INTEGRACIÓ DE FUNCIONS DE VÀRIES VARIABLES

1 Usant el teorema del valor mitjà per a integrals proveu les següents desigualtats.

(a) $4e^5 \le \iint_A e^{x^2+y^2}\,dxdy \le 4e^{25}$, on $A = [1,3]\times[2,4]$

> Teo. valor mitjà per a integrals (en 2 variables):
> D domini elemental, f integrable sobre D
> (a) si $m \le f(x,y) \le M \quad \forall (x,y)\in D$
> $\Rightarrow m \cdot \text{àrea}(D) \le \int_D f \le M \cdot \text{àrea}(D)$
> (b) si f contínua sobre D, i D és connex,
> $\Rightarrow \int_D f = f(p)\cdot \text{àrea}(D)$, per algun $p \in D$

$f(x,y) = e^{x^2+y^2}$

Si $(x,y)\in A$, tenim

$\underbrace{1^2+2^2}_{=5} \le x^2+y^2 \le \underbrace{3^2+4^2}_{=25}$

$\Rightarrow e^5 \le f(x,y) \le e^{25}$.

Tenim àrea $(A) = 4$

$\Rightarrow 4e^5 \le \int_A f \le 4e^{25}$

(b) $\frac{1}{e} \le \frac{1}{4\pi^2}\iint_A e^{\sin(x+y)}\,dxdy \le e$, on $A = [-\pi,\pi]\times[-\pi,\pi]$.

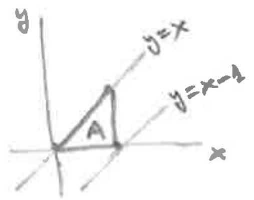
$f(x,y) = e^{\sin(x+y)}$

Si $(x,y)\in A$, tenim $-1 \le \sin(x+y) \le 1$ (màxim quan $x+y = \frac{\pi}{2}, -\frac{3\pi}{2}$; mínim quan $x+y = -\frac{\pi}{2}, \frac{3\pi}{2}$.)

$e^{-1} \le f(x,y) \le e$

àrea $(A) = (2\pi)^2 = 4\pi^2 \rightsquigarrow 4\pi^2 e^{-1} \le \int_A f \le 4\pi^2 e$

(c) $\frac{1}{6} \le \iint_A \frac{dx\,dy}{y-x+3} \le \frac{1}{4}$, on A és el triangle de vèrtexs $(0,0)$, $(1,1)$ i $(1,0)$.



$f(x,y) = \frac{1}{y-x+3}$

Si $(x,y)\in A$, $\quad -1 \le y-x \le 0$

$0 < 2 \le y-x+3 \le 3$

$\frac{1}{3} \le f(x,y) \le \frac{1}{2}$

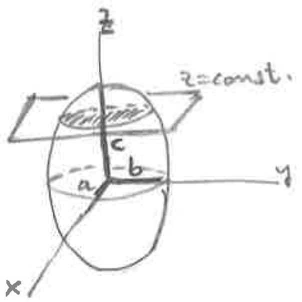
àrea $(A) = \frac{1}{2}$

$\Rightarrow \frac{1}{6} \le \int_A f \le \frac{1}{4}$

3 Apliqueu el principi de Cavalieri per calcular els següents volums a partir de l'àrea de seccions amb plans paral·lels als plans coordenats (triades de forma adequada).

