1 Aplicant la definició calculeu (si existeixen) les derivades parcials primeres de les següents funcions en el $(0,0)$

(a) $f(x,y)=\dfrac{xy}{x^2+y^2}$ si $(x,y)\neq(0,0)$, i $f(0,0)=0$.

Podem usar: $\dfrac{\partial f}{\partial x}(0,0)=\lim\limits_{h\to0}\dfrac{f(h,0)-f(0,0)}{h}$, $\dfrac{\partial f}{\partial y}(0,0)=\lim\limits_{h\to0}\dfrac{f(0,h)-f(0,0)}{h}$,

o bé $\dfrac{\partial f}{\partial x}(0,0)=\dfrac{d}{dx}f(x,0)\Big|_{x=0}$, $\dfrac{\partial f}{\partial y}(0,0)=\dfrac{d}{dy}f(0,y)\Big|_{y=0}$.

Tenim $f(x,0)=0 \quad \forall x\in\mathbb{R} \Rightarrow \dfrac{\partial f}{\partial x}(0,0)=0$

$f(0,y)=0 \quad \forall y\in\mathbb{R} \Rightarrow \dfrac{\partial f}{\partial y}(0,0)=0$

(b) $f(x,y)=x\ln(x^2+y^2)$ si $(x,y)\neq(0,0)$, i $f(0,0)=0$.

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{h\ln(h^2)-0}{h}=\lim_{h\to0}\ln(h^2)=\ln 0=-\infty \rightsquigarrow \nexists\frac{\partial f}{\partial x}(0,0)$$

$\dfrac{\partial f}{\partial y}(0,0)=0$ ja que $f(0,y)=0 \quad \forall y\in\mathbb{R}$. (f no derivable en $(0,0)$)

(c) $f(x,y)=(x^2+y^2)^x$ si $(x,y)\neq(0,0)$, i $f(0,0)=1$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{(h^2)^h-1}{h}=\lim_{h\to0}\frac{(h^2)^h(\ln(h^2)+2)}{1}=1\cdot(-\infty+2)=-\infty$$

$\nexists\dfrac{\partial f}{\partial x}(0,0)$

$\lim\limits_{h\to0}(h^2)^h=e^{\lim\limits_{h\to0}h\ln(h^2)}=e^0=1 \rightsquigarrow$ tenim $\dfrac{1-1}{0}=\dfrac{0}{0}\rightsquigarrow$ apliquem L'Hôp.

$$\frac{d}{dh}\left[(h^2)^h\right]=\frac{d}{dh}\left[e^{h\ln(h^2)}\right]=e^{h\ln(h^2)}\left(\ln(h^2)+h\cdot\frac{2h}{h^2}\right)=(h^2)^h\cdot(\ln(h^2)+2)$$

$\dfrac{\partial f}{\partial y}(0,0)=0$ ja que $f(0,y)=1 \quad \forall y\in\mathbb{R}$.

2 Calculeu $\dfrac{\partial f}{\partial y}(0,\pi)$, essent $f(x,y)=\dfrac{(e^{xy}-1)\cdot\arcsin(y^x)+y^4\cdot\cos(\pi+xy)}{x^2y^2+x^2+y^2}$

Podríem calcular $\dfrac{\partial f}{\partial y}(x,y)\ \forall(x,y)$, i substituir $(x,y)=(0,\pi)$, però l'expressió és molt complicada.

De fet, només ens cal conèixer f sobre la recta vertical per $(0,\pi)$:

$$f(0,y)=\frac{(e^0-1)\arcsin(y^0)+y^4\cdot\cos(\pi+0)}{0+0+y^2}=-y^2 \Rightarrow \frac{\partial f}{\partial y}(0,\pi)=\frac{d}{dy}f(0,y)\Big|_{y=\pi}=-2y\Big|_{y=\pi}=\underline{\underline{-2\pi}}$$

4 Calculeu les derivades parcials primeres de $f(x,y)=x^2\cdot\operatorname{tg}\dfrac{y^2}{x^2+y^2}$ si $(x,y)\neq(0,0)$, i comproveu que $x\dfrac{\partial f}{\partial x}(x,y)+y\dfrac{\partial f}{\partial y}(x,y)=2f(x,y)\quad\forall(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$

$$\frac{\partial f}{\partial x}=2x\cdot\operatorname{tg}\frac{y^2}{x^2+y^2}+x^2\left(1+\operatorname{tg}^2\frac{y^2}{x^2+y^2}\right)\cdot\frac{-2xy^2}{(x^2+y^2)^2}=2x\operatorname{tg}\frac{y^2}{x^2+y^2}-\frac{2x^3y^2}{(x^2+y^2)^2}\left(1+\operatorname{tg}^2\frac{y^2}{x^2+y^2}\right)$$

← derivem respecte x suposant y const.

$$\frac{\partial f}{\partial y}=x^2\left(1+\operatorname{tg}^2\frac{y^2}{x^2+y^2}\right)\cdot\frac{2y(x^2+y^2)-y^2\cdot2y}{(x^2+y^2)^2}=\frac{2x^4y}{(x^2+y^2)^2}\left(1+\operatorname{tg}^2\frac{y^2}{x^2+y^2}\right)$$

← derivem respecte y suposant x const.

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=2x^2\operatorname{tg}\frac{y^2}{x^2+y^2}-\cancel{\frac{2x^4y^2}{(x^2+y^2)^2}\left(1+\operatorname{tg}^2\frac{y^2}{x^2+y^2}\right)}+\cancel{\frac{2x^4y^2}{(x^2+y^2)^2}\left(1+\operatorname{tg}^2\frac{y^2}{x^2+y^2}\right)}=2f(x,y).$$

(10) Considerem la funció $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$ si $(x,y) \neq (0,0)$, i $f(0,0)=0$.

(a) Aplicant la definició de derivada parcial calculem $\frac{\partial f}{\partial x}(0,0)$.

(b) Calculem $\frac{\partial f}{\partial x}(x,y)$ si $(x,y) \neq (0,0)$.

(c) Demostrem que $\nexists \lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y)$.

(d) Quin és el subconjunt $D \subset \mathbb{R}^2$ més gran on tenim $f \in C^1(D)$.

(a) $\frac{\partial f}{\partial x}(0,0)=0$ ja que $f(x,0)=0 \quad \forall x \in \mathbb{R}$

(b)
$$\frac{\partial f}{\partial x}(x,y) = \frac{y\sqrt{x^2+y^2} - xy\cdot\frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y(x^2+y^2)-x^2y}{(x^2+y^2)^{3/2}} = \frac{y^3}{(x^2+y^2)^{3/2}} \quad \forall (x,y)\neq(0,0)$$

(c) Fem el límit de $\frac{\partial f}{\partial x}$ en $(0,0)$, al llarg de la recta $x=0$:

$$\frac{\partial f}{\partial x}(0,y) = \frac{y^3}{(y^2)^{3/2}} = \frac{y^3}{\sqrt{y^6}} = \frac{y^3}{|y|^3} = \begin{cases} 1 & \text{si } y>0 \\ -1 & \text{si } y<0 \end{cases} \Rightarrow \nexists \lim_{y\to 0} \frac{\partial f}{\partial x}(0,y)$$

(límits laterals diferents)

Per tant, $\nexists \lim_{(0,0)} \frac{\partial f}{\partial x}(x,y)$, i per tant $\frac{\partial f}{\partial x}$ no és contínua en $(0,0)$.

(d) $f \in C^1(D)$ si $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ són funcions contínues en D.

Per generació, veiem que f és C^1 en $\underline{D = \mathbb{R}^2 - \{(0,0)\}}$, ja que és quocient de funcions C^1 amb denominador $\neq 0$. En canvi, f no pot ser C^1 a tot $\mathbb{R}^2$ ja que hem vist que $\frac{\partial f}{\partial x}$ no és contínua en $(0,0)$.

(11) Calculem les derivades parcials segones de les següents funcions i donem el seu hessià.

(a) $f(x,y) = \sin x \cdot \sin^2 y$.

$$\frac{\partial f}{\partial x} = \cos x \cdot \sin^2 y, \quad \frac{\partial f}{\partial y} = \sin x \cdot 2\sin y \cdot \cos y = \sin x \cdot \sin 2y.$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin x \cdot \sin^2 y, \quad \frac{\partial^2 f}{\partial x \partial y} = \cos x \cdot \sin 2y, \quad \frac{\partial^2 f}{\partial y^2} = 2\sin x \cdot \cos 2y$$

[<u>Nota</u>. Tenim $\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ ja que f és C^2 (derivades parcials segones contínues)

Matriu hessiana o de derivades segones:
(si f és $C^2 \Rightarrow H_f$ és <u>simètrica</u>)

$$H_f = D^2 f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -\sin x \cdot \sin^2 y & \cos x \cdot \sin 2y \\ \cos x \cdot \sin 2y & 2\sin x \cdot \cos 2y \end{pmatrix}$$

(b) $f(x,y) = \sin(x^2-3xy)$

$$\frac{\partial f}{\partial x} = (2x-3y)\cos(x^2-3xy), \quad \frac{\partial f}{\partial y} = -3x\cos(x^2-3xy).$$

$$\frac{\partial^2 f}{\partial x^2} = 2\cos(x^2-3y) - (2x-3y)^2 \sin(x^2-3y), \quad \frac{\partial^2 f}{\partial x \partial y} = -3\cos(x^2-3y) - 3x(2x-3y)\sin(x^2-3xy),$$

$$\frac{\partial^2 f}{\partial y^2} = 9x^2\cos(x^2-3xy) \longrightarrow H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

(c) $f(x,y) = x \operatorname{arctg} \dfrac{x}{y}$

$$\frac{\partial f}{\partial x} = \operatorname{arctg}\frac{x}{y} + x\cdot\frac{1/y}{1+\left(\frac{x}{y}\right)^2} = \operatorname{arctg}\frac{x}{y} + \frac{xy}{x^2+y^2}, \qquad \frac{\partial f}{\partial y} = x\cdot\frac{-x/y^2}{1+\left(\frac{x}{y}\right)^2} = -\frac{x^2}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1/y}{1+\left(\frac{x}{y}\right)^2} + \frac{y(x^2+y^2) - xy\cdot 2x}{(x^2+y^2)^2} = \frac{y}{x^2+y^2} + \frac{y(y^2-x^2)}{(x^2+y^2)^2} = \frac{2y^3}{(x^2+y^2)^2}$$

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\frac{2x(x^2+y^2)-x^2\cdot 2x}{(x^2+y^2)^2} = -\frac{2xy^2}{(x^2+y^2)^2} \\ \frac{\partial^2 f}{\partial y^2} &= -\frac{x^2(-2y)}{(x^2+y^2)^2} = \frac{2x^2y}{(x^2+y^2)^2} \end{aligned}\right\} \quad H_f = \frac{2y}{(x^2+y^2)^2}\begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}$$

