# Analytic properties of one and a half degrees of freedom Hamiltonian Systems and exponentially small splitting of separatrices

**Inmaculada Baldomá, Ernest Fontich, Marcel Guardia, Tere M-Seara**

## Integrable Hamiltonian Systems with a fast periodic perturbation

Non-autonomous fast periodic perturbation of a one degree of freedom hamiltonian

$$H\left(x, y, \frac{t}{\varepsilon}\right) = H_0(x, y) + \mu H_1\left(x, y, \frac{t}{\varepsilon}\right)$$

such that

- The Hamiltonian is analytic.
- $H_0(x, y) = \dfrac{y^2}{2} + V(x)$.
- $\varepsilon > 0$ is a small parameter.
- $\mu$ is a parameter not necessarily small.
- The perturbation is $2\pi\varepsilon$ periodic in time and has zero average

$$\int_0^{2\pi} H_1(x, y, \tau) d\tau = 0$$

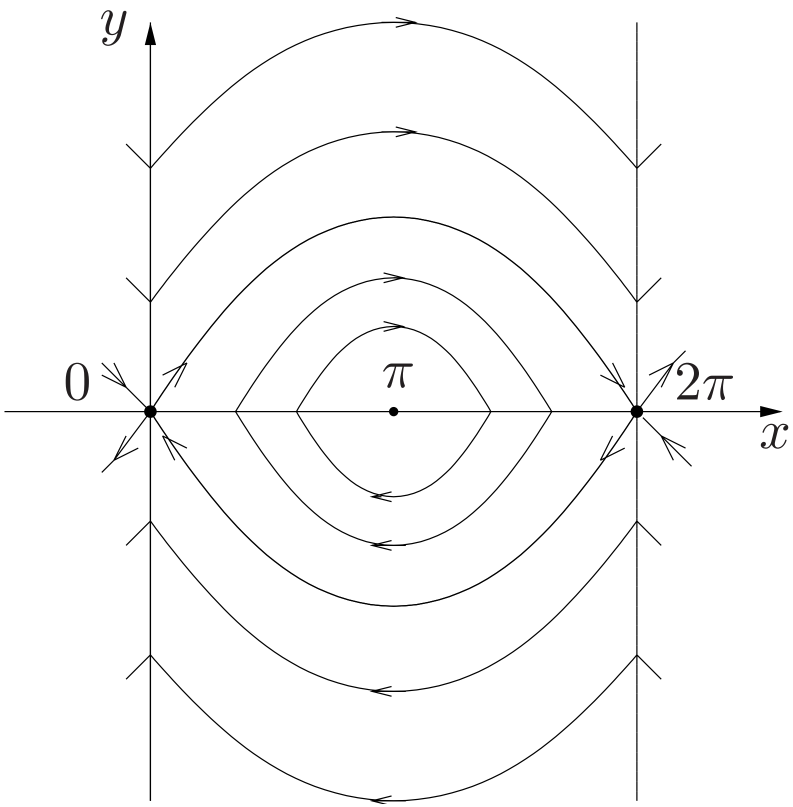
## Features of the integrable Hamiltonian Systems

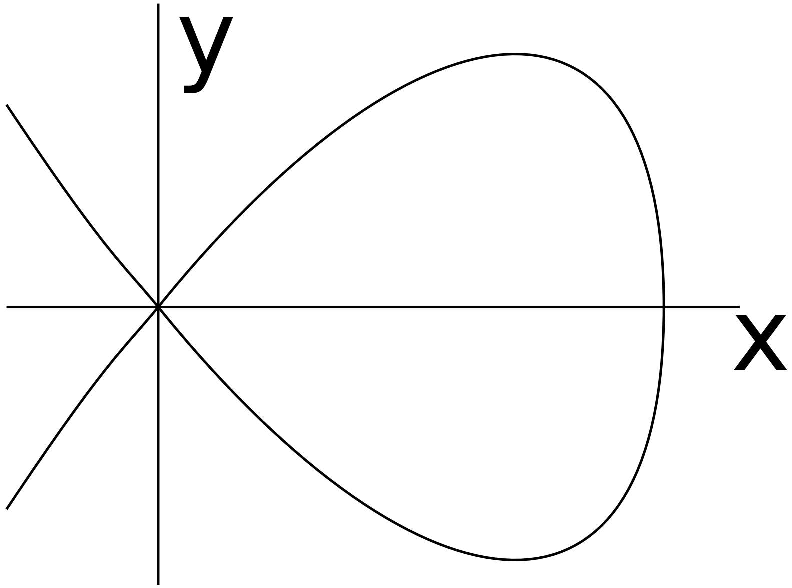
Assume that the integrable system

$$H_0(x, y) = \frac{y^2}{2} + V(x)$$

- Has a hyperbolic critical point at $(0, 0)$.
- Its stable and unstable invariant manifolds coincide along a separatrix.