(a) Volum envoltat per l'el·lipsoide $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

(a, b, c : semieixos)



$D = \{(x,y,z) : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1\}$

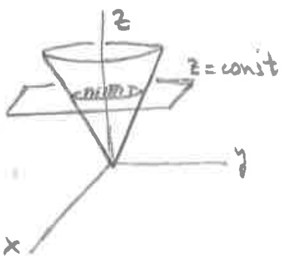
La secció per un pla $z = \text{const}$ $(-c \le z \le c)$, que té x, y com a coordenades, ens dóna una el·lipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1-\frac{z^2}{c^2} \rightsquigarrow \frac{x^2}{\alpha(z)^2}+\frac{y^2}{\beta(z)^2} \le 1$,

amb semieixos $\alpha(z) = a\sqrt{1-\frac{z^2}{c^2}}$, $\beta(z) = b\sqrt{1-\frac{z^2}{c^2}}$ $\Rightarrow$ Àrea secció: $A(z) = \pi\alpha(z)\beta(z) = \pi ab\left(1-\frac{z^2}{c^2}\right)$

Pel principi de Cavalieri,

$$\text{vol}(D) = \int_{-c}^{c} A(z)\,dz = \pi ab\int_{-c}^{c}\left(1-\frac{z^2}{c^2}\right)dz = \pi ab\left(z-\frac{z^3}{3c^2}\right)\Big]_{-c}^{c} = \frac{4}{3}\pi abc.$$

(b) Volum envoltat pel con invertit de base el·líptica $\frac{x^2}{a^2}+\frac{y^2}{b^2}=z^2$, amb $0\le z\le h$.



Secció per $z=\text{const}$ $(0\le z\le h)$: el·lipse $\frac{x^2}{(az)^2}+\frac{y^2}{(bz)^2}=1$.

Àrea secció:

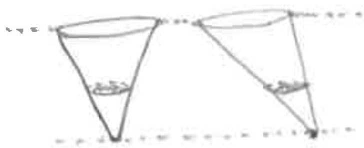
$$A(z)=\pi\cdot az\cdot bz=\pi ab z^2.$$

$$\longrightarrow \text{Volum} = \int_0^h A(z)\,dz = \pi ab\int_0^h z^2\,dz = \frac{\pi ab h^3}{3}.$$

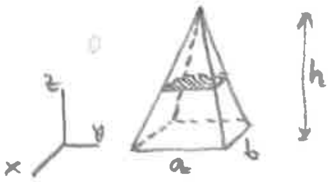
Obs. [La base del con, que obtenim per a $z=h$, és l'el·lipse $\frac{x^2}{(ah)^2}+\frac{y^2}{(bh)^2}=1$.

$\longrightarrow$ àrea de la base: $B=\pi abh^2$, alçada: h, volum $=\frac{Bh}{3}$.

Obs. [Qualsevol con (oblic) de la mateixa base i alçada, tindrà el mateix volum (ja que les seccions són les mateixes).



(c) Volum de la piràmide de base rectangular de costats a,b i alçada h.



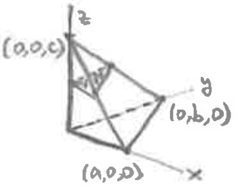
La secció per $z=\text{const.}$ ens dóna un rectangle de costats $\alpha(z)$, $\beta(z)$, que depenen linealment de z, amb $\alpha(0)=a$, $\beta(0)=b$, $\alpha(h)=\beta(h)=0$.

$$\longrightarrow \alpha(z)=a\left(1-\frac{z}{h}\right),\quad \beta(z)=b\left(1-\frac{z}{h}\right)$$

Àrea secció: $A(z)=\alpha(z)\beta(z)=ab\left(1-\frac{z}{h}\right)^2$

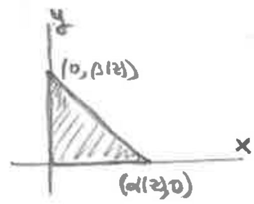
$$\text{Volum} = \int_0^h A(z)\,dz = ab\int_0^h\left(1-\frac{z}{h}\right)^2 dz = -abh\left.\frac{\left(1-\frac{z}{h}\right)^3}{3}\right]_0^h = \frac{abh}{3}$$

(d) Volum del tetràedre limitat pels plans $x=0$, $y=0$, $z=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ $(a,b,c>0)$