(d) $f(x,y) = \exp\left(-\dfrac{1}{x^2+y^2}\right)$

$$\frac{\partial f}{\partial x} = \exp\left(-\frac{1}{x^2+y^2}\right)\frac{2x}{(x^2+y^2)^2}, \qquad \frac{\partial f}{\partial y} = \exp\left(-\frac{1}{x^2+y^2}\right)\frac{2y}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \exp\left(-\frac{1}{x^2+y^2}\right)\cdot\left[\frac{4x^2}{(x^2+y^2)^4} + \frac{2(x^2+y^2)^2 - 2x\cdot 2(x^2+y^2)2x}{(x^2+y^2)^4}\right] = \exp\left(-\frac{1}{x^2+y^2}\right)\cdot\frac{4x^2+2(x^2+y^2)(y^2-3x^2)}{(x^2+y^2)^4}$$

$$\frac{\partial^2 f}{\partial y^2} = \exp\left(-\frac{1}{x^2+y^2}\right)\cdot\frac{4y^2+2(x^2+y^2)(x^2-3y^2)}{(x^2+y^2)^4} \quad \text{(per simmetria)}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \exp\left(-\frac{1}{x^2+y^2}\right)\left[\frac{4xy}{(x^2+y^2)^4} - \frac{2x\cdot 2(x^2+y^2)2y}{(x^2+y^2)^4}\right] = \exp\left(-\frac{1}{x^2+y^2}\right)\cdot\frac{4xy(1-2x^2-2y^2)}{(x^2+y^2)^4}$$

$$H_f = \exp\left(-\frac{1}{x^2+y^2}\right)\cdot\frac{2}{(x^2+y^2)^4}\begin{pmatrix} 2x^2+(x^2+y^2)(y^2-3x^2) & 2xy(1-2x^2-2y^2) \\ 2xy(1-2x^2-2y^2) & 2y^2+(x^2+y^2)(x^2-3y^2) \end{pmatrix}$$

(e) $f(x,y,z) = xy^2z^3e^x$

$$\frac{\partial f}{\partial x} = (1+x)y^2z^3e^x, \quad \frac{\partial f}{\partial y} = 2xyz^3e^x, \quad \frac{\partial f}{\partial z} = 3xy^2z^2e^x$$

$$\frac{\partial^2 f}{\partial x^2} = (2+x)y^2z^3e^x, \quad \frac{\partial^2 f}{\partial x\partial y} = 2(1+x)yz^3e^x, \quad \frac{\partial^2 f}{\partial x\partial z} = 3(1+x)y^2z^2e^x, \quad \frac{\partial^2 f}{\partial y^2} = 2xz^3e^x,$$

$$\frac{\partial^2 f}{\partial y\partial z} = 6xyz^2e^x, \quad \frac{\partial^2 f}{\partial z^2} = 6xy^2ze^x \qquad H_f = ze^x\begin{pmatrix} (2+x)y^2z^2 & 2(1+x)yz^2 & 3(1+x)y^2z \\ 2(1+x)yz^2 & 2xz^2 & 6xyz \\ 3(1+x)y^2z & 6xyz & 6xy^2 \end{pmatrix}$$

(f) $f(x,y,z) = x^2y + xy^2 + yz^2$

$$\frac{\partial f}{\partial x} = 2xy + y^2, \quad \frac{\partial f}{\partial y} = x^2+2xy+z^2, \quad \frac{\partial f}{\partial z} = 2yz$$

$$\frac{\partial^2 f}{\partial x^2} = 2y, \quad \frac{\partial^2 f}{\partial x\partial y} = 2x+2y, \quad \frac{\partial^2 f}{\partial x\partial z} = 0, \quad \frac{\partial^2 f}{\partial y^2} = 2x, \quad \frac{\partial^2 f}{\partial y\partial z} = 2z, \quad \frac{\partial^2 f}{\partial z^2} = 2y$$

$$H_f = 2\begin{pmatrix} y & x+y & 0 \\ x+y & x & z \\ x+y & z & y \end{pmatrix}$$

27 Considerem la funció $f(x,y) = xy \cdot \dfrac{x^2-y^2}{x^2+y^2}$ si $(x,y) \neq (0,0)$, i $f(0,0) = 0$.

(a) Calculeu $\dfrac{\partial f}{\partial x}(x,y)$ i $\dfrac{\partial f}{\partial y}(x,y)$, si $(x,y) \neq (0,0)$

(b) Usant la definició calculeu $\dfrac{\partial f}{\partial x}(0,0)$ i $\dfrac{\partial f}{\partial y}(0,0)$.

(c) Usant la definició calculeu $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ i $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$.

[Ind. $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)(0,0) = \lim\limits_{h \to 0} \dfrac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h}$]

(d) Quin és el subconjunt $D \subset \mathbb{R}^2$ més gran on tenim $f \in C^2(D)$?

(e) Quin és el subconjunt $D \subset \mathbb{R}^2$ més gran on tenim $f \in C^\infty(D)$?

(a) $f(x,y) = \dfrac{x^3y - xy^3}{x^2+y^2}$ si $(x,y) \neq (0,0)$

$$\longrightarrow \frac{\partial f}{\partial x} = \frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)\cdot 2x}{(x^2+y^2)^2} = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x^3 - 3xy^2)(x^2+y^2) - (x^3y - xy^3)\,2y}{(x^2+y^2)^2} = \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}$$

[també podem usar l'antisimetria:
$f(x,y) = -f(y,x)\ \forall (x,y) \Rightarrow \dfrac{\partial f}{\partial y}(x,y) = -\dfrac{\partial f}{\partial x}(y,x)$]

(b) $\dfrac{\partial f}{\partial x}(0,0) = 0$ ja que $f(x,0) = 0\ \forall x \in \mathbb{R}$

$\dfrac{\partial f}{\partial y}(0,0) = 0$ ja que $f(0,y) = 0\ \forall y \in \mathbb{R}$.

(c) $$\frac{\partial f}{\partial x}(0,y) = \begin{cases} -\dfrac{y^5}{y^4} = -y & \text{si } y \neq 0 \\ 0 & \text{si } y = 0 \end{cases} \Rightarrow \frac{\partial f}{\partial x}(0,y) = -y\ \forall y \in \mathbb{R} \Rightarrow \frac{\partial^2 f}{\partial y \partial x}(0,0) = -1.$$

$$\frac{\partial f}{\partial y}(x,0) = \begin{cases} \dfrac{x^5}{x^4} = x & \text{si } x \neq 0 \\ 0 & \text{si } x = 0 \end{cases} \Rightarrow \frac{\partial f}{\partial y}(x,0) = x\ \forall x \in \mathbb{R} \Rightarrow \frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$$

(d) Per generació, veiem que f és C^2 en $D = \mathbb{R}^2 \setminus \{(0,0)\}$, ja que és quocient de funcions C^2 amb denominador $\neq 0$.

En canvi, f no pot ser C^2 a tot $\mathbb{R}^2$ ja que $\dfrac{\partial^2 f}{\partial y \partial x}(0,0) \neq \dfrac{\partial^2 f}{\partial x \partial y}(0,0)$.

(e) Per generació, f és C^∞ en $D = \mathbb{R}^2 \setminus \{(0,0)\}$ ja que és quocient de funcions C^∞ (polinomis) amb denominador $\neq 0$.

17 Siguin $g(s,t)$ i $h(s,t)$ dues funcions tals que $g(0,0)=0$, $h(0,0)=\frac{\pi}{2}$, $\frac{\partial g}{\partial s}(0,0)=1$, $\frac{\partial g}{\partial t}(0,0)=-1$, $\frac{\partial h}{\partial s}(0,0)=0$ i $\frac{\partial h}{\partial t}(0,0)=2$.
Si $f(x,y)=\operatorname{arctg}(e^x \sin y)$ i definim $F(s,t)$ fent les substitucions $x=g(s,t)$ i $y=h(s,t)$, és a dir, $F(s,t)=f(g(s,t),h(s,t))$, calculeu $\frac{\partial F}{\partial s}(0,0)$ i $\frac{\partial F}{\partial t}(0,0)$

Usem: $\frac{\partial F}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$, $\frac{\partial F}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$, on $x=g(s,t)$, $y=h(s,t)$

Avaluant en $(s,t)=(0,0)$, tenim $(x,y)=(g(0,0),h(0,0))=(0,\frac{\pi}{2})$, i llavors:

$$\frac{\partial F}{\partial s}(0,0)=\frac{\partial f}{\partial x}(0,\tfrac{\pi}{2})\cdot\frac{\partial g}{\partial s}(0,0)+\frac{\partial f}{\partial y}(0,\tfrac{\pi}{2})\,\frac{\partial h}{\partial s}(0,0)=\frac{1}{2}\cdot 1+0\cdot 0=\boxed{\frac{1}{2}}$$
$$\frac{\partial F}{\partial t}(0,0)=\frac{\partial f}{\partial x}(0,\tfrac{\pi}{2})\,\frac{\partial g}{\partial t}(0,0)+\frac{\partial f}{\partial y}(0,\tfrac{\pi}{2})\,\frac{\partial h}{\partial t}(0,0)=\frac{1}{2}\cdot(-1)+0\cdot 2=\boxed{-1/2}$$

$$\frac{\partial f}{\partial x}(0,\pi/2)=\frac{e^x\sin y}{1+(e^x\sin y)^2}\bigg|_{(0,\frac{\pi}{2})}=\frac{1}{2},\quad \frac{\partial f}{\partial y}(0,\tfrac{\pi}{2})=\frac{e^x\cos y}{1+(e^x\sin y)^2}\bigg|_{(0,\frac{\pi}{2})}=0$$

Si usem les matrius jacobianes, definim una funció vectorial $G(s,t)=(g(s,t),h(s,t))$, i tenim $F=f\circ G$. Per tant,

$$DF(0,0)=Df(G(0,0))\cdot DG(0,0)=\begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\end{pmatrix}\bigg|_{(0,\pi/2)}\cdot\begin{pmatrix}\frac{\partial g}{\partial s} & \frac{\partial g}{\partial t}\\ \frac{\partial h}{\partial s} & \frac{\partial h}{\partial t}\end{pmatrix}\bigg|_{(0,0)}=$$
$$=\begin{pmatrix}1/2 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & -1/2\end{pmatrix}\Rightarrow \frac{\partial F}{\partial s}(0,0)=\frac{1}{2},\ \frac{\partial F}{\partial t}(0,0)=-\frac{1}{2}.$$

19 Donada una funció $f=f(u,v,w)$ de classe C^1, calculeu mitjançant la regla de la cadena expressions per a les derivades o derivades parcials primeres de la funció h en termes de les de α, β i γ en cadascun dels casos següents.