## Examples



## What happens when we add the perturbation?

- The parameter $\mu$ is not small and then the perturbation has the same size as the unperturbed system.
- *A priori* we cannot say anything about the perturbed system.
- Nevertheless, since the perturbation is fast periodic and has zero average, we can do one step of averaging.
- This change of variables is $\varepsilon$-close to the identity and transforms

$$H\left(x, y, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} + V(x) + \mu H_1\left(x, y, \frac{t}{\varepsilon}\right)$$

into

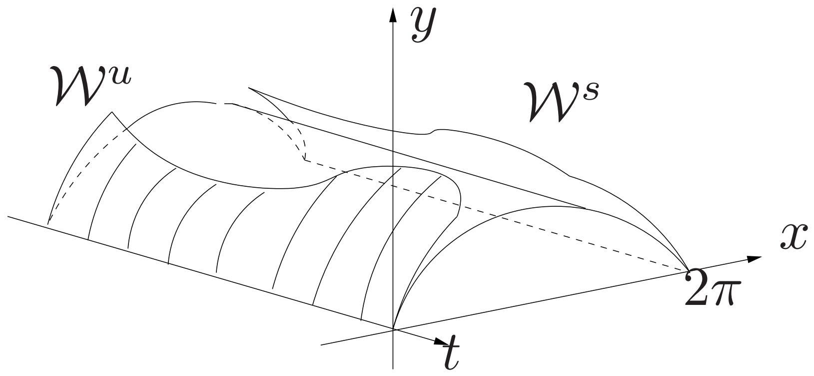
$$\widetilde{H}\left(\tilde{x}, \tilde{y}, \frac{t}{\varepsilon}\right) = \frac{\tilde{y}^2}{2} + V(\tilde{x}) + \mu\varepsilon \widetilde{H}_1\left(\tilde{x}, \tilde{y}, \frac{t}{\varepsilon}, \varepsilon\right)$$

- Now, the perturbation has size $\mu\varepsilon$ and therefore is small.
- Classical perturbation theory ensures that:
  - There exist a hyperbolic $2\pi\varepsilon$-periodic orbit $\mu\varepsilon$-close to $(0,0)$.
  - Its stable and unstable invariant manifolds are $\mu\varepsilon$-close to the unperturbed separatrix.
- Question: Do the invariant manifolds of the perturbed periodic orbit still coincide or not?