Secció per $z=\text{const.}$ $(0\le z\le c)$: triangle limitat per les rectes

$x=0,\ y=0,\ \frac{x}{a}+\frac{y}{b}=1-\frac{z}{c}$



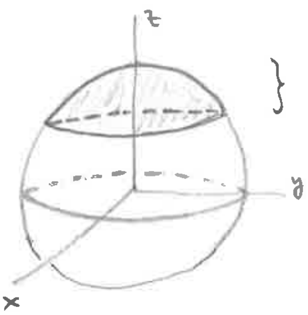
$$\frac{x}{\alpha(z)}+\frac{y}{\beta(z)}=1,\quad \text{amb } \alpha(z)=a\left(1-\frac{z}{c}\right),\ \beta(z)=b\left(1-\frac{z}{c}\right)$$

$\longrightarrow$ Àrea secció: $A(z)=\frac{ab}{2}\left(1-\frac{z}{c}\right)^2$.

$$\text{Volum} = \int_0^c A(z)\,dz = \frac{ab}{2}\int_0^c\left(1-\frac{z}{a}\right)^2 dz = \frac{abh}{6}$$

(Obs: volum $=\frac{Bc}{3}$, $B=\frac{ab}{2}$ àrea base, c alçada)

(e) Volum envoltat pel casquet esfèric determinat per l'esfera $x^2+y^2+z^2=R^2$ i la condició $R-h\le z\le R$.



$D=\{(x,y,z):\ x^2+y^2+z^2\le R^2,\ R-h\le z\le R\}$

Secció per $z=\text{const.}$ $(R-h\le z\le R)$: $x^2+y^2\le R^2-z^2$, cercle de radi $\alpha(z)=\sqrt{R^2-z^2}$

Àrea secció: $A(z)=\pi\alpha(z)^2=\pi(R^2-z^2)$

$$\text{vol}(D)=\int_{R-h}^{R}A(z)\,dz=\pi\left[R^2z-\frac{z^3}{3}\right]_{R-h}^{R}=\pi\left(R^3-\frac{R^3}{3}-R^2(R-h)+\frac{(R-h)^3}{3}\right)=$$

$$=\pi\left(R^2h+\frac{-3R^2h+3Rh^2-h^3}{3}\right)=\frac{\pi}{3}h^2(3R-h)$$

⑤ Troben les següents integrals dobles en els rectangles que s'indiquen.

(a) $\iint_R x^2 y \, dx\, dy$, $R=[0,1]\times[0,1]$.

$$I = \int_0^1 x^2 dx \cdot \int_0^1 y\, dy = \left[\frac{x^3}{3}\right]_{x=0}^{x=1} \cdot \left[\frac{y^2}{2}\right]_{y=0}^{y=1} = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

Nota. En general, sempre que la funció té "variables separades" $f(x,y)=g(x)\cdot h(y)$, i si el domini és un rectangle, $R=[a,b]\times[c,d]$, podem calcular la integral doble com un producte d'integrals simples:

$$\iint_R f(x,y)\,dx\,dy = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy.$$

(b) $\iint_R \frac{x^2}{1+y^2}\, dx\, dy$, $R=[0,1]\times[0,1]$.

$$I = \int_0^1 x^2 dx \cdot \int_0^1 \frac{dy}{1+y^2} = \left[\frac{x^3}{3}\right]_{x=0}^{x=1} \cdot \left[\operatorname{arctg} y\right]_{y=0}^{y=1} = \frac{1}{3}\cdot\frac{\pi}{4} = \frac{\pi}{12}$$

(c) $\iint_R y \ln x \, dx\, dy$, $R=[1,e]\times[1,e]$.

$$I = \int_1^e \ln x\, dx \cdot \int_1^e y\, dy = \left[x(\ln x - 1)\right]_{x=1}^{x=e} \cdot \left[\frac{y^2}{2}\right]_{y=1}^{y=e} = \frac{e^2-1}{2}$$

(d) $\iint_R (x^2+y)\,dx\,dy$, $R=[0,1]\times[0,2]$.

$$I = \iint_R x^2\,dx\,dy + \iint_R y\,dx\,dy = \int_0^1 x^2 dx \cdot \int_0^2 dy + \int_0^1 dx \cdot \int_0^2 y\,dy = \frac{1}{3}\cdot 2 + 1\cdot 2 = \frac{8}{3}$$

(e) $\iint_R \frac{1}{(x+2y)^2}\,dx\,dy$, $R=[2,5]\times[1,3]$.