(a) $h(x)=f(x,\alpha(x),\beta(x))$

$$h'(x)=\frac{\partial f}{\partial u}\underbrace{\frac{\partial u}{\partial x}}_{=1}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\,\frac{\partial w}{\partial x}=\frac{\partial f}{\partial u}(x,\alpha(x),\beta(x))+\frac{\partial f}{\partial v}(x,\alpha(x),\beta(x))\cdot\alpha'(x)+$$
$$+\frac{\partial f}{\partial w}(x,\alpha(x),\beta(x))\cdot\beta'(x).$$

(b) $h(x,y)=f(y,\alpha(x,y),\beta(x))$

$$\frac{\partial h}{\partial x}=\frac{\partial f}{\partial u}\underbrace{\frac{\partial u}{\partial x}}_{=0}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\,\frac{\partial w}{\partial x}=\frac{\partial f}{\partial v}(y,\alpha(x,y),\beta(x))\cdot\frac{\partial\alpha}{\partial x}(x,y)+\frac{\partial f}{\partial w}(\ldots\ldots)\cdot\beta'(x)$$

$$\frac{\partial h}{\partial y}=\frac{\partial f}{\partial u}\underbrace{\frac{\partial u}{\partial y}}_{=1}+\frac{\partial f}{\partial v}\,\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\underbrace{\frac{\partial w}{\partial y}}_{=0}=\frac{\partial f}{\partial u}(\ldots\ldots)+\frac{\partial f}{\partial v}(\ldots\ldots)\cdot\frac{\partial\alpha}{\partial y}(x,y).$$

(c) $h(x,y,z)=f(\alpha(z),\beta(y,z),\gamma(x,y,z))$

$$\frac{\partial h}{\partial x}=\frac{\partial f}{\partial u}\underbrace{\frac{\partial u}{\partial x}}_{=0}+\frac{\partial f}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{=0}+\frac{\partial f}{\partial w}\,\frac{\partial w}{\partial x}=\frac{\partial f}{\partial w}(\alpha(z),\beta(y,z),\gamma(x,y,z))\cdot\frac{\partial\gamma}{\partial x}(x,y,z)$$

$$\frac{\partial h}{\partial y}=\frac{\partial f}{\partial u}\underbrace{\frac{\partial u}{\partial y}}_{=0}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\,\frac{\partial w}{\partial y}=\frac{\partial f}{\partial v}(\ldots\ldots)\,\frac{\partial\beta}{\partial y}(y,z)+\frac{\partial f}{\partial w}(\ldots\ldots)\cdot\frac{\partial\gamma}{\partial y}(x,y,z).$$

$$\frac{\partial h}{\partial z}=\frac{\partial f}{\partial u}\,\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z}+\frac{\partial f}{\partial w}\,\frac{\partial w}{\partial z}=\frac{\partial f}{\partial u}(\ldots\ldots)\cdot\alpha'(z)+\frac{\partial f}{\partial v}(\ldots)\,\frac{\partial\beta}{\partial z}(y,z)+\frac{\partial f}{\partial w}(\ldots)\cdot\frac{\partial\gamma}{\partial z}(x,y,z).$$

(34) Sigui $f(x,y,z) = (e^{2y}+e^{2z}, e^{2x}-e^{2z}, x-y)$. Proveu que f té una inversa global i calculeu la seva matriu derivada.

Teo. f. inversa (local). $f: A\subset\mathbb{R}^n \longrightarrow \mathbb{R}^n$ de classe C^1, $a\in A$.

Si $Jf(a) = \det Df(a) \neq 0 \Rightarrow f$ és localment invertible prop de a

($\exists$ funció inversa f^{-1}, definida prop de $b=f(a)$)

Per a la funció inversa, tenim: $Df^{-1}(y) = Df(x)^{-1}$ si $y=f(x)$.

Teo. f. inversa global. $f: A\subset\mathbb{R}^n\longrightarrow\mathbb{R}^n$ de classe C^1.

Si f és injectiva, amb $Jf(x)\neq 0$ $\forall x\in A \Rightarrow f$ és globalment invertible

($\exists$ funció inversa f^{-1}, definida a tot $B=f(A)$)

Llavors, diem que f és un canvi de coordenades entre els dominis A i $B=f(A)$

Tenim $f:\mathbb{R}^3\longrightarrow\mathbb{R}^3$, de classe C^1.

Jacobià:

$$Df(x,y,z) = \begin{pmatrix} 0 & 2e^{2y} & 2e^{2z} \\ 2e^{2x} & 0 & -2e^{2z} \\ 1 & -1 & 0 \end{pmatrix}, \quad Jf(x,y,z) = -4e^{2y+2z} - 4e^{2x+2z} \neq 0 \quad \forall (x,y,z)\in\mathbb{R}^3$$

Comprovem que f és injectiva: si $f(x,y,z) = f(\tilde{x},\tilde{y},\tilde{z}) \overset{?}{\Rightarrow} (x,y,z) = (\tilde{x},\tilde{y},\tilde{z})$

$$\left.\begin{cases} e^{2y}+e^{2z} = e^{2\tilde{y}}+e^{2\tilde{z}} \\ e^{2x}-e^{2z} = e^{2\tilde{x}}-e^{2\tilde{z}} \\ x-y = \tilde{x}-\tilde{y} \end{cases}\right] \longrightarrow e^{2x}+e^{2y} = e^{2\tilde{x}}+e^{2\tilde{y}}$$

$$e^{2y}(\cancel{e^{2x-2y}}+1) = e^{2\tilde{y}}(\cancel{e^{2\tilde{x}-2\tilde{y}}}+1)$$

$y=\tilde{y}$, i deduïm que $x=\tilde{x}$, $z=\tilde{z}$.

$\Rightarrow f$ té inversa global, definida en $B=f(\mathbb{R}^3)$

Donat $(u,v,w) = f(x,y,z)\in B$, tindrem:

$$Df^{-1}(u,v,w) = Df(x,y,z)^{-1} = -\frac{1}{4e^{2z}(e^{2x}+e^{2y})}\begin{pmatrix} -2e^{2z} & -2e^{2y} & -4e^{2y+2z} \\ -2e^{2x} & -2e^{2z} & 4e^{2x+2z} \\ -2e^{2x} & 2e^{2y} & -4e^{2x+2y} \end{pmatrix} \quad (*)$$

Nota. Per comprovar que f és injectiva, també podem veure que, donat $(u,v,w)\in\mathbb{R}^3$, $\exists$ com a molt una solució (x,y,z) del sistema $f(x,y,z)=(u,v,w)$.

$$\left.\begin{cases} e^{2y}+e^{2z}=u \\ e^{2x}-e^{2z}=v \\ x-y=w \end{cases}\right] \longrightarrow e^{2x}+e^{2y} = u+v$$

$$e^{2y}(e^{2x-2y}+1) \qquad e^{2y}(e^{2w}+1)$$

$$e^{2y} = \frac{u+v}{e^{2w}+1} \Rightarrow y = \frac{1}{2}\ln\frac{u+v}{e^{2w}+1}$$

$$e^{2x} = u+v-e^{2y} = (u+v)\cdot\frac{e^{2w}}{e^{2w}+1} \Rightarrow x = \dots$$

$$e^{2z} = u-e^{2y} = u-\frac{u+v}{e^{2w}+1} \Rightarrow z = \dots$$

Per tant, la solució és única (si $\exists$),

i tenim $B = \left\{(u,v,w): u+v>0,\ u-\frac{u+v}{e^{2w}+1}>0\right\}$

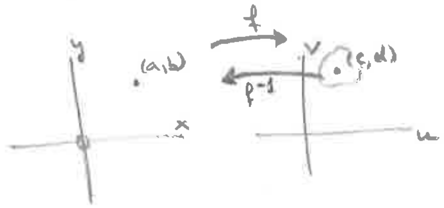
A més, podem trobar $(x,y,z) = f^{-1}(u,v,w)$ explícitament, i escriure la matriu jacobiana (*) en funció de (u,v,w).

(35) Sigui $f:\mathbb{R}^2\setminus\{(0,0)\}\longrightarrow\mathbb{R}^2$ definida per $f(x,y)=(x^2-y^2,\,xy)$. Provem que f té una inversa local en cada punt del seu domini i calculem la seva matriu jacobiana. Estudiem si f té una inversa global.

- f és de classe C^1

Donat un punt $(a,b)\in\mathbb{R}^2\setminus\{(0,0)\}$, calculem el seu jacobià:

$$Df(a,b)=\begin{pmatrix}2x&-2y\\ y&x\end{pmatrix}\Bigg|_{(a,b)}=\begin{pmatrix}2a&-2b\\ b&a\end{pmatrix},\quad Jf(a,b)=2a^2+2b^2\neq0$$



$\Rightarrow$ $\exists$ funció inversa local f^{-1}, definida prop de $(c,d)=f(a,b)$

La seva matriu jacobiana, en un punt $(u,v)=f(x,y)$,

$$Df^{-1}(u,v)=Df(x,y)^{-1}=\frac{1}{2x^2+2y^2}\begin{pmatrix}x&2y\\ -y&2x\end{pmatrix}$$

(es pot expressar en funció de (u,v) si coneixem $(u,v)=f^{-1}(x,y)$)

- La funció f no té inversa global a $\mathbb{R}^2\setminus\{(0,0)\}$, ja que no és injectiva: $f(x,y)=f(-x,-y)$.

Estudiem si és injectiva en un domini $A\subset\mathbb{R}^2\setminus\{(0,0)\}$ (el més gran possible):

$$f(x,y)=f(\tilde x,\tilde y)\overset{?}{\Longrightarrow}(x,y)=(\tilde x,\tilde y)$$

$$\begin{cases}x^2-y^2=\tilde x^2-\tilde y^2\\ xy=\tilde x\tilde y\end{cases}\longrightarrow\quad \tilde y=\frac{xy}{\tilde x}\ \text{ si }\ \tilde x\neq0$$

$\left[\text{Si } \tilde x=0,\quad xy=0\Rightarrow\begin{cases}x=0\Rightarrow -y^2=-\tilde y^2\Rightarrow\tilde y=\pm y\\ y=0\Rightarrow x^2=-\tilde y^2\Rightarrow x=\tilde y=0\end{cases}\right]$

$$x^2-y^2=\tilde x^2-\frac{x^2y^2}{\tilde x^2}$$

$$\tilde x^4-(x^2-y^2)\tilde x^2-x^2y^2=0$$

$$\tilde x^2=\frac{x^2-y^2\pm\sqrt{(x^2-y^2)^2+4x^2y^2}}{2}=\frac{x^2-y^2\pm(x^2+y^2)}{2}=\begin{cases}x^2\\ -y^2\ \text{(no pot ser, si } \tilde x\neq0)\end{cases}$$

$$\Rightarrow\tilde x=\begin{cases}x\longrightarrow\tilde y=y,\ (\tilde x,\tilde y)=(x,y)\\ -x\longrightarrow\tilde y=-y,\ (\tilde x,\tilde y)=(-x,-y)\end{cases}$$

Si restringim f al conjunt $A=\{(x,y):x>0\}$, llavors és injectiva, i per tant té inversa global

Nota: També podem comprovar que, donat $(u,v)\in\mathbb{R}^2$, $\exists$ com a molt una solució $(x,y)\in A$ del sistema $f(x,y)=(u,v)$.

$$\begin{cases}x^2-y^2=u\\ xy=v\end{cases}\longrightarrow y=\frac{v}{x}\longrightarrow x^2-\frac{v^2}{x^2}=u$$

$$x^4-ux^2-v^2=0$$

$$x^2=\frac{u+\sqrt{u^2+4v^2}}{2}\quad\longleftarrow\ (\text{signe } \oplus\text{, per tenir } x^2>0)$$

$$x=\sqrt{\frac{u+\sqrt{u^2+4v^2}}{2}}\quad\longleftarrow\ (\text{signe } \oplus\text{, per tenir } (x,y)\in A)$$

$$y=\frac{v}{x}=v\cdot\sqrt{\frac{2}{u+\sqrt{u^2+4v^2}}}$$

$\Rightarrow$ La solució és única, i compleix $x>0$ excepte si $v=0,\ u\leq0$.

Així tenim $(x,y)=f^{-1}(u,v)=\left(\sqrt{\frac{u+\sqrt{u^2+4v^2}}{2}},\ v\cdot\sqrt{\frac{2}{u+\sqrt{u^2+4v^2}}}\right)$, definida a

$$B=f(A)=\mathbb{R}^2\setminus\{(u,v):\ v=0,\ u\leq0\}$$

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El sistema
$$\begin{cases} x + yv + e^{yu} + e^{xv} = 3 \\ y - xv + e^{xu} + e^{yv} = 3 \end{cases}$$
determina dues funcions $u(x,y)$ i $v(x,y)$ que satisfan $u(1,1) = v(1,1) = 0$.
Sigui ara $g(x,y) = (u(x,y), v(x,y))$, calculeu $Dg(1,1)$ i $Dg^{-1}(0,0)$.