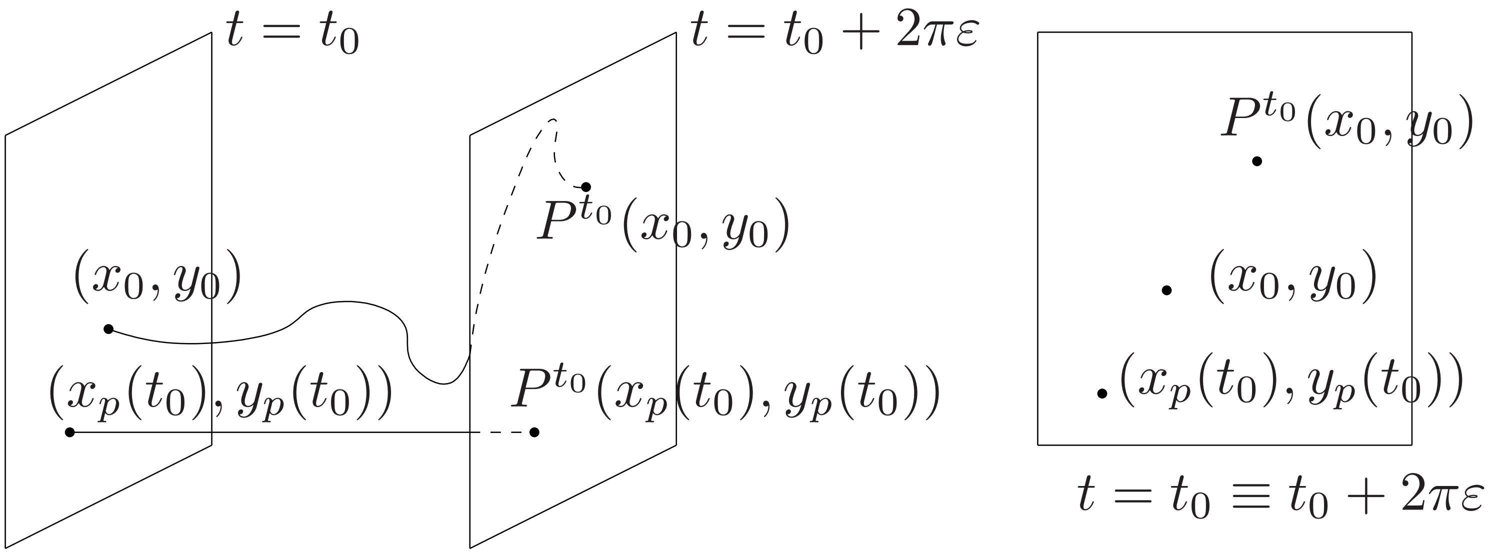
## 3-dimensional phase space

Recall

$$H\left(x, y, \frac{t}{\varepsilon}\right) = H_0(x, y) + \mu H_1\left(x, y, \frac{t}{\varepsilon}\right)$$

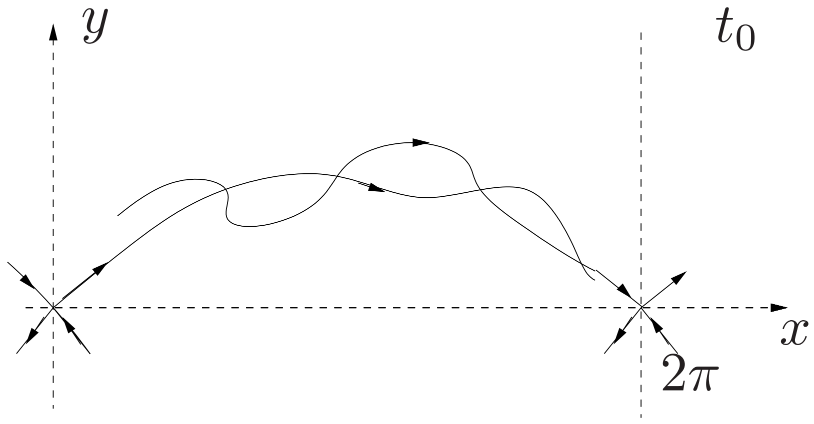


## The $2\pi\varepsilon$-time Poincaré map formulation




From the perturbed system it can be derived a discrete dynamical system considering the $2\pi\varepsilon$-time Poincaré map.

## The splitting of separatrices in the Poincaré map




- Considering the Poincaré map, we obtain this picture.
- We can measure several quantities to study the splitting.

- Since we want to prove the existence of transversal homoclinic points, a natural quantity to measure would be the angle between the invariant manifolds at the homoclinic point.
- Nevertheless, the angle depend on the homoclinic point and it is not a symplectic invariant.
- The distance between the invariant manifolds also depends on the chosen coordinates and is neither a symplectic invariant.
- Between transversal homoclinic points, the invariant manifolds create lobes.
- The area of these lobes is invariant by iteration of the Poincaré map due to the symplectic structure.
- We will measure the splitting in terms of the area of these lobes.

## Perturbative approach in $\varepsilon$

To understand what is happening, we can look for parameterizations of the invariant manifolds

$$x^u(r,\varepsilon), \ \ x^s(r,\varepsilon)$$

$\rightarrow$ Since $\varepsilon$ is small, we can look for formal solutions as a power series of $\varepsilon$:

$$x^\alpha(r,\varepsilon) = x_0(r) + \varepsilon x_1^\alpha(r) + \varepsilon^2 x_2^\alpha(r) + \dots \quad \text{for } \alpha = u, s$$

where we have omitted the dependence on $\mu$.