Pel teo. de Fubini, ho podem calcular mitjançant una integral iterada:

$$I = \int_1^3 dy \int_2^5 \frac{dx}{(x+2y)^2} = \int_1^3 dy \cdot \left[-\frac{1}{x+2y}\right]_{x=2}^{x=5} =$$

$$= \int_1^3 \left(\frac{1}{2+2y} - \frac{1}{5+2y}\right) dy = \left.\frac{\ln(1+y) - \ln(\frac{5}{2}+y)}{2}\right]_1^3 = \frac{1}{2}\ln\frac{14}{11}.$$

Notació (integrals iterades)

$$\int_a^b \left(\int_c^d f(x,y)\,dy\right) dx = \int_a^b dx \int_c^d f(x,y)\,dy$$

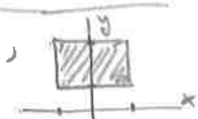
$$\int_c^d \left(\int_a^b f(x,y)\,dx\right) dy = \int_c^d dy \int_a^b f(x,y)\,dx$$

(f) $\iint_R e^y \sin\frac{x}{y}\,dx\,dy$, $R=[-\frac{\pi}{2},\frac{\pi}{2}]\times[1,2]$

$$I = \int_1^2 dy \int_{-\pi/2}^{\pi/2} e^y \sin\frac{x}{y}\,dx = \int_1^2 e^y\,dy \int_{-\pi/2}^{\pi/2} \sin\frac{x}{y}\,dx = \int_1^2 e^y dy \left[-y\cos\frac{x}{y}\right]_{x=-\pi/2}^{x=\pi/2} = 0,$$

ja que $\cos\frac{\pi}{2y} = \cos\left(-\frac{\pi}{2y}\right)$.

Obs. De fet, la integral és 0 ja que $f(x,y)=e^y \sin\frac{x}{y}$ és una funció senar en x, i el domini R és simètric respecte l'eix X.



(g) $\iint_R (x+y)^{27}\,dx\,dy$, $R=[-1,1]\times[-1,1]$

$$I = \int_{-1}^1 dx \int_{-1}^1 (x+y)^{27} dy = \int_{-1}^1 dx \left[\frac{(x+y)^{28}}{28}\right]_{y=-1}^{y=1} = \frac{1}{28}\int_{-1}^1 \left((x+1)^{28} - (x-1)^{28}\right) dx = 0$$

funció senar

⑥ Calculeu $\iint_R x^y\,dx\,dy$ en $R=[0,1]\times[a,b]$, essent $0<a<b$, i deduïu el valor de la integral $\int_0^1 \frac{x^b-x^a}{\ln x}dx$.

$$I=\iint_R x^y\,dx\,dy=\int_a^b dy\int_0^1 x^y\,dx=\int_a^b dy\left[\frac{x^{y+1}}{y+1}\right]_{x=0}^{x=1}=\int_a^b\frac{dy}{y+1}=\ln(y+1)\Big]_a^b=\ln\frac{b+1}{a+1}$$

(Obs. $y\neq -1$ a l'interval $[a,b]$) ($0^{y+1}=0$ ja que $y+1>0$)

D'altra banda,

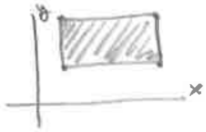
$$I=\int_0^1 dx\int_a^b x^y\,dy=\int_0^1 dx\left[\frac{x^y}{\ln x}\right]_{y=a}^{y=b}=\int_0^1\frac{x^b-x^a}{\ln x}dx=$$