El sistema és
$$\begin{cases} F_1(x,y,u,v) = x + yv + e^{yu} + e^{xv} - 3 = 0 \\ F_2(x,y,u,v) = y - xv + e^{xu} + e^{yv} - 3 = 0, \end{cases}$$
tenim així una funció $F = (F_1, F_2) : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$.

Volem veure que podem aïllar $u = u(x,y)$, $v = v(x,y)$ localment, prop del punt $(x,y,u,v) = (1,1,0,0)$. Comprovem:

* F és de classe C^1.
* el punt $(1,1,0,0)$ és solució del sistema.
* $J_{(u,v)}F(1,1,0,0) = \det \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix}\Bigg|_{(1,1,0,0)} = \det \begin{pmatrix} ye^{yu} & y + xe^{xv} \\ xe^{xu} & -x + ye^{yv} \end{pmatrix}\Bigg|_{(1,1,0,0)} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \neq 0$.

Aplicant el teo. f. implícita, el sistema determina funcions implícites $u = u(x,y)$, $v = v(x,y)$, definides prop de $(x,y) = (1,1)$, que satisfan:

* $u(1,1) = 0$, $v(1,1) = 0$
* $F_1(x,y,u(x,y),v(x,y)) = F_2(x,y,u(x,y),v(x,y)) = 0$.

A més, podem calcular la derivada de les funcions implícites:
Escrivint $g(x,y) = (u(x,y), v(x,y))$,
$$Dg(x,y) = -D_{(u,v)}F(x,y,u,v)^{-1} \cdot D_{(x,y)}F(x,y,u,v).$$

$$\Rightarrow \boxed{Dg(1,1)} = -D_{(u,v)}F(1,1,0,0)^{-1} \cdot D_{(x,y)}F(1,1,0,0) =$$
$$= -\begin{pmatrix} ye^{yu} & y + xe^{xv} \\ xe^{xu} & -x + ye^{yv} \end{pmatrix}\Bigg|_{(1,1,0,0)}^{-1} \cdot \begin{pmatrix} 1 + ve^{xv} & v + ue^{yu} \\ -v + ue^{xu} & 1 + ve^{yv} \end{pmatrix}\Bigg|_{(1,1,0,0)} = -\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 & -1 \\ -1/2 & 1/2 \end{pmatrix}}$$

Com que $g(1,1) = (0,0)$ i tenim $Jg(1,1) \neq 0$, pel teo. f. inversa obtenim:
$$Dg^{-1}(0,0) = Dg(1,1)^{-1} = \begin{pmatrix} -1 & -2 \\ -1 & 0 \end{pmatrix}$$

* Alternativament, per calcular $Dg(1,1) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\Bigg|_{(1,1)}$ podem considerar $u = u(x,y)$, $v = v(x,y)$ al sistema inicial i derivar implícitament:

$$\frac{\partial}{\partial x} \longrightarrow \begin{cases} 1 + y\frac{\partial v}{\partial x} + y\frac{\partial u}{\partial x} e^{yu} + \left(v + x\frac{\partial v}{\partial x}\right)e^{xv} = 0 \\ -\left(v + x\frac{\partial v}{\partial x}\right) + \left(u + x\frac{\partial u}{\partial x}\right)e^{xu} + y\frac{\partial v}{\partial x} e^{yv} = 0 \end{cases}$$

$\rightarrow$ substituint $(x,y,u,v) = (1,1,0,0)$, $\begin{cases} 1 + 2\frac{\partial v}{\partial x}(1,1) + \frac{\partial u}{\partial x}(1,1) = 0 \\ \frac{\partial u}{\partial x}(1,1) = 0 \end{cases} \Rightarrow \boxed{\begin{array}{l} \frac{\partial u}{\partial x}(1,1) = 0 \\ \frac{\partial v}{\partial x}(1,1) = -1/2 \end{array}}$

$\frac{\partial}{\partial y} \longrightarrow$ anàlogament, obtenim $\boxed{\frac{\partial u}{\partial y}(1,1) = -1, \ \frac{\partial v}{\partial y}(1,1) = 1/2}$.

(20) Useu la fórmula del gradient per calcular les següents derivades direccionals.

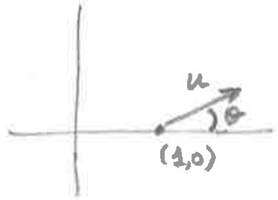
(a) $D_u f(1,0)$ si $f(x,y)=\ln\sqrt{x^2+y^2}$ i $u=\left(\frac{2}{\sqrt5},\frac{1}{\sqrt5}\right)$.

La funció f és C^1 per a $(x,y)\neq(0,0)$ $\Rightarrow$ podem aplicar la fórmula del gradient.

$$\nabla f(1,0)=\left(\frac{\partial f}{\partial x}(1,0),\frac{\partial f}{\partial y}(1,0)\right)=\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right)\Big|_{(1,0)}=(1,0).$$

$f(x,y)=\frac12\ln(x^2+y^2)$

$$\boxed{D_u f(1,0)}=\langle\nabla f(1,0),u\rangle=\left\langle(1,0),\left(\tfrac{2}{\sqrt5},\tfrac{1}{\sqrt5}\right)\right\rangle=\boxed{\frac{2}{\sqrt5}}=0.894427.$$



Nota: $u=(\cos\theta,\sin\theta)$, amb $\theta=\operatorname{arctg}\frac12=0.463648$ $(\simeq 27^\circ)$

Nota. Aplicant la definició de derivada direccional, faríem:

$$D_u f(1,0)=\frac{d}{dt}\left[f((1,0)+tu)\right]\Big|_{t=0}=\frac{d}{dt}\left[f\left(1+\tfrac{2t}{\sqrt5},\tfrac{t}{\sqrt5}\right)\right]\Big|_{t=0}=\frac{d}{dt}\left[\ln\sqrt{\left(1+\tfrac{2t}{\sqrt5}\right)^2+\left(\tfrac{t}{\sqrt5}\right)^2}\right]\Big|_{t=0}=$$

$$=\frac12\frac{d}{dt}\left[\ln\left(1+\frac{4t}{\sqrt5}+t^2\right)\right]\Big|_{t=0}=\frac12\cdot\frac{\frac{4}{\sqrt5}+2t}{1+\frac{4t}{\sqrt5}+t^2}\Big|_{t=0}=\frac{2}{\sqrt5}.$$

(b) $D_u f(0,-1)$ si $f(x,y)=e^x\cos\pi y$ i $u=\left(-\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)$

$$\nabla f(0,-1)=\left(e^x\cos\pi y,-\pi e^x\sin\pi y\right)\Big|_{(0,-1)}=(-1,0)\ \longrightarrow\ D_u f(0,-1)=\langle\nabla f(0,-1),u\rangle=\frac{1}{\sqrt2}$$

(c) $D_u f(1,0,0)$ si $f(x,y,z)=x^2e^{-yz}$ i $u=\left(\frac{1}{\sqrt3},-\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right)$

$$\nabla f(1,0,0)=\left(2xe^{-yz},-x^2ze^{-yz},-x^2ye^{-yz}\right)\Big|_{(1,0,0)}=(2,0,0)$$

$$\longrightarrow D_u f(1,0,0)=\langle\nabla f(1,0,0),u\rangle=\frac{2}{\sqrt3}$$

(21) Perfil muntanya: $z=h(x,y)=5000-0.01x^2-0.02y^2$.

(a) Un muntanyenc al punt $(x,y)=(10,10)$ $\longrightarrow$ direcció per pujar més ràpidament? pendent?

(b) Quina direcció ha de seguir perquè el pendent sigui del 40%?

(a) Ha de seguir la direcció i sentit del vector gradient:

$$\nabla h(10,10)=(-0.02x,-0.04y)\Big|_{(10,10)}=(-0.2,-0.4)=\left(-\tfrac15,-\tfrac25\right),\ \text{amb}\ \|\nabla h(10,10)\|=\frac{\sqrt5}{5}=\frac{1}{\sqrt5}$$

Per a $\boxed{v^*}=\dfrac{\nabla h(10,10)}{\|\nabla h(10,10)\|}=\boxed{\left(-\frac{1}{\sqrt5},-\frac{2}{\sqrt5}\right)}$, el pendent és $\boxed{D_{v^*}h(10,10)=\|\nabla h(10,10)\|=\frac{1}{\sqrt5}}$ $(\simeq 45\%)$

(b) Busquem un vector $v=(\alpha,\beta)$ unitari, tal que $D_{(\alpha,\beta)}h(10,10)=0.4=\frac25$

Per la fórmula del gradient, $D_{(\alpha,\beta)}h(10,10)=\langle\nabla h(10,10),(\alpha,\beta)\rangle=-\frac{\alpha}{5}-\frac{2\beta}{5}$

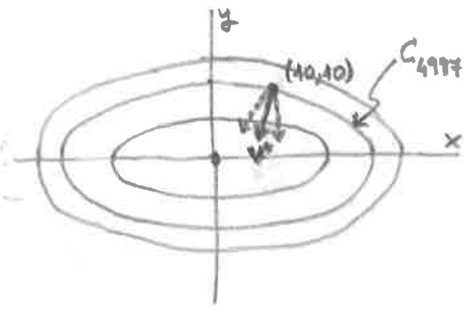
Com que el vector ha de ser unitari, resolem:

$$\begin{cases}-\frac{\alpha}{5}-\frac{2\beta}{5}=\frac25 & \longrightarrow\ \beta=-\frac{\alpha+2}{2}\\ \alpha^2+\beta^2=1\end{cases}$$

$$\alpha^2+\beta^2=\alpha^2+\left(\frac{\alpha+2}{2}\right)^2=\frac{5\alpha^2+4\alpha+4}{4}=1$$

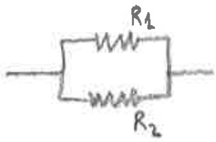
$$\Rightarrow 5\alpha^2+4\alpha=0\ \Rightarrow\ \alpha=0\ \text{o}\ \alpha=-\frac45.$$

Vectors: $\boxed{(\alpha,\beta)=\begin{cases}(0,-1)\\ \left(-\frac45,-\frac35\right)\end{cases}}$



(23) La resistència total R corresponent a dues resistències R_1 i R_2 connectades en paral·lel satisfà
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$
Usant l'aproximació lineal, estimeu la variació del valor de R si incrementem el valor de R_1 de $10\,\Omega$ a $10.5\,\Omega$ i decreixem el valor de R_2 de $15\,\Omega$ a $13\,\Omega$.

Tenim una funció $R = f(R_1,R_2) = \dfrac{1}{\frac{1}{R_1}+\frac{1}{R_2}} = \dfrac{R_1R_2}{R_1+R_2}$



Per a $R_1=10$ i $R_2=15$, tenim $R = f(10,15) = 6$.

Increments: $\Delta R_1 = 0.5$, $\Delta R_2 = -2$.

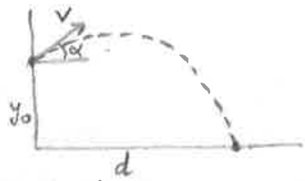
Podem estimar l'increment de la resistència total R usant l'aproximació lineal de la funció f en el punt $(10,15)$:
$$\Delta R = f(10+\Delta R_1, 15+\Delta R_2) - f(10,15) \simeq \frac{\partial f}{\partial R_1}(10,15)\,\Delta R_1 + \frac{\partial f}{\partial R_2}(10,15)\,\Delta R_2$$

Calculem: $\dfrac{\partial f}{\partial R_1} = \dfrac{R_2(R_1+R_2)-R_1R_2}{(R_1+R_2)^2} = \left(\dfrac{R_2}{R_1+R_2}\right)^2$, i per simetria $\dfrac{\partial f}{\partial R_2} = \left(\dfrac{R_1}{R_1+R_2}\right)^2$

Avaluant, $\dfrac{\partial f}{\partial R_1}(10,15) = 0.36$, $\dfrac{\partial f}{\partial R_2}(10,15) = 0.16$
$$\Rightarrow \boxed{\Delta R \simeq 0.36\cdot\Delta R_1 + 0.16\cdot\Delta R_2 = -0.14\,\Omega}$$

(comparem amb el valor exacte: $f(10+\Delta R_1, 15+\Delta R_2) = 5.80851 \Rightarrow \Delta R = 5.80851-6 = -0.191489$)

(25) Projectil:



$$d(\alpha,v) = \frac{v^2\sin 2\alpha}{2g}\left(1+\sqrt{1+\frac{2gy_0}{v^2\sin^2\alpha}}\right),$$ fem $g = 10\,m/s^2$, $y_0 = 2m$

Suposem que inicialment $v_0 = 8\,m/s$, $\alpha_0 = \pi/4$

(a) Si fem petits increments $\Delta\alpha$, Δv, useu l'aproximació lineal per aproximar $d(\alpha_0+\Delta\alpha, v_0+\Delta v)$

(b) Usant la fórmula del gradient, deduïu la relació entre $\Delta\alpha$ i Δv per tal que el projectil assoleixi la màxima distància possible, usant l'aproximació lineal.

(a) Aproximació lineal: $d(\frac{\pi}{4}+\Delta\alpha, 8+\Delta v) \simeq d(\frac{\pi}{4},8) + \frac{\partial d}{\partial\alpha}(\frac{\pi}{4},8)\,\Delta\alpha + \frac{\partial d}{\partial v}(\frac{\pi}{4},8)\,\Delta v$

$$\frac{\partial d}{\partial\alpha} = \frac{v^2\cos 2\alpha}{g}\left(1+\sqrt{\quad}\right) - \frac{2y_0}{tg^2\alpha\cdot\sqrt{\quad}}, \qquad \frac{\partial d}{\partial v} = \frac{v\sin 2\alpha}{g}\left(1+\sqrt{\quad}\right) - \frac{2y_0}{v\,tg\alpha\sqrt{\quad}}.$$

Avaluem, posant $v=8$, $\alpha=\pi/4$, $g=10$, $y_0=2$:
$$\sqrt{(\dots)} = \sqrt{1+\frac{2\cdot10\cdot2}{8^2\cdot\left(\frac{\sqrt2}{2}\right)^2}} = \frac{3}{2} \longrightarrow d(\tfrac{\pi}{4},8) = \frac{8^2\cdot1}{2\cdot10}\left(1+\tfrac32\right) = 8, \quad \frac{\partial d}{\partial\alpha}(\tfrac{\pi}{4},8) = 0 - \frac{2\cdot2}{1^2\cdot3/2} = -\frac{8}{3}$$
$$\frac{\partial d}{\partial v}(\tfrac{\pi}{4},8) = \frac{8\cdot1}{10}\left(1+\tfrac32\right) - \frac{2\cdot2}{8\cdot1\cdot\frac32} = \frac{5}{3}$$

Obtenim
$$d(\tfrac{\pi}{4}+\Delta\alpha, 8+\Delta v) \simeq 8 - \tfrac{8}{3}\Delta\alpha + \tfrac{5}{3}\Delta v$$

(b) Si $(\Delta\alpha,\Delta v) = \lambda u$ amb u unitari, $\lambda = \sqrt{(\Delta\alpha)^2+(\Delta v)^2} > 0$,

$d(\frac{\pi}{4}+\Delta\alpha, 8+\Delta v) \simeq 8 + \langle \nabla d(\frac{\pi}{4},8), \lambda u\rangle = 8 + \lambda D_u d(\frac{\pi}{4},8)$, ja que $\nabla d(\frac{\pi}{4},8) = \left(-\frac{8}{3}, \frac{5}{3}\right)$.

Sabem que la derivada direccional $D_u d(\frac{\pi}{4},8)$ és màxima si $u = \frac{1}{b}\nabla d(\frac{\pi}{4},8)$, on $b = \|\nabla d(\frac{\pi}{4},8)\| = \frac{\sqrt{89}}{3}$.

$\Rightarrow$ cal $\dfrac{\Delta v}{\Delta\alpha} = \dfrac{5/3}{-8/3} = -\dfrac{5}{8}$ amb $\Delta\alpha<0$, $\Delta v>0$, i llavors: $d(\frac{\pi}{4}+\Delta\alpha, 8+\Delta v) = 8 + \frac{\sqrt{89}}{3}\lambda$

Nota. La trajectòria del projectil és $x(t) = (v\cos\alpha)t$, $y(t) = y_0 + (v\sin\alpha)t - \frac{g}{2}t^2$.
Trobant $t^*>0$ tal que $y(t^*)=0$, obtenim la fórmula per a $x(t^*) = d(\alpha,v)$.

(26) $V(x,y,z) = xyz$, volum d'una capsa de costats x, y, z.
Si en un cas concret mesurem $x = 0.9 \pm 0.1$, $y = 0.4 \pm 0.1$, $z = 2.4 \pm 0.1$, useu el teorema del valor mitjà per acotar rigorosament l'error màxim que podem cometre si aproximem el volum per $V_0 = V(0.9, 0.4, 2.4)$. Error relatiu màxim?
[Ind. s'ha d'acotar les derivades parcials de V si $(x,y,z) \in [0.8, 1] \times [0.3, 0.5] \times [2.3, 2.5]$]

Els valors reals (i desconeguts) dels costats es poden escriure:

$$\bar{x} = 0.9 + \Delta x, \quad \bar{y} = 0.4 + \Delta y, \quad \bar{z} = 2.4 + \Delta z, \quad \text{amb } |\Delta x|, |\Delta y|, |\Delta z| \leq 0.1$$

El volum real és $\bar{V} = V(\bar{x}, \bar{y}, \bar{z})$, que aproximem per $V_0 = V(0.9, 0.4, 2.4) = 0.864 \text{ m}^3$.

L'error absolut és $\Delta V = \bar{V} - V_0$, i l'error relatiu és $\dfrac{\Delta V}{V_0}$.

Pel teo. del valor mitjà, $\Delta V = \dfrac{\partial V}{\partial x}(x^*, y^*, z^*) \Delta x + \dfrac{\partial V}{\partial y}(x^*, y^*, z^*) \Delta y + \dfrac{\partial V}{\partial z}(x^*, y^*, z^*) \Delta z$,

on (x^*, y^*, z^*) és algun punt del segment entre $(0.9, 0.4, 2.4)$ i $(\bar{x}, \bar{y}, \bar{z})$.

Notem que $(x^*, y^*, z^*) \in [0.8, 1] \times [0.3, 0.5] \times [2.3, 2.5]$.

Fitem les derivades parcials:

$$\left|\frac{\partial V}{\partial x}(x^*, y^*, z^*)\right| = y^* z^* \leq 0.5 \cdot 2.5 = 1.25 \quad (\text{obs. } x^*, y^*, z^* > 0)$$

$$\left|\frac{\partial V}{\partial y}(\text{---})\right| = x^* z^* \leq 1 \cdot 2.5 = 2.5$$

$$\left|\frac{\partial V}{\partial z}(\text{---})\right| = x^* y^* \leq 1 \cdot 0.5 = 0.5$$

Fita de l'error absolut: $|\Delta V| \leq 1.25 \cdot |\Delta x| + 2.5\, |\Delta y| + 0.5\, |\Delta z| \underset{|\Delta x|, |\Delta y|, |\Delta z| \leq 0.1}{\leq} (1.25 + 2.5 + 0.5) \cdot 0.1 = 0.425$

Fita de l'error relatiu: $\left|\dfrac{\Delta V}{V_0}\right| \leq \dfrac{0.425}{0.864} \simeq \underline{0.49}$ $(\to 49\%)$

(28) Calculeu totes les derivades parcials fins a ordre 2 de les següents funcions i doneu el seu desenvolupament de Taylor fins a termes de grau 2 (inclosos) entorn del punt que s'indica en cada cas.

(a) $f(x,y) = \sin xy$, entorn del punt $(1, \pi/2)$.

$$f(x,y) = f(1,\pi/2) + \frac{\partial f}{\partial x}(1,\pi/2)(x-1) + \frac{\partial f}{\partial y}(1,\tfrac{\pi}{2})(y - \pi/2) +$$
$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(1,\pi/2)(x-1)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(1,\pi/2)(x-1)(y-\pi/2) + \frac{\partial^2 f}{\partial y^2}(1,\pi/2)(y-\pi/2)^2\right] + R_2(x,y).$$

Calculem: $\dfrac{\partial f}{\partial x} = y \cos xy$, $\dfrac{\partial f}{\partial y} = x \cos xy$, $\dfrac{\partial^2 f}{\partial x^2} = -y^2 \sin xy$, $\dfrac{\partial^2 f}{\partial x \partial y} = \cos xy - xy \sin xy$, $\dfrac{\partial^2 f}{\partial y^2} = -x^2 \sin xy$

Avaluem en $(1, \pi/2)$:

$f(1,\pi/2) = 1$, $\dfrac{\partial f}{\partial x}(1,\pi/2) = 0$, $\dfrac{\partial f}{\partial y}(\text{---}) = 0$, $\dfrac{\partial^2 f}{\partial x^2}(\text{---}) = -\dfrac{\pi^2}{4}$, $\dfrac{\partial^2 f}{\partial x \partial y}(\text{---}) = -\dfrac{\pi}{2}$, $\dfrac{\partial^2 f}{\partial y^2}(\text{---}) = -1$.

Taylor: $f(x,y) = 1 - \dfrac{\pi^2}{8}(x-1)^2 - \dfrac{\pi}{2}(x-1)(y-\pi/2) - \dfrac{1}{2}(y-\pi/2)^2 + R_2(x,y)$.

(b) $f(x,y) = x^y$, entorn del punt $(1,1)$.

$$\frac{\partial f}{\partial x} = y x^{y-1}, \quad \frac{\partial f}{\partial y} = x^y \ln x, \quad \frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2}, \quad \frac{\partial^2 f}{\partial x \partial y} = x^{y-1}(1 + y \ln x), \quad \frac{\partial^2 f}{\partial y^2} = x^y (\ln x)^2$$

$f(1,1) = 1$, $\dfrac{\partial f}{\partial x}(1,1) = 1$, $\dfrac{\partial f}{\partial y}(\text{---}) = 0$, $\dfrac{\partial^2 f}{\partial x^2}(\text{---}) = 0$, $\dfrac{\partial^2 f}{\partial x \partial y}(\text{---}) = 1$, $\dfrac{\partial^2 f}{\partial y^2}(\text{---}) = 0$

Taylor: $f(x,y) = 1 + (x-1) + (x-1)(y-1) + R_2(x,y)$.

(c) $f(x,y) = e^{x/y}$, entorn del punt $(0,1)$.

$$\frac{\partial f}{\partial x} = \frac{1}{y} e^{x/y}, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2} e^{x/y}, \quad \frac{\partial^2 f}{\partial x^2} = \frac{1}{y^2} e^{x/y}, \quad \frac{\partial^2 f}{\partial x \partial y} = -\frac{x+y}{y^3} e^{x/y}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{2xy+x^2}{y^4} e^{x/y}$$

$$f(0,1) = 1, \quad \frac{\partial f}{\partial x}(0,1) = 1, \quad \frac{\partial f}{\partial y}(\dots) = 0, \quad \frac{\partial^2 f}{\partial x^2}(\dots) = 1, \quad \frac{\partial^2 f}{\partial x \partial y}(\dots) = -1, \quad \frac{\partial^2 f}{\partial y^2}(\dots) = 0$$

Taylor: $f(x,y) = 1 + x + \frac{1}{2}x^2 - x(y-1) + R_2(x,y)$.

(d) $f(x,y,z) = e^{-x} \sin yz$, entorn del punt $(0,1,\pi)$.

$$\frac{\partial f}{\partial x} = -f, \quad \frac{\partial f}{\partial y} = z e^{-x} \cos yz, \quad \frac{\partial f}{\partial z} = y e^{-x} \cos yz$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\partial f}{\partial x} = f, \quad \frac{\partial^2 f}{\partial x \partial y} = -\frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial z} = -\frac{\partial f}{\partial z}, \quad \frac{\partial^2 f}{\partial y^2} = -z^2 e^{-x} \sin yz,$$

$$\frac{\partial^2 f}{\partial y \partial z} = e^{-x}(\cos yz - yz \sin yz), \quad \frac{\partial^2 f}{\partial z^2} = -y^2 e^{-x} \sin yz.$$

$$f(0,1,\pi) = 0, \quad \frac{\partial f}{\partial x}(0,1,\pi) = 0, \quad \frac{\partial f}{\partial y}(\dots) = -\pi, \quad \frac{\partial f}{\partial z}(\dots) = -1,$$

$$\frac{\partial^2 f}{\partial x^2}(\dots) = 0, \quad \frac{\partial^2 f}{\partial x \partial y}(\dots) = \pi, \quad \frac{\partial^2 f}{\partial x \partial z}(\dots) = 1, \quad \frac{\partial^2 f}{\partial y^2}(\dots) = 0, \quad \frac{\partial^2 f}{\partial y \partial z}(\dots) = -1, \quad \frac{\partial^2 f}{\partial z^2}(\dots) = 0.$$

Taylor:
$$f(x,y,z) = -\pi(y-1) - (z-\pi) + \pi x(y-1) + x(z-\pi) - (y-1)(z-\pi) + R_2(x,y,z).$$

29. Mitjançant l'ús de desenvolupaments de Taylor de funcions d'una variable (coneguts a priori), calculeu els desenvolupaments de Taylor a l'origen fins a termes de grau 2 (inclosos) de les funcions següents:

(a) $f(x,y) = e^{xy} \ln(1+x+y)$.

Calcularem el desenvolupament de Taylor per generació (usant desenvolupaments coneguts).

En primer lloc, desenvolupem cada factor:

$$e^{xy} = 1 + xy + \underbrace{\frac{(xy)^2}{2!} + \dots}_{(\text{grau} \geq 4)} = 1 + xy + O_4(x,y)$$

$$\ln(1+x+y) = x + y - \frac{(x+y)^2}{2} + O_3(x,y) =$$
$$= x + y - \frac{x^2}{2} - xy - \frac{y^2}{2} + O_3(x,y)$$

Fem el producte, i seleccionem els termes fins a grau 2:

$$f(x,y) = (1 + xy + O_4(x,y)) \cdot \left(x + y - \frac{x^2}{2} - xy - \frac{y^2}{2} + O_3(x,y)\right) =$$
$$= x + y - \frac{x^2}{2} - xy - \frac{y^2}{2} + O_3(x,y).$$

Desenvolupaments de Taylor bàsics d'una variable:

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^k}{k!} + O_{k+1}(x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + O_{2k+3}(x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^k \frac{x^{2k}}{(2k)!} + O_{2k+2}(x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{k+1} \frac{x^k}{k} + O_{k+1}(x)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \dots + (-1)^k x^k + O_{k+1}(x)$$

$$(1+x)^\alpha = 1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \dots + \binom{\alpha}{k}x^k + O_{k+1}(x)$$

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + (-1)^k \frac{x^{2k+1}}{2k+1} + O_{2k+3}(x)$$

(b) $f(x,y) = e^x \cdot \cos y = \left(1 + x + \frac{x^2}{2!} + O_3(x)\right) \cdot \left(1 - \frac{y^2}{2!} + O_4(y)\right) = 1 + x + \frac{x^2}{2} - \frac{y^2}{2} + O_3(x,y)$

(c)

$$f(x,y)=\frac{1}{1+x+y}=1-(x+y)+(x+y)^2+o_3(x,y)=1-x-y+x^2+2xy+y^2+o_3(x,y).$$

(d)

$$f(x,y,z)=e^{x+y}\cdot\sqrt{1+x}\cdot\cos(x+y+z)$$

$$e^{x+y}=1+x+y+\frac{(x+y)^2}{2!}+o_3(x,y)=1+x+y+\frac{x^2}{2}+xy+\frac{y^2}{2}+o_3(x,y).$$

$$\sqrt{1+x}=(1+x)^{1/2}=1+\binom{1/2}{1}x+\binom{1/2}{2}x^2+o_3(x)=1+\frac{x}{2}-\frac{x^2}{8}+o_3(x).$$

$$\binom{\alpha}{k}=\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!},$$
$$\Rightarrow \binom{1/2}{1}=1,\ \binom{1/2}{2}=-\frac{1}{8}.$$

$$\cos(x+y+z)=1-\frac{(x+y+z)^2}{2!}+o_4(x,y,z)=$$
$$=1-\frac{x^2}{2}-xy-xz-\frac{y^2}{2}-yz-\frac{z^2}{2}+o_4(x,y,z).$$

Fent el producte dels 3 factors, obtenim:

$$f(x,y,z)=1+\frac{3}{2}x+y+\frac{3}{8}x^2+\frac{xy}{2}-xz-yz-\frac{z^2}{2}+o_3(x,y,z)$$

30 Calculeu el desenvolupament de Taylor a l'origen de les següents funcions fins l'ordre que s'indica en cada cas i doneu el valor de totes les derivades parcials de la funció en el $(0,0)$ corresponents a l'ordre màxim fins al qual s'ha desenvolupat (p.ex., si desenvolupem fins a ordre 5 volem $\frac{\partial^5 f}{\partial x^n\partial y^m}(0,0)$ amb $n+m=5$)

(a) $f(x,y)=\ln(1+x^2-y)$ fins a ordre 3.

$$f(x,y)=(x^2-y)-\frac{(x^2-y)^2}{2}+\frac{(x^2-y)^3}{3}+o_4(x,y)=x^2-y-\frac{x^4-2x^2y+y^2}{2}+\frac{x^6-3x^4y+3x^2y^2-y^3}{3}+o_4(x,y)=$$
$$=-y+x^2-\frac{y^2}{2}+\underbrace{x^2y-\frac{y^3}{3}}+o_4(x,y)$$

la part de grau 3 és:

$$\frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}(0,0)x^3+3\frac{\partial^3 f}{\partial x^2\partial y}(0,0)x^2y+3\frac{\partial^3 f}{\partial x\partial y^2}(0,0)xy^2+\frac{\partial^3 f}{\partial y^3}(0,0)y^3\right)$$

amb els coeficients $\binom{3}{0}=1,\ \binom{3}{1}=3,\ \binom{3}{2}=3,\ \binom{3}{3}=1.$

$$\Rightarrow \frac{\partial^3 f}{\partial x^3}(0,0)=0,\quad \frac{\partial^3 f}{\partial x^2\partial y}(0,0)=2,\quad \frac{\partial^3 f}{\partial x\partial y^2}(0,0)=0,\quad \frac{\partial^3 f}{\partial y^3}(0,0)=-2$$

(b) $f(x,y)=\cos xy$ fins a ordre 8.

$$f(x,y)=1-\frac{(xy)^2}{2!}+\frac{(xy)^4}{4!}+o_{12}(x,y)=1-\frac{x^2y^2}{2}+\frac{x^4y^4}{24}+o_{12}(x,y)$$

L'únic terme no nul de grau 8 és $\frac{x^4y^4}{24}=\frac{1}{8!}\binom{8}{4}\frac{\partial^8 f}{\partial x^4\partial y^4}(0,0)\cdot x^4y^4=\frac{1}{24^2}\frac{\partial^8 f}{\partial x^4\partial y^4}(0,0)$

$$\Rightarrow \frac{\partial^8 f}{\partial x^4\partial y^4}(0,0)=\underline{24},$$ i les altres derivades d'ordre 8 valen 0.

(c) $f(x,y) = e^{x^2-y^2}$ fins a ordre 8

$$f(x,y) = 1 + (x^2-y^2) + \frac{(x^2-y^2)^2}{2!} + \frac{(x^2-y^2)^3}{3!} + \frac{(x^2-y^2)^4}{4!} + O_{10}(x,y).$$

Els termes de grau 8 són:

$$\frac{(x^2-y^2)^4}{4!} = \frac{1}{24}(x^8 - 4x^6y^2 + 6x^4y^4 - 4x^2y^6 + y^8) =$$

$$= \frac{1}{8!}\left[\binom{8}{0}\frac{\partial^8 f}{\partial x^8}(0,0)\cdot x^8 + \binom{8}{1}\frac{\partial^8 f}{\partial x^7 \partial y}(0,0)\, x^7 y + \cdots\cdots + \binom{8}{8}\frac{\partial^8 f}{\partial y^8}(0,0)\, y^8\right].$$

$\Rightarrow$ les derivades d'ordre 8 no nul·les són:

$$\left[\begin{array}{l} \frac{\partial^8 f}{\partial x^8}(0,0) = \frac{1}{24}\frac{8!}{\binom{8}{0}} = \frac{8!}{24} = \underline{1680}, \quad \frac{\partial^8 f}{\partial x^6 \partial y^2}(0,0) = -\frac{4}{24}\cdot\frac{8!}{\binom{8}{2}} = -\frac{1}{6}\cdot 2!\,6! = \underline{-240}, \\ \frac{\partial^8 f}{\partial x^4 \partial y^4}(0,0) = \frac{6}{24}\frac{8!}{\binom{8}{4}} = \frac{1}{4}\cdot 4!\,4! = \underline{144}, \quad \frac{\partial^8 f}{\partial x^2 \partial y^6}(0,0) = -\frac{4}{24}\frac{8!}{\binom{8}{6}} = \underline{-240}, \quad \frac{\partial^8 f}{\partial y^8}(0,0) = \frac{1}{24}\frac{8!}{\binom{8}{0}} = \underline{1680} \end{array}\right.$$

32 Determineu el valor de $\lambda \in \mathbb{R}$ per tal que

$$\lim_{(x,y)\to(0,0)} \frac{\operatorname{arctg}(x^2+y) - x^2 - y - \lambda y^3}{(x^2+y^2)^{3/2}} = 0.$$

Usem: si f és C^k i considerem el desenvolupament de Taylor de f al $(0,0)$ fins a grau k, $f(x,y) = P_k(x,y) + R_k(x,y)$, llavors la resta o residu compleix:

$$\lim_{(0,0)} \frac{R_k(x,y)}{\left(\sqrt{x^2+y^2}\right)^k} = 0.$$

Trobem el desenvolupament de Taylor fins a ordre 3 de la funció:

$$f(x,y) = \operatorname{arctg}(x^2+y) = (x^2+y) - \frac{(x^2+y)^3}{3} + O_5(x,y) = \underbrace{y + x^2 - \frac{y^3}{3}}_{P_3(x,y)} + \underbrace{O_4(x,y)}_{R_3(x,y)}.$$

- Si $\underline{\lambda = -1/3}$, tenim: $\displaystyle\lim_{(0,0)} \frac{\operatorname{arctg}(x^2+y) - x^2 - y + \frac{y^3}{3}}{(x^2+y^2)^{3/2}} = \lim_{(0,0)} \frac{R_3(x,y)}{(x^2+y^2)^{3/2}} = \underline{0}.$

- Si $\lambda \neq -1/3$, $\displaystyle\lim_{(0,0)} \frac{\operatorname{arctg}(x^2+y) - x^2 - y - \lambda y^3}{(x^2+y^2)^{3/2}} = \lim_{(0,0)} \frac{R_3(x,y) - \left(\frac{1}{3}+\lambda\right)y^3}{(x^2+y^2)^{3/2}} =$

$$= -\left(\frac{1}{3}+\lambda\right)\lim_{(0,0)} \frac{y^3}{(x^2+y^2)^{3/2}} \quad \nexists,$$ ja que si fem el límit sobre la recta $x=0$, tenim: $\displaystyle\frac{y^3}{(y^2)^{3/2}} = \frac{y^3}{|y|^3} = \begin{cases} 1 & \text{si } y>0 \\ -1 & \text{si } y<0 \end{cases}$

(33) Troben els candidats a extrems relatius de les funcions següents i classifiqueu-los.

(a) $f(x,y)=x^4+y^4-4xy+1$, és una funció C^∞ a tot $\mathbb{R}^2$.

Punts crítics (candidats a extrems relatius):

$$\begin{cases} \dfrac{\partial f}{\partial x}=4x^3-4y=0 \longrightarrow y=x^3 \\ \dfrac{\partial f}{\partial y}=4y^3-4x=0 \longrightarrow x=y^3=x^9 \longrightarrow \begin{cases} x=0 \longrightarrow y=0 \\ x^8=1 \longrightarrow x=\pm 1 \to y=\pm 1 \end{cases} \end{cases}$$

Obtenim els punts $(0,0),(1,1),(-1,-1)$.

Calculem la matriu hessiana: $H_f=\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}=\begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$

Tenim:

$Hf(0,0)=\begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix}$, $\det<0 \longrightarrow$ punt de sella

$Hf(1,1)=Hf(-1,-1)=\begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix}$, $\det>0, \operatorname{tr}>0 \to$ mínims relatius.

(b) $f(x,y)=x^2+xy+y^3-y^2-3x-2y+1$

$$\begin{cases} \dfrac{\partial f}{\partial x}=2x+y-3=0 \longrightarrow y=3-2x \\ \dfrac{\partial f}{\partial y}=x+3y^2-2y-2=0 \longrightarrow x+3(3-2x)^2-2(3-2x)-2=0. \end{cases}$$

$12x^2-31x+19=0$

$x=\begin{cases} 19/12 \to y=-\frac{1}{6} \\ 1 \to y=1 \end{cases}$

Punts crítics: $\left(\frac{19}{12},-\frac{1}{6}\right),(1,1)$

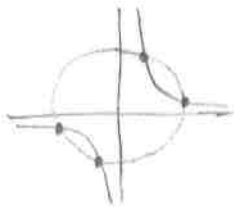
$H_f=\begin{pmatrix} 2 & 1 \\ 1 & 6y-2 \end{pmatrix}$,

$Hf\left(\frac{19}{12},-\frac{1}{6}\right)=\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$, $\det<0 \longrightarrow$ punt de sella

$Hf(1,1)=\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$, $\det>0, \operatorname{tr}>0 \to$ mínim relatiu.

(c) $f(x,y)=x^3+3xy^2-15x-12y$.

$$\begin{cases} \dfrac{\partial f}{\partial x}=3x^2+3y^2-15=0 \longrightarrow x^2+y^2=5 \\ \dfrac{\partial f}{\partial y}=6xy-12=0 \longrightarrow xy=2. \end{cases}$$



Fent $y=\frac{2}{x} \to x^2+\left(\frac{2}{x}\right)^2=5$

$x^4-5x^2+4=0$

$x^2=\begin{cases} 4 \to x=\pm 2 \to y=\pm 1 \\ 1 \to x=\pm 1 \to y=\pm 2 \end{cases}$

Tenim 4 punts crítics: $(2,1),(-2,-1),(1,2),(-1,-2)$.

$H_f=\begin{pmatrix} 6x & 6y \\ 6y & 6x \end{pmatrix} \longrightarrow H_f(\pm 2,\pm 1)=\begin{pmatrix} \pm 12 & \pm 6 \\ \pm 6 & \pm 12 \end{pmatrix}$, $\det>0$, $\begin{cases} \operatorname{tr}>0 \to (2,1) \text{ mínim relatiu} \\ \operatorname{tr}<0 \to (-2,-1) \text{ màxim relatiu} \end{cases}$

$H_f(\pm 1,\pm 2)=\begin{pmatrix} \pm 6 & \pm 12 \\ \pm 12 & \pm 6 \end{pmatrix}$, $\det<0 \longrightarrow (\pm 1,\pm 2)$ punts de sella.

(d) $f(x,y) = 8xy + \frac{1}{x} + \frac{1}{y}$ si $x,y>0$.

$$\begin{cases} \frac{\partial f}{\partial x} = 8y - \frac{1}{x^2} = 0 \longrightarrow y = \frac{1}{8x^2} \\ \frac{\partial f}{\partial y} = 8x - \frac{1}{y^2} = 0 \longrightarrow 8x - (8x^2)^2 = 0 \end{cases}$$

$$8x^4 - x = 0$$

$$x = \begin{cases} 0 \text{ (no pertany al domini)} \\ 1/2 \to y = \frac{1}{2} \Rightarrow \text{Punt crític: } \left(\frac{1}{2}, \frac{1}{2}\right). \end{cases}$$

$$H_f\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \begin{pmatrix} 2/x^3 & 8 \\ 8 & 2/y^3 \end{pmatrix}\Bigg|_{\left(\frac{1}{2},\frac{1}{2}\right)} = \begin{pmatrix} 16 & 8 \\ 8 & 16 \end{pmatrix}, \ \det>0,\ tr>0 \to \underline{\text{mínim relatiu}}$$

(e) $f(x,y) = xy + \frac{50}{x} + \frac{20}{y}$ si $x,y>0$

$$\begin{cases} \frac{\partial f}{\partial x} = y - \frac{50}{x^2} = 0 \longrightarrow y = \frac{50}{x^2} \\ \frac{\partial f}{\partial y} = x - \frac{20}{y^2} = 0 \longrightarrow x - 20\left(\frac{x^2}{50}\right)^2 = 0 \end{cases}$$

$$x - \frac{x^4}{125} = 0$$

$$x = \begin{cases} 0 \ (\underline{\text{no}}) \\ 5 \to y = 2 \Rightarrow \text{Punt crític: } (5,2) \end{cases}$$

$$H_f(5,2) = \begin{pmatrix} \frac{100}{x^3} & 1 \\ 1 & \frac{40}{y^3} \end{pmatrix}\Bigg|_{(5,2)} = \begin{pmatrix} 4/5 & 1 \\ 1 & 5 \end{pmatrix}, \ \det>0,\ tr>0 \to \underline{\text{mínim relatiu}}$$

(f) $f(x,y) = xy\ln(x^2+y^2)$, si $(x,y) \neq (0,0)$.

$$\left.\begin{cases} \frac{\partial f}{\partial x} = y\ln(x^2+y^2) + \frac{2x^2y}{x^2+y^2} = 0 \\ \frac{\partial f}{\partial y} = x\ln(x^2+y^2) + \frac{2xy^2}{x^2+y^2} = 0 \end{cases}\right\} \to x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} = \frac{2xy(x^2-y^2)}{x^2+y^2} = 0.$$

$\longrightarrow x=0$, o $y=0$, o $y=x$, o $y=-x$

- si $\underline{x=0}$, $y\ln(y^2) = 0 \longrightarrow \begin{cases} y=0 \ (\underline{\text{no}}) \\ \text{o} \\ \ln(y^2)=0 \to y^2=1 \to y=\pm1 \to \text{punts } (0,\pm1) \end{cases}$
- si $\underline{y=0}$, $x\ln(x^2)=0 \longrightarrow$ punts $(\pm1, 0)$
- si $\underline{y=x}$, $x\ln(2x^2) + \frac{2x^3}{2x^2} = 0$

  $x(1+\ln(2x^2)) = 0 \longrightarrow \begin{cases} x=0 \to y=0 \ (\underline{\text{no}}) \\ \ln(2x^2) = -1 \to x = \pm\frac{1}{\sqrt{2e}} \to \text{punts } \left(\pm\frac{1}{\sqrt{2e}}, \pm\frac{1}{\sqrt{2e}}\right) \end{cases}$
- si $\underline{y=-x}$, $-x\ln(2x^2) - \frac{2x^3}{2x^2} = 0 \longrightarrow x = \pm\frac{1}{\sqrt{2e}} \longrightarrow$ punts $\left(\pm\frac{1}{\sqrt{2e}}, \mp\frac{1}{\sqrt{2e}}\right)$

Tenim 8 punts crítics: $(0,\pm1)$, $(\pm1,0)$, $\left(\pm\frac{1}{\sqrt{2e}}, \pm\frac{1}{\sqrt{2e}}\right)$, $\left(\pm\frac{1}{\sqrt{2e}}, \mp\frac{1}{\sqrt{2e}}\right)$.

$$H_f = \begin{pmatrix} \dfrac{2xy(x^2+3y^2)}{(x^2+y^2)^2} & \ln(x^2+y^2) + \dfrac{2(x^4+y^4)}{(x^2+y^2)^2} \\ \ln(x^2+y^2) + \dfrac{2(x^4+y^4)}{(x^2+y^2)^2} & \dfrac{2xy(3x^2+y^2)}{(x^2+y^2)^2} \end{pmatrix}$$

$$H_f(0,\pm 1) = H_f(\pm 1, 0) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \det < 0 \longrightarrow \underline{\text{punts de sella}}$$

$$H_f\left(\pm \frac{1}{\sqrt{2e}}, \pm \frac{1}{\sqrt{2e}}\right) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \longrightarrow \underline{\text{mínims relatius}}$$

$$H_f\left(\pm \frac{1}{\sqrt{2e}}, \mp \frac{1}{\sqrt{2e}}\right) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \longrightarrow \underline{\text{màxims relatius}}$$

(g) $f(x,y,z) = \dfrac{x^2}{2} + xyz - z - y$

$$\begin{cases} \dfrac{\partial f}{\partial x} = x + yz = 0 \longrightarrow x = -yz \\ \dfrac{\partial f}{\partial y} = xz - 1 = 0 \\ \dfrac{\partial f}{\partial z} = xy - 1 = 0 \end{cases} \longrightarrow \begin{cases} yz^2 + 1 = 0 \longrightarrow y = -\dfrac{1}{z^2} \\ y^2 z + 1 = 0 \longrightarrow \dfrac{1}{z^3} + 1 = 0 \longrightarrow z = -1, \ y = -1, \ x = -1. \end{cases}$$

Punt crític: $(-1,-1,-1)$.

$$H_f(-1,-1,-1) = \begin{pmatrix} 1 & z & y \\ z & 0 & x \\ y & x & 0 \end{pmatrix}\Bigg|_{(-1,-1,-1)} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \quad \begin{matrix} \Delta_1 = 1 \\ \Delta_2 = -1 \\ \Delta_3 = -3 \end{matrix} \longrightarrow \underline{\text{punt de sella}}$$

(també podem usar que els vaps són $1, \pm\sqrt{3}$)

(h) $f(x,y,z) = x^2 + y^2 + z^2 + xy$.

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x + y = 0 \\ \dfrac{\partial f}{\partial y} = 2y + x = 0 \\ \dfrac{\partial f}{\partial z} = 2z = 0 \end{cases} \longrightarrow \text{Punt crític: } (0,0,0)$$

$$H_f(0,0,0) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \Delta_1, \Delta_2, \Delta_3 > 0 \longrightarrow \underline{\text{mínim relatiu}}$$

(els vaps són $1, 3, 2$)

48. Trobeu els màxims i mínims absoluts (si existeixen) per a les següents funcions de dues variables definides en els dominis D que s'indiquen. Justifiqueu la resposta en cada cas.

(a) $f(x,y) = (x^2+2y^2)\,e^{-(x^2+y^2)}$, $D=\mathbb{R}^2$.

Candidats a extrems absoluts (si n'hi ha):
- punts crítics a l'interior
- punts on la funció no és derivable
- extrems a la frontera.

La funció f és derivable a tot $\mathbb{R}^2$, i el domini $D=\mathbb{R}^2$ no té frontera ($\partial D=\emptyset$) $\longrightarrow$ els únics candidats són els punts crítics.

$$\begin{cases} \dfrac{\partial f}{\partial x} = \left(2x - 2x(x^2+2y^2)\right) e^{-(x^2+y^2)} = 0 \longrightarrow x=0 \text{ o } x^2+2y^2=1 \\ \dfrac{\partial f}{\partial y} = \left(4y - 2y(x^2+2y^2)\right) e^{-(x^2+y^2)} = 0 \longrightarrow y=0 \text{ o } x^2+2y^2=2 \end{cases}$$

- si $x=y=0 \longrightarrow$ el $(0,0)$
- si $x^2+2y^2=1$, $y=0 \longrightarrow (\pm1,0)$
- si $x=0$, $x^2+2y^2=2 \longrightarrow (0,\pm1)$
- si $x^2+2y^2=1$, $x^2+2y^2=2 \longrightarrow$ cap punt.

$\longrightarrow$ Tenim 5 punts crítics (no cal esbrinar si són extrems relatius o no).

Avaluem f en cadascun dels punts:
- $f(0,0)=0 \longrightarrow$ possible mínim absolut.
- $f(\pm1,0)=e^{-1}$
- $f(0,\pm1)=2e^{-1} \longrightarrow$ possibles màxims absoluts.

Hem d'esbrinar si els punts trobats són realment els extrems absoluts.

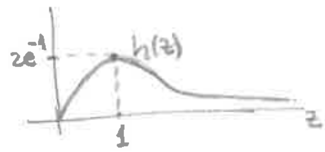
- f té mínim absolut en $(0,0)$, ja que $f(0,0)=0$, $f(x,y)>0 \;\; \forall (x,y)\neq(0,0)$

- Podem esperar que f tingui $(0,\pm1)$ com a màxims absoluts, ja que $f(x,y) \longrightarrow 0$ quan $(x,y)\to\infty$, i tenim $f(0,\pm1)>0$.

  Per comprovar-ho rigorosament, notem que per a tot $(x,y)\in\mathbb{R}^2$ tenim:

$$f(x,y) \underset{(1)}{\leq} 2(x^2+y^2)\,e^{-(x^2+y^2)} = h(z) \underset{(2)}{\leq} 2e^{-1} = f(0,\pm1), \text{ amb } z=x^2+y^2,$$

  i definint la funció d'una variable $h(z)=2ze^{-z}$



  En (1) i (2) només tenim igualtats ($=$), quan $x=0$ i $z=1$, és a dir per a $(x,y)=(0,\pm1)$.

  Per tant, f té màxims absoluts en $(0,\pm1)$.

(b) $f(x,y) = x^2+y^2-x-y+1$, $D=\mathbb{R}^2$

Punts crítics: $\begin{cases} \frac{\partial f}{\partial x} = 2x-1 = 0 \\ \frac{\partial f}{\partial y} = 2y-1 = 0 \end{cases} \longrightarrow (1/2,1/2)$

mínim relatiu, ja que $H_f(1/2,1/2) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (vaps >0)

Escrivint $f(x,y) = (x-\frac{1}{2})^2+(y-\frac{1}{2})^2+\frac{1}{2}$, veiem que $f(1/2,1/2) = 1/2$

$f(x,y) > 1/2$ si $(x,y) \neq (1/2,1/2)$

$\Rightarrow$ f té el mínim absolut en $(1/2,1/2)$.

D'altra banda, $\nexists$ màxim absolut, ja que p.ex. $f(x,0) = x^2-x+1 \longrightarrow \infty$ si $x \to \pm\infty$

(c) $f(x,y) = x^2+y^2-x-y+1$, $D = \{(x,y) \in \mathbb{R}^2 ; x^2+y^2 \leq 1\}$.

Tenim f contínua i D compacte $\Rightarrow$ $\exists$ mínim i màxim absoluts de f sobre D.

Candidats a extrems absoluts:

- Punts crítics: $(1/2,1/2)$ (per l'apartat (b)) $\longrightarrow$ és mínim absolut, ja que ho és a tot $\mathbb{R}^2$

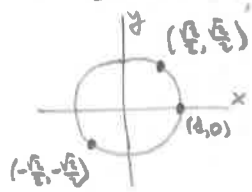
- Frontera: és la circumferència $x^2+y^2=1$.

  Podem estudiar la funció f a la frontera, considerant una funció d'una variable. Escrivint els punts de la frontera com a $x=\cos t$, $y = \sin t$, definim $h(t) := f(\cos t, \sin t) = \cos^2 t + \sin^2 t - \cos t - \sin t + 1 = 2 - \cos t - \sin t$, $0 \leq t \leq 2\pi$.

  Busquem possibles extrems de $h(t)$ a l'interval $[0, 2\pi]$:

  $h'(t) = \sin t - \cos t = 0 \longrightarrow t = \frac{\pi}{4}, \frac{5\pi}{4} \longrightarrow$ punts $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$

  i hi afegim $t=0$ i $t=2\pi$ $\longrightarrow$ punt $(1,0)$



Avaluem:

$f(1/2,1/2) = 1/2 \rightarrow$ mínim absolut.

$f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 2-\sqrt{2}$

$f(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = 2+\sqrt{2} \rightarrow$ màxim absolut

$f(1,0) = 1$

Obs. $f(x,y) = g(x,y)^2 + \frac{1}{2}$, essent $g(x,y) = \sqrt{(x-\frac{1}{2})^2+(y-\frac{1}{2})^2}$ = dist. de (x,y) a $(1/2,1/2)$

$\Rightarrow$ hem trobat que el punt més llunyà a $(1/2,1/2)$, en el disc D, és $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

(d) $f(x,y) = \sin x + \cos y$, $D = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le 2\pi,\ 0 \le y \le 2\pi\}$

f contínua, D compacte $\Rightarrow$ $\exists$ mínim i màxim absoluts.

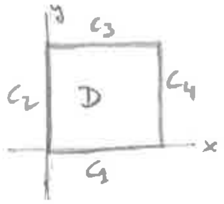
- Punts crítics (a l'interior: $0 < x < 2\pi$, $0 < y < 2\pi$).

$$\left\{ \begin{array}{l} \dfrac{\partial f}{\partial x} = \cos x = 0 \longrightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \\ \dfrac{\partial f}{\partial y} = -\sin y = 0 \longrightarrow y = \pi \end{array} \right\}$$

Punts: $\left(\frac{\pi}{2}, \pi\right)$, $\left(\frac{3\pi}{2}, \pi\right)$

(no cal esbrinar si són extrems relatius).

- Frontera: està formada per 4 segments, $\partial D = C_1 \cup C_2 \cup C_3 \cup C_4$



$C_1 = \{y = 0,\ 0 \le x \le 2\pi\}$
$C_2 = \{x = 0,\ 0 \le y \le 2\pi\}$
$C_3 = \{y = 2\pi,\ 0 \le x \le 2\pi\}$
$C_4 = \{x = 2\pi,\ 0 \le y \le 2\pi\}$

Hem de buscar possibles extrems sobre cada segment, usant funcions d'una variable. En aquest cas, com que f és 2π-periòdica en x i en y, n'hi haurà prou amb 2 funcions.

C_1, C_3: $h_1(x) = f(x,0) = f(x,2\pi) = \sin x + 1$, $0 \le x \le 2\pi$.

$h_1'(x) = \cos x = 0 \longrightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \longrightarrow$ punts $\left(\frac{\pi}{2},0\right), \left(\frac{3\pi}{2},0\right), \left(\frac{\pi}{2},2\pi\right), \left(\frac{3\pi}{2},2\pi\right)$.

Afegim $x = 0, 2\pi \longrightarrow$ els 4 vèrtexs $(0,0), (2\pi,0), (0,2\pi), (2\pi,2\pi)$

C_2, C_4: $h_2(y) = f(0,y) = f(2\pi,y) = \cos y$, $0 \le y \le 2\pi$.

$h_2'(y) = -\sin y = 0 \longrightarrow y = \pi \longrightarrow$ punts $(0,\pi), (2\pi,\pi)$.

Afegint $y = 0, 2\pi \longrightarrow$ els 4 vèrtexs.

Avaluem: $f\left(\frac{\pi}{2},\pi\right) = 0$, $f\left(\frac{3\pi}{2},\pi\right) = -2$ | $f\left(\frac{\pi}{2},0\right) = f\left(\frac{\pi}{2},2\pi\right) = 2$, $f\left(\frac{3\pi}{2},0\right) = f\left(\frac{3\pi}{2},2\pi\right) = 0$ | $f(0,\pi) = f(2\pi,\pi) = -1$.

$f(0,0) = f(2\pi,0) = f(0,2\pi) = f(2\pi,2\pi) = 1$

$\Rightarrow$ f té mínim absolut en $\left(\frac{3\pi}{2},\pi\right)$, i màxims absoluts en $\left(\frac{\pi}{2},0\right)$, $\left(\frac{\pi}{2},2\pi\right)$.

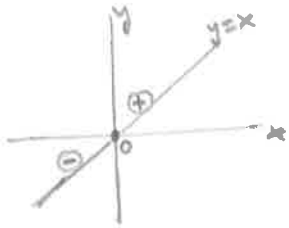
49 Per a la funció $f(x,y) = x^2y + y^2x$ veiem que $(0,0)$ és un candidat a extrem relatiu però que el mètode del hessià no permet caracteritzar-lo. A quina conclusió arribem si restringim els valors de (x,y) als de la recta $y = x$?

$\frac{\partial f}{\partial x} = 2xy + y^2$, $\frac{\partial f}{\partial y} = x^2 + 2yx \leadsto \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0 \longrightarrow f$ té un punt crític en $(0,0)$.

$$H_f(0,0) = \left.\begin{pmatrix} 2y & 2x+2y \\ 2x+2y & 2x \end{pmatrix}\right|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \det = 0 \Rightarrow$$ no podem classificar el punt crític usant només la matriu hessiana.

Restringint f a la recta $y = x$,

$f(0,0) = 0$, $\quad f(x,x) = 2x^3 \begin{cases} > f(0,0) & \text{si } x > 0 \\ < f(0,0) & \text{si } x < 0 \end{cases}$



$\Rightarrow$ El punt crític $(0,0)$ no és extrem relatiu.