For these problems which are analytic and have a fast periodic perturbation:

$$x_k^u(r) = x_k^s(r) \ \ \forall k \in \mathbb{N}$$

Conclusion:

$$x^u(r,\varepsilon) - x^s(r,\varepsilon) = \mathcal{O}(\varepsilon^k) \ \ \forall k \in \mathbb{N}$$

$\rightarrow$ Proceeding formally we see that their difference is beyond all orders.

## What is happening?

Two options:

1 Both manifolds coincide also in the perturbed case (the perturbed system is also integrable) $\rightarrow$ the power series in $\varepsilon$ is convergent:

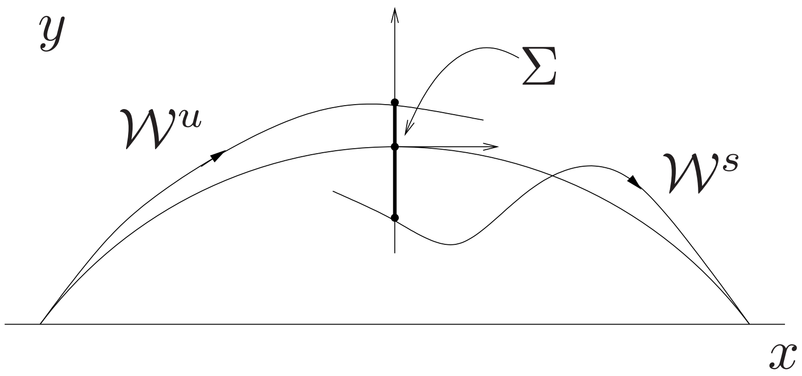
2 Both manifolds do not coincide $\rightarrow$ the power series in $\varepsilon$ is divergent and the difference between manifolds has to be flat with respect $\varepsilon$.

Generically is happening the second option
$\rightarrow$ In fact, we will see that generically the difference is exponentially small with respect $\varepsilon$.

## Perturbative approach in $\mu$: Classical Poincaré-Melnikov Theory

- For a moment, let's forget that $\varepsilon$ is a small parameter and let us consider $\mu$ as an arbitrarily small parameter.
- In that case, we can consider a perturbative approach in $\mu$.
- This approach was initially considered by Poincaré and is usually called Poincaré-Melnikov Method.

## Poincaré-Melnikov Theory (I)




1. We fix $\Sigma$, a transversal section to the unperturbed separatrix in order to measure in it the splitting.
2. We consider a parameterization $\gamma$ of the unperturbed separatrix such that $\gamma(0)$ belongs to this section.

## Poincaré-Melnikov Theory (II)

3. We define the Melnikov function as:

$$M(s,\varepsilon) = \int_{-\infty}^{+\infty} \{H_0, H_1\}\left(\gamma(t-s), \frac{t}{\varepsilon}\right) dt$$

where

– $s$ corresponds to the time evolution through the separatrix.

– $\{H_0, H_1\}$ is the Poisson bracket:

$$\{H_0, H_1\} = \frac{\partial H_0}{\partial x}\frac{\partial H_1}{\partial y} - \frac{\partial H_1}{\partial x}\frac{\partial H_0}{\partial y}$$

$\rightarrow$ $M$ can be computed since $H_0$, $H_1$ and $\gamma$ are known.

## Poincaré-Melnikov Theory (III)

Then:

- The distance between both invariant manifolds for $\mu > 0$ is given by:

$$d(s, \mu, \varepsilon) = \mu \frac{M(s, \varepsilon)}{\|DH_0(\gamma(-s))\|} + \mathcal{O}\left(\mu^2\right)$$

- If there exists $s_0$ such that

$$(i)\ M(s_0, \varepsilon) = 0 \quad (ii)\ \left.\frac{\partial M}{\partial s}\right|_{s=s_0} \neq 0$$

  Then the invariant manifolds intersect transversally in a point which is close to $\gamma(s_0)$.