(en $x=0$, seria 0) (no és impròpia en $x=0$ ja que $\lim_{x\to 0^+}=0$)

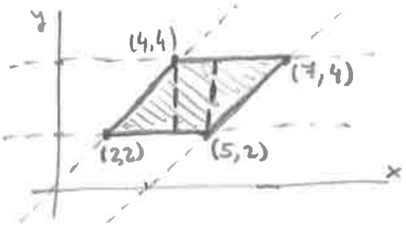
⑧ Per a les regions $A\subset\mathbb{R}^2$ indicades escriviu la integral doble $\iint_A f(x,y)\,dx\,dy$ en termes d'integrals iterades preses en diferents ordres, $\int\left(\int f\,dx\right)dy$ i $\int\left(\int f\,dy\right)dx$, donant quins són els extrems d'integració per a x i y en cada cas.

(a) A rectangle de vèrtexs $(1,2)$, $(5,2)$, $(5,4)$ i $(1,4)$.

$$\int_A f=\int_1^5 dx\int_2^4 f(x,y)\,dy=\int_2^4 dy\int_1^5 f(x,y)\,dx$$



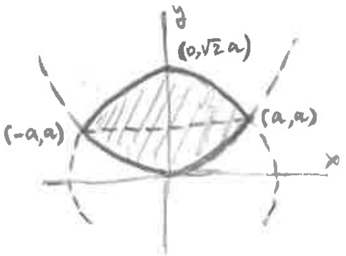
(b) A paral·lelogram limitat per les rectes $y=x$, $y=x-3$, $y=2$, $y=4$



$$\int_A f=\int_2^4 dy\int_y^{y+3} f(x,y)\,dx=$$

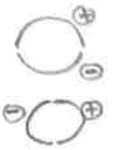
$$=\int_2^4 dx\int_2^x f(x,y)\,dy+\int_4^5 dx\int_2^4 f(x,y)\,dy+\int_5^7 dx\int_{x-3}^4 f(x,y)\,dy.$$

(c) A regió limitada per les corbes $x^2+y^2=2a^2$, $x^2=ay$ $(y\geq 0,\ a>0)$



Posem les corbes com a gràfiques $y(x)$ o $x(y)$:

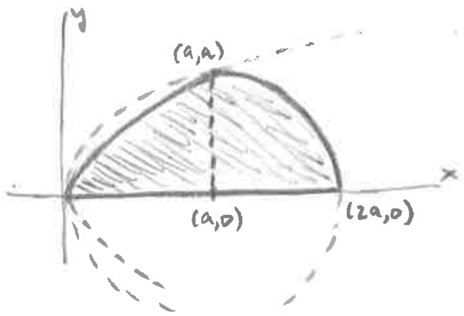
* circumferència: $x^2+y^2=2a^2$ ⟶ $y=\pm\sqrt{2a^2-x^2}$, $x=\pm\sqrt{2a^2-y^2}$



* paràbola: $x^2=ay$ ⟶ $y=x^2/a$, $x=\pm\sqrt{ay}$

$$\int_A f=\int_{-a}^a dx\int_{x^2/a}^{\sqrt{2a^2-x^2}} f(x,y)\,dy=\int_0^a dy\int_{-\sqrt{ay}}^{\sqrt{ay}} f(x,y)\,dx+\int_a^{\sqrt2 a} dy\int_{-\sqrt{2a^2-y^2}}^{\sqrt{2a^2-y^2}} f(x,y)\,dx$$

(d) A regió limitada per les corbes $y^2=ax$, $x^2+y^2=2ax$, $y=0$ $(y\geq 0,\ a>0)$



* paràbola: $y^2=ax \longrightarrow y=\pm\sqrt{ax}$

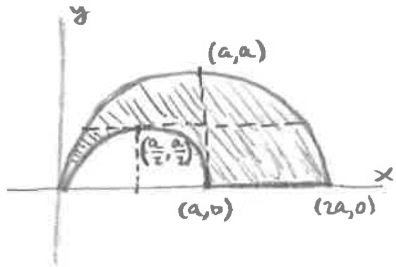
$\searrow x = y^2/a$.

* circumferència: $\left[\begin{array}{l} x^2+y^2=2ax \longrightarrow y=\pm\sqrt{2ax-x^2} \\ (x-a)^2+y^2=a^2 \longrightarrow x = a\pm\sqrt{a^2-y^2} \end{array}\right.$

[Obs. La paràbola queda a la dreta de la part esquerra de la circumferència: $a-\sqrt{a^2-y^2} = \dfrac{y^2}{a+\sqrt{a^2-y^2}} < \dfrac{y^2}{a}$, si $0<y<a$]

$$\int_A f = \int_0^a dy \int_{y^2/a}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx = \int_0^a dx \int_0^{\sqrt{ax}} f(x,y)\,dy + \int_a^{2a} dx \int_0^{\sqrt{2ax-x^2}} f(x,y)\,dy.$$

(e) A regió limitada per les corbes $x^2+y^2=ax$, $x^2+y^2=2ax$, $y=0$ $(y\geq 0,\ a>0)$



* circumferència: $\left[\begin{array}{l} x^2+y^2=ax \longrightarrow y=\pm\sqrt{ax-x^2} \\ \left(x-\frac{a}{2}\right)^2+y^2=\left(\frac{a}{2}\right)^2 \longrightarrow x=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-y^2} \end{array}\right.$

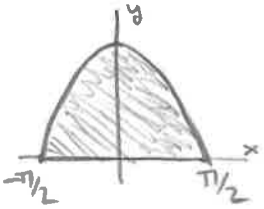
* circumferència: $\left[\begin{array}{l} x^2+y^2=2ax \longrightarrow y=\pm\sqrt{2ax-x^2} \\ (x-a)^2+y^2=a^2 \longrightarrow x=a\pm\sqrt{a^2-y^2}. \end{array}\right.$

$$\int_A f = \int_0^a dx \int_{\sqrt{ax-x^2}}^{\sqrt{2ax-x^2}} f(x,y)\,dy + \int_a^{2a} dx \int_0^{\sqrt{2ax-x^2}} f(x,y)\,dy =$$

$$= \int_0^{a/2} dy \left[\int_{a-\sqrt{a^2-y^2}}^{\frac{a}{2}-\sqrt{\frac{a^2}{4}-y^2}} f(x,y)\,dx + \int_{\frac{a}{2}+\sqrt{\frac{a^2}{4}-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx \right] + \int_{a/2}^{a} dy \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dx$$

⑨ Calculeu les següents integrals dobles en els dominis de $\mathbb{R}^2$ que s'indiquen.

(a) $\displaystyle\iint_A y^3\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2:\ \underbrace{-\tfrac{\pi}{2}\le x\le\tfrac{\pi}{2}}_{\text{(interval de x)}},\ \underbrace{0\le y\le 2\cos x}_{\text{(entre 2 gràfiques: } y=0 \text{ i } y=2\cos x)}\}$

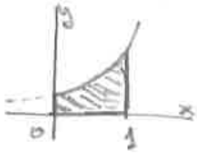


$$I=\int_{-\pi/2}^{\pi/2}dx\int_0^{2\cos x}y^3\,dy=\int_{-\pi/2}^{\pi/2}dx\cdot\left[\frac{y^4}{4}\right]_{y=0}^{y=2\cos x}=4\int_{-\pi/2}^{\pi/2}\cos^4 x\,dx=4\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)^2dx=$$

$$=\int_{-\pi/2}^{\pi/2}\Big(1+2\cos 2x+\underbrace{\cos^2 2x}_{=\frac{1+\cos 4x}{2}}\Big)dx=\int_{-\pi/2}^{\pi/2}\left(\frac{3}{2}+2\cos 2x+\frac{\cos 4x}{2}\right)dx=$$

$$=\frac{3}{2}x+\sin 2x+\frac{\sin 4x}{8}\Big]_{-\pi/2}^{\pi/2}=\frac{3\pi}{2}$$

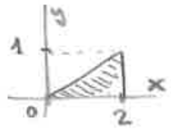
(b) $\displaystyle\iint_A x\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2:\ 0\le x\le 1,\ 0\le y\le e^x\}$



$$I=\int_0^1 dx\int_0^{e^x}(x)\,dy=\int_0^1 x\,dx\,[y]_{y=0}^{y=e^x}=\int_0^1 xe^x\,dx=\left[xe^x\right]_0^1-\int_0^1 e^x\,dx=1$$

(parts: $u=x,\ dv=e^x\,dx$)

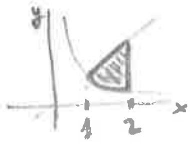
(c) $\displaystyle\iint_A xy\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2:\ 0\le x\le 2,\ 0\le y\le \frac{x}{2}\}$



$$I=\int_0^2 x\,dx\int_0^{x/2}y\,dy=\int_0^2 x\,dx\cdot\left[\frac{y^2}{2}\right]_{y=0}^{y=x/2}=\frac{1}{8}\int_0^2 x^3\,dx=\frac{1}{2}$$

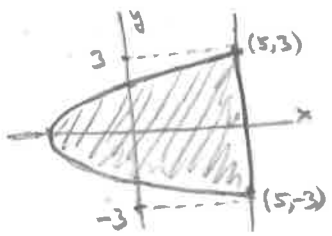
$\left(\text{o bé: } I=\int_0^1 y\,dy\int_{2y}^2 x\,dx=\cdots\right)$

(d) $\displaystyle\iint_A \frac{x^2}{y^2}\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2:\ 1\le x\le 2,\ \frac{1}{x}\le y\le x\}$



$$I=\int_1^2 x^2\,dx\int_{1/x}^{x}\frac{dy}{y^2}=\int_1^2 x^2\,dx\left[-\frac{1}{y}\right]_{y=1/x}^{y=x}=\int_1^2(x^3-x)\,dx=\frac{9}{4}$$

(e) $\displaystyle\iint_A (x+2y)\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2:\ -3\le y\le 3,\ y^2-4\le x\le 5\}$



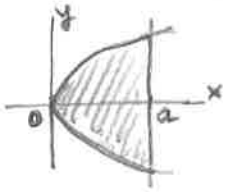
$$I=\int_{-3}^{3}dy\int_{y^2-4}^{5}(x+2y)\,dx=\int_{-3}^{3}dy\left[\frac{x^2}{2}+2xy\right]_{x=y^2-4}^{x=5}=$$

$$=\int_{-3}^{3}\left(\frac{25}{2}+10y-\frac{(y^2-4)^2}{2}-2(y^2-4)y\right)dy=\int_{-3}^{3}\left(\frac{9}{2}+18y+4y^2-2y^3-\frac{y^4}{2}\right)dy=$$

(la integral dels termes de grau senar és 0)

$$=\frac{9}{2}y+\frac{4}{3}y^3-\frac{y^5}{10}\Big]_{-3}^{3}=\frac{252}{5}$$

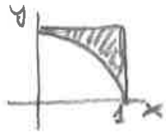
(f) $\iint_A y^3\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2:\ 0\le x\le a,\ y^2\le 2px\}$, $a>0$, $p>0$.



$$I=\int_0^a dx\int_{-\sqrt{2px}}^{\sqrt{2px}} y^3\,dy=\int_0^a dx\left[\frac{y^4}{4}\right]_{y=-\sqrt{2px}}^{y=\sqrt{2px}}=\underline{\underline{0}}$$

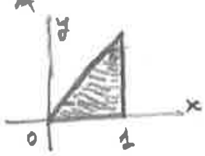
Obs. [Podem dir que la integral és 0 ja que $f(x,y)=y^3$ és una funció senar en y, i el domