results**

Consider

$$H \left( x, y, \frac{t}{\varepsilon} \right) = H_0(x, y) + \mu H_1 \left( x, y, \frac{t}{\varepsilon} \right)$$

Assuming the same hypotheses as us and taking $\mu = \varepsilon^p$:

- V. Gelfreich (1997) proved that Melnikov function predicted correctly the splitting provided $p$ was big enough.

- A. Delshams and T. Seara (1997) improved the result to $p > \ell$. 
• Results in the case $\mu$ constant independent of $\varepsilon$ have only been obtained for particular examples.


• In particular, in all these cases the perturbation did not depend on $y$.

• In all these cases it is seen that the splitting is exponentially small but the first asymptotic order does not coincide with Melnikov.
Our results

- We generalize the previous results, giving an asymptotic formula for the area of the lobes for wider ranges in $\mu$.
- In some cases we are able to give results for fixed $\mu$ independent of $\varepsilon$.
- In this case the unperturbed system and perturbation have the same size.
- In some other cases we have to restrict ourselves taking $\mu$ polynomially small with respect to $\varepsilon$.
- We see that in general, Melnikov does not predict the splitting correctly.
- In this talk, we show them focusing on an example: The perturbed Duffing equation
The perturbed Duffing equation

\[ H \left( x, y, \frac{t}{\varepsilon} \right) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} + \mu x^n \sin \frac{t}{\varepsilon} \]

Order of the perturbation:

\[ (x_0(u))^n \sim \frac{1}{(u \pm i\pi/2)^n} \]

That is \( \ell = n \).

Melnikov prediction:

\[ \mathcal{A} = \mu \frac{|f_0|}{\varepsilon^{n-1}} e^{-\frac{\pi}{2\varepsilon}} + \mathcal{O}(\mu^2) \]

The constant \( f_0 \) can be explicitly computed and satisfies \( f_0 \neq 0 \).
True asymptotic formula

We obtain different results depending on $n$.

First case: $n < 4$ (namely $\ell - 2r < 0$)

For any $\mu$ independent of $\varepsilon$ and $\varepsilon$ small enough:

$$A = \mu \frac{|f_0|}{\varepsilon^{n-1}} e^{-\frac{\pi}{2\varepsilon}} \left( 1 + O \left( \frac{1}{\ln |\varepsilon|^\nu} \right) \right)$$

for some $\nu > 0$.

Conclusion: In this case, Melnikov predicts correctly even if the perturbation has the same size as the unperturbed system.
Second case: \( n = 4 \) (namely \( \ell - 2r = 0 \))

For any \( \mu \) independent of \( \varepsilon \) and \( \varepsilon \) small enough, there exists a function \( f(\mu) \), such that, if \( f(\mu) \neq 0 \),

\[
\mathcal{A} = \mu \frac{|f(\mu)|}{\varepsilon^{n-1}} e^{-\frac{\pi}{2\varepsilon}} \left( 1 + \mathcal{O} \left( \frac{1}{|\ln \varepsilon|} \right) \right)
\]

The function \( f(\mu) \) is analytic and satisfies \( f(\mu) = f_0 + \mathcal{O}(\mu) \).

Conclusions:

- As in the previous case, even if perturbation and unperturbed system have the same size, the splitting is exponentially small.
- If \( \mu \) is small, Melnikov predicts correctly the splitting.
- If \( \mu \) is independent of \( \varepsilon \), Melnikov fails to predict the splitting.
Third case: \( n > 4 \) (namely \( \ell - 2r > 0 \))

- We are not able to give results for any \( \mu \) independent of \( \varepsilon \). We have to take \( \mu \sim \varepsilon^p \) with \( p \geq n - 4 \).

- Take \( \mu = \delta \varepsilon^{n-4} \). Namely,

\[
H \left( x, y, \frac{t}{\varepsilon} \right) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} + \mu x^n \sin \frac{t}{\varepsilon}
\]

- Then, for any \( \delta \) independent of \( \varepsilon \) and \( \varepsilon \) small enough, there exists a function \( f(\delta) \), such that, if \( f(\delta) \neq 0 \),

\[
A = \delta \frac{|f(\delta)|}{\varepsilon^3} e^{-\frac{\pi}{2\varepsilon}} \left( 1 + \mathcal{O} \left( \frac{1}{|\ln \varepsilon|^{n-4}} \right) \right)
\]

The function \( f(\delta) \) is analytic and satisfies \( f(\delta) = f_0 + \mathcal{O}(\delta) \).
Conclusions:

• Since $f(\delta) = f_0 + O(\delta)$, for $\delta$ small Melnikov predicts correctly.

• In the original parameter, if $\mu \sim \epsilon^p$ with $p > n - 4$ Melnikov predicts correctly.

• If $\delta$ is independent of $\epsilon$, that is $\mu \sim \epsilon^{n-4}$, Melnikov fails to predict the splitting.

• The true asymptotic formula for $\mu$ independent of $\epsilon$ (unperturbed system and perturbation of the same size) has probably additional correcting terms in the exponential.

• There is not any result dealing with this case.
Perturbations depending on $y$

• If the perturbation depends on $y$, the first order also changes.

• Example:

$$H\left(x, y, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} + \mu \left(\frac{y^2 \cos \frac{t}{\varepsilon}}{\varepsilon} + x^4 \sin \frac{t}{\varepsilon}\right)$$

• Then, for $\mu$ independent of $\varepsilon$ and $\varepsilon$ small enough,

$$A = \mu \frac{|f(\mu)|}{\varepsilon^3} e^{-\frac{\pi}{2\varepsilon} + \mu^2 b \ln \frac{1}{\varepsilon}} \left(1 + \mathcal{O}\left(\frac{1}{|\ln \varepsilon|^{n-4}}\right)\right)$$

$$= \mu \frac{|f(\mu)|}{\varepsilon^{3+\mu^2 b}} e^{-\frac{\pi}{2\varepsilon}} \left(1 + \mathcal{O}\left(\frac{1}{|\ln \varepsilon|^{n-4}}\right)\right)$$
Conclusions

- For $\mu$ independent of $\varepsilon$, Melnikov does not predict correctly even the power of $\varepsilon$ in front of the exponentially

- Namely, in these cases Melnikov only predicts correctly the coefficient in the exponential.

- Nevertheless, if $\mu \ll \frac{1}{\sqrt{|\ln \varepsilon|}}$, Melnikov predicts correctly.
Splitting of separatrices for a meromorphic perturbation

- The results given apply when $V$ and $H_1$ are polynomials or trigonometric polynomials in $x$ and polynomials in $y$.

- Nevertheless, in many models, for instance in Celestial Mechanics, the Hamiltonian functions are not entire but have a finite strip of analyticity.

- Sometimes, the strip of analyticity $\sigma$ is large of the form $\sigma \sim \ln \alpha$ with $\alpha \ll 1$. 
Questions

• Which is the size of the Melnikov function for this kind of Hamiltonian Systems?

• When does the Melnikov function predict correctly the size of the splitting of separatrices?

• When the strip is of the form $\sigma \sim \ln \alpha$ with $\alpha \ll 1$, can we expand the perturbation in $\alpha$ and just consider the first order? or all the orders make a contribution to the first order of the difference between manifolds?
A simple model

• Example:

\[ \ddot{x} = \sin x + \mu \frac{\sin x}{(1 - \alpha \sin x)^2} \sin \frac{t}{\varepsilon} \]

where \( \alpha \in (0, 1) \).

• It has Hamiltonian

\[ H(x, y, t) = \frac{y^2}{2} + \cos x - 1 + \mu m(x) \sin \frac{t}{\varepsilon} \]

where \( m \) is the primitive of \( \frac{\sin x}{(1 - \alpha \sin x)^2} \).

• \((0, 0)\) is a hyperbolic periodic orbit even for the perturbed system.
Computation of the Melnikov function

• Melnikov function:

\[ M(t_0) = \int_{-\infty}^{+\infty} y(u) \frac{\sin x(u)}{(1 - \alpha \sin x(u))^2} \sin \left( \frac{u + t_0}{\varepsilon} \right) du \]

\[ = 4 \int_{-\infty}^{+\infty} \frac{\sinh u \cosh u}{(\cosh^2 u - 2\alpha \sinh u)^2} \sin \left( \frac{u + t_0}{\varepsilon} \right) du \]

• The first order of this integral can be computed using residues theorem.
Computation of the Melnikov function (II)

• If $\alpha = O(\varepsilon^q)$ with $q > 2$: Since the integral is uniformly convergent in the reals, we can expand $M(t_0)$ in power series of $\alpha$ and split the integral

$$M(t_0) = 4 \sum_{k=0}^{\infty} (k + 1)2^k \alpha^k \int_{-\infty}^{+\infty} \frac{\sinh^{k+1} u}{\cosh^{2k+3} u} \sin \left( \frac{u + t_0}{\varepsilon} \right) du$$

• Its first term gives the bigger contribution to the splitting

$$M(t_0) \sim 4\pi \varepsilon^{-2} e^{-\frac{\pi}{2\varepsilon}}$$

• In that case, the exponential coefficient is given by the complex singularity of the separatrix.

• Conclusion: If the analyticity strip of the perturbation is big enough ($\alpha \ll \varepsilon^2$), the size of the Melnikov function is given as in the entire perturbation case.
Computation of the Melnikov function (III)

- If $\alpha = \mathcal{O}(\varepsilon^q)$ with $q \in [0, 2]$, the integral of the summands is bigger as $k$ increases.
- We look for the singularities of the integrand.
- Consider $u^* = \eta \pm i\rho$ singularities of the integrand closest to the reals.
- If $\alpha$ is small $\rho = \pm \left( \frac{\pi}{2} - \sqrt{\alpha} + \mathcal{O}(\alpha) \right)$.
- If $\alpha$ is fixed and independent of $\varepsilon$, $\rho$ is also independent of $\varepsilon$ and is unrelated to the singularities of the separatrix.
Computation of the Melnikov function (IV)

- Then, Melnikov is given by

\[ M(t_0) \sim e^{-\frac{\rho}{\varepsilon}} \left( \frac{\varepsilon^{p-1}}{\sqrt{\alpha}} + \text{smaller terms} \right) \]

- If, for instance, one takes \( \alpha = \varepsilon \),

\[ M(t_0) \sim e^{-\frac{\pi - 2\sqrt{\varepsilon}}{2\varepsilon}} \left( \varepsilon^{p - \frac{3}{2}} + \text{smaller terms} \right) \]

- In these cases, even if \( \alpha \) is small, the first order of the Melnikov function is given by the full jet in \( \alpha \) of the perturbation.
Validity of the Melnikov prediction

- If $\alpha = \mathcal{O}(\varepsilon^q)$ with $q > 2$ and $\varepsilon$ is small enough: the Melnikov function predicts correctly the splitting provided $\mu = \mathcal{O}(\varepsilon^p)$ with $p > 0$.

- The limit case $\mu$ fixed and independent of $\varepsilon$ (integrable system and perturbation of the same order) is expected to have exponentially small splitting of separatrices which is not well predicted by the Melnikov function.

- If $\alpha = \mathcal{O}(\varepsilon^q)$ with $q \in [0, 2]$, $\varepsilon$ is small enough and $\alpha < 1$: Melnikov function predicts correctly the splitting provided $p + \frac{q}{2} - 1 > 0$.

- The limit case $p + \frac{q}{2} - 1 = 0$ is expected to have exponentially small splitting of separatrices which is not well predicted by the Melnikov function.
Non-exponentially small splitting of separatrices

Recall

\[ \ddot{x} = \sin x + \mu \frac{\sin x}{(1 - \alpha \sin x)^2} \sin \frac{t}{\varepsilon} \]

If we take \( \alpha = 1 - \mathcal{O}(\varepsilon^r) \):

- The strip of analyticity in \( x \) of the Hamiltonian is \( \mathcal{O}(\varepsilon^{\frac{r}{2}}) \).
- If \( r > 2 \), the Melnikov function satisfies
  \[ M(t_0) \sim \varepsilon^{-1} \]
- If \( \mu = \varepsilon^p \) with \( p > 1 \), Melnikov predicts correctly the difference between the invariant manifolds, which is not exponentially small.