- If $s_0$ y $s_1$ are two consecutive simple zeros of the Melnikov function, the area of the corresponding lobe is given by:

$$\mathcal{A} = \mu \int_{s_0}^{s_1} M(s, \varepsilon) ds + \mathcal{O}\left(\mu^2\right)$$

Conclusion: Poincaré-Melnikov theory allows to

- Prove the existence of transversal homoclinic orbits
- Compute the distance between manifolds, and therefore to compute asymptotically for $\mu \to 0$ the region of the phase space where chaos is confined.
- Nevertheless, these results are for $\mu$ arbitrarily small and $\varepsilon$ fixed.

- To see whether Poincaré-Melnikov Theory is valid also for small $\varepsilon$ we have to study the dependence on $\varepsilon$ of the Melnikov function.
- The dependence on $\varepsilon$ of the Melnikov functions is extremely sensitive on the analyticity properties of the Hamiltonian.
- So, we need to impose strong conditions on the Hamiltonian to compute this dependence.

## Hypotheses on the Hamiltonian

Assume

- $V(x)$ is either a polynomial or a trigonometric polynomial.
- $H_1(x, y, t/\varepsilon)$ is
  - A polynomial in $y$.
  - Either a polynomial or a trigonometric polynomial on $x$.
  - Any dependence on $t$.

## Hypotheses on the separatrix

- Parameterization of the separatrix $\gamma(u) = (x_0(u), y_0(u))$.
- $(x_0(u), y_0(u))$ must have singularities in the complex plane. Why?
  - $(x_0(u), y_0(u))$ are analytic functions bounded at infinity.
  - If they do not have singularities, they are entire, and then by Liouville Theorem they must be constant.
  - Therefore, they must have some singularities.

- Let us assume that
  - $y_0(u)$ is analytic in a complex strip $\{|\mathrm{Im}\, u| < a\}$.
  - $y_0(u)$ has only one singularity in the lines $\{\mathrm{Im}\, u = \pm a\}$. We choose $\gamma$ such that they are at $u = \pm ia$.
- Then, $y_0(u)$ satisfies

$$y_0(u) \sim \frac{1}{(u \mp ia)^r}$$

for certain $r \geq 1$.

# Example satisfying these hypotheses

The perturbed Duffing equation:

$$H\left(x,y\right)=\frac{y^2}{2}-\frac{x^2}{2}+\frac{x^4}{4}$$

The unperturbed separatrix is

$$\gamma(u)=\left(\frac{\sqrt{2}}{\cosh t},-\sqrt{2}\frac{\sinh t}{\cosh^2 t}\right)$$

has singularities at $u=i\frac{\pi}{2}+k\pi$, $k\in\mathbb{Z}$. Moreover

$$-\sqrt{2}\frac{\sinh t}{\cosh^2 t}\sim\frac{1}{(u\pm i\pi/2)^2}$$

That is, $r=2$.

## The order of the perturbation

There exists $\ell > 0$ such that

$$H_1\left(x_0(u), y_0(u), \frac{t}{\varepsilon}\right) \sim \frac{1}{(u \mp ia)^\ell}$$

## Poincaré-Melnikov prediction

- With these (strong) hypotheses we can compute the Melnikov function using Residuums Theory or other complex analytic techniques.
- The area of the lobes is given by,

$$\mathcal{A} = \mu \frac{|f_0|}{\varepsilon^{\ell-1}} e^{-\frac{a}{\varepsilon}} + \mathcal{O}\left(\mu^2\right)$$

  where

  - $a$ is the imaginary part of the singularity of the unperturbed separatrix.
  - $\ell$ is the order of the perturbation.
  - $f_0 \in \mathbb{C}$ is a constant independent of $\mu$ and $\varepsilon$ given by the Melnikov integral, which generically satisfies $f_0 \neq 0$.
- The first order in $\mu$ is exponentially small with respect to $\varepsilon$.

- If we take $\mu = \varepsilon^p$ for $p \geq 0$ (which is the natural relation):

$$\mathcal{A}(\varepsilon) = |f_0| e^{-\frac{a}{\varepsilon}} \varepsilon^{p-\ell+1} + \mathcal{O}\left(\varepsilon^{2p}\right).$$

  Therefore, if $\mu = \varepsilon^p$, the remainder is bigger than the Melnikov function prediction.

- $\mu$ has to be exponentially small with respect to $\varepsilon$ to apply Poincar'e-Melinkov Theory.

## Previous results

Consider

$$H\left(x, y, \frac{t}{\varepsilon}\right) = H_0(x, y) + \mu H_1\left(x, y, \frac{t}{\varepsilon}\right)$$

Assuming the same hypotheses as us and taking $\mu = \varepsilon^p$:

- V. Gelfreich (1997) proved that Melnikov function predicted correctly the splitting provided $p$ was big enough.
- A. Delshams and T. Seara (1997) improved the result to $p > \ell$.

- Results in the case $\mu$ constant independent of $\varepsilon$ have only been obtained for particular examples.
- Only for the pendulum with certain perturbations: D. Treschev (1997), V. Gelfreich (2001), C. Olivé (2006), M. G., C. Olivé, T. M. Seara (2010).
- In particular, in all these cases the perturbation did not depend on $y$.
- In all these cases it is seen that the splitting is exponentially small but the first asymptotic order does not coincide with Melnikov.

## Our results

- We generalize the previous results, giving an asymptotic formula for the area of the lobes for wider ranges in $\mu$.
- In some cases we are able to give results for fixed $\mu$ independent of $\varepsilon$.
- In this case the unperturbed system and perturbation have the same size.
- In some other cases we have to restrict ourselves taking $\mu$ polynomially small with respect to $\varepsilon$.
- We see that in general, Melnikov does not predict the splitting correctly.
- In this talk, we show them focusing on an example: The perturbed Duffing equation

## The perturbed Duffing equation

$$H\left(x, y, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} + \mu x^n \sin\frac{t}{\varepsilon}$$

Order of the perturbation:

$$(x_0(u))^n \sim \frac{1}{(u \pm i\pi/2)^n}$$

That is $\ell = n$.

Melnikov prediction:

$$\mathcal{A} = \mu \frac{|f_0|}{\varepsilon^{n-1}} e^{-\frac{\pi}{2\varepsilon}} + \mathcal{O}\left(\mu^2\right)$$

The constant $f_0$ can be explicitly computed and satisfies $f_0 \neq 0$.

## True asymptotic formula

We obtain different results depending on $n$.

First case: $n < 4$ (namely $\ell - 2r < 0$)

For any $\mu$ independent of $\varepsilon$ and $\varepsilon$ small enough:

$$\mathcal{A} = \mu \frac{|f_0|}{\varepsilon^{n-1}} e^{-\frac{\pi}{2\varepsilon}} \left(1 + \mathcal{O}\left(\frac{1}{|\ln \varepsilon|^{\nu}}\right)\right)$$

for some $\nu > 0$.

Conclusion: In this case, Melnikov predicts correctly even if the perturbation has the same size as the unperturbed system.

Second case: $n = 4$ (namely $\ell - 2r = 0$)

For any $\mu$ independent of $\varepsilon$ and $\varepsilon$ small enough, there exists a function $f(\mu)$, such that, if $f(\mu) \neq 0$,

$$\mathcal{A} = \mu \frac{|f(\mu)|}{\varepsilon^{n-1}} e^{-\frac{\pi}{2\varepsilon}} \left(1 + \mathcal{O}\left(\frac{1}{|\ln \varepsilon|}\right)\right)$$

The function $f(\mu)$ is analytic and satisfes $f(\mu) = f_0 + \mathcal{O}(\mu)$.

Conclusions:

- As in the previous case, even if perturbation and unperturbed system have the same size, the splitting is exponentially small.
- If $\mu$ is small, Melnikov predicts correctly the splitting.
- If $\mu$ is independent of $\varepsilon$, Melnikov fails to predict the splitting.

Third case: $n > 4$ (namely $\ell - 2r > 0$)

- We are not able to give results for any $\mu$ independent of $\varepsilon$. We have to take $\mu \sim \varepsilon^p$ with $p \geq n-4$.
- Take $\mu = \delta\varepsilon^{n-4}$. Namely,

$$H\left(x, y, \frac{t}{\varepsilon}\right) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4} + \mu x^n \sin\frac{t}{\varepsilon}$$

- Then, for any $\delta$ independent of $\varepsilon$ and $\varepsilon$ small enough, there exists a function $f(\delta)$, such that, if $f(\delta) \neq 0$,

$$\mathcal{A} = \delta\frac{|f(\delta)|}{\varepsilon^3}e^{-\frac{\pi}{2\varepsilon}}\left(1 + \mathcal{O}\left(\frac{1}{|\ln\varepsilon|^{n-4}}\right)\right)$$

The function $f(\delta)$ is analytic and satisfes $f(\delta) = f_0 + \mathcal{O}(\delta)$.

Conclusions:

- Since $f(\delta) = f_0 + \mathcal{O}(\delta)$, for $\delta$ small Melnikov predicts correctly.
- In the original parameter, if $\mu \sim \varepsilon^p$ with $p > n - 4$ Melnikov predicts correctly.
- If $\delta$ is independent of $\varepsilon$, that is $\mu \sim \varepsilon^{n-4}$, Melnikov fails to predict the splitting.
- The true asymptotic formula for $\mu$ independent of $\varepsilon$ (unperturbed system and perturbation of the same size) has probably aditional correcting terms in the exponential.
- There is not any result dealing with this case.

## Perturbations depending on $y$

- If the perturbation depends on $y$, the first order also changes.
- Example:

$$H\left(x,y,\frac{t}{\varepsilon}\right)=\frac{y^2}{2}-\frac{x^2}{2}+\frac{x^4}{4}+\mu\left(y^2\cos\frac{t}{\varepsilon}+x^4\sin\frac{t}{\varepsilon}\right)$$

- Then, for $\mu$ independent of $\varepsilon$ and $\varepsilon$ small enough,

$$\begin{aligned}\mathcal{A}&=\mu\frac{|f(\mu)|}{\varepsilon^3}e^{-\frac{\pi}{2\varepsilon}+\mu^2 b\ln\frac{1}{\varepsilon}}\left(1+\mathcal{O}\left(\frac{1}{|\ln\varepsilon|^{n-4}}\right)\right)\\&=\mu\frac{|f(\mu)|}{\varepsilon^{3+\mu^2 b}}e^{-\frac{\pi}{2\varepsilon}}\left(1+\mathcal{O}\left(\frac{1}{|\ln\varepsilon|^{n-4}}\right)\right)\end{aligned}$$

Conclusions

- For $\mu$ independent of $\varepsilon$, Melnikov does not predict correctly even the power of $\varepsilon$ in front of the exponentially
- Namely, in these cases Melnikov only predicts correctly the coefficient in the exponential.
- Nevertheless, if $\mu \ll \dfrac{1}{\sqrt{|\ln \varepsilon|}}$, Melnikov predicts correctly.

## Splitting of separatrices for a meromorphic perturbation

- The results given apply when $V$ and $H_1$ are polynomials or trigonometric polynomials in $x$ and polynomials in $y$.
- Nevertheless, in many models, for instance in Celestial Mechanics, the Hamiltonian functions are not entire but have a finite strip of analyticity.
- Sometimes, the strip of analyticity $\sigma$ is large of the form $\sigma \sim \ln \alpha$ with $\alpha \ll 1$.

## Questions

- Which is the size of the Melnikov function for this kind of Hamiltonian Systems?
- When does the Melnikov function predict correctly the size of the splitting of separatrices?
- When the strip is of the form $\sigma \sim \ln \alpha$ with $\alpha \ll 1$, can we expand the perturbation in $\alpha$ and just consider the first order? or all the orders make a contribution to the first order of the difference between manifolds?

## A simple model

- Example:

$$\ddot{x} = \sin x + \mu \frac{\sin x}{(1 - \alpha \sin x)^2} \sin \frac{t}{\varepsilon}$$

  where $\alpha \in (0, 1)$.

- It has Hamiltonian

$$H(x, y, t) = \frac{y^2}{2} + \cos x - 1 + \mu m(x) \sin \frac{t}{\varepsilon}$$

  where $m$ is the primitive of $\dfrac{\sin x}{(1 - \alpha \sin x)^2}$.

- $(0, 0)$ is a hyperbolic periodic orbit even for the perturbed system.

## Computation of the Melnikov function

- Melnikov function:

$$
\begin{aligned}
M(t_0) &= \int_{-\infty}^{+\infty} y(u) \frac{\sin x(u)}{\left(1 - \alpha \sin x(u)\right)^2} \sin\left(\frac{u+t_0}{\varepsilon}\right) du \\
&= 4 \int_{-\infty}^{+\infty} \frac{\sinh u \cosh u}{\left(\cosh^2 u - 2\alpha \sinh u\right)^2} \sin\left(\frac{u+t_0}{\varepsilon}\right) du
\end{aligned}
$$

- The first order of this integral can be computed using residues theorem.

## Computation of the Melnikov function (II)

- If $\alpha = \mathcal{O}\left(\varepsilon^q\right)$ with $q > 2$: Since the integral is uniformly convergent in the reals, we can expand $M(t_0)$ in power series of $\alpha$ and split the integral

$$M(t_0) = 4\sum_{k=0}^{\infty}(k+1)2^k\alpha^k \int_{-\infty}^{+\infty} \frac{\sinh^{k+1} u}{\cosh^{2k+3} u} \sin\left(\frac{u+t_0}{\varepsilon}\right) du$$

- Its first term gives the bigger contribution to the splitting

$$M(t_0) \sim 4\pi\varepsilon^{-2}e^{-\frac{\pi}{2\varepsilon}}$$

- In that case, the exponential coefficient is given by the complex singularity of the separatrix.
- Conclusion: If the analyticity strip of the perturbation is big enough ($\alpha \ll \varepsilon^2$), the size of the Melnikov function is given as in the entire perturbation case.

## Computation of the Melnikov function (III)

- If $\alpha = \mathcal{O}(\varepsilon^q)$ with $q \in [0, 2]$, the integral of the summands is bigger as $k$ increases.
- We look for the singularities of the integrand.
- Consider $u^* = \eta \pm i\rho$ singularities of the integrand closest to the reals.
- If $\alpha$ is small $\rho = \pm\left(\frac{\pi}{2} - \sqrt{\alpha} + \mathcal{O}(\alpha)\right)$.
- If $\alpha$ is fixed and independent of $\varepsilon$, $\rho$ is also independent of $\varepsilon$ and is unrelated to the singularities of the separatrix.

## Computation of the Melnikov function (IV)

- Then, Melnikov is given by

$$M(t_0) \sim e^{-\frac{\rho}{\varepsilon}} \left( \frac{\varepsilon^{p-1}}{\sqrt{\alpha}} + \text{smaller terms} \right)$$

- If, for instance, one takes $\alpha = \varepsilon$,

$$M(t_0) \sim e^{-\frac{\pi - 2\sqrt{\varepsilon}}{2\varepsilon}} \left( \varepsilon^{p-\frac{3}{2}} + \text{smaller terms} \right)$$

- In these cases, even if $\alpha$ is small, the first order of the Melnikov function is given by the full jet in $\alpha$ of the perturbation.

## Validity of the Melnikov prediction

- If $\alpha = \mathcal{O}(\varepsilon^q)$ with $q > 2$ and $\varepsilon$ is small enough: the Melnikov function predicts correctly the splitting provided $\mu = \mathcal{O}\left(\varepsilon^p\right)$ with $p > 0$.

- The limit case $\mu$ fixed and independent of $\varepsilon$ (integrable system and perturbation of the same order) is expected to have exponentially small splitting of separatrices which is not well predicted by the Melnikov function.

- If $\alpha = \mathcal{O}(\varepsilon^q)$ with $q \in [0, 2]$, $\varepsilon$ is small enough and $\alpha < 1$: Melnikov function predicts correctly the splitting provided $p + \dfrac{q}{2} - 1 > 0$.

- The limit case $p + \dfrac{q}{2} - 1 = 0$ is expected to have exponentially small splitting of separatrices which is not well predicted by the Melnikov function.

## Non-exponentially small splitting of separatrices

Recall

$$\ddot{x} = \sin x + \mu \frac{\sin x}{(1 - \alpha \sin x)^2} \sin \frac{t}{\varepsilon}$$

If we take $\alpha = 1 - \mathcal{O}(\varepsilon^r)$:

- The strip of analyticity in $x$ of the Hamiltonian is $\mathcal{O}\left(\varepsilon^{\frac{r}{2}}\right)$.
- If $r > 2$, the Melnikov function satisfies

$$M(t_0) \sim \varepsilon^{-1}$$

- If $\mu = \varepsilon^p$ with $p > 1$, Melnikov predicts correctly the difference between the invariant manifolds, which is not exponentially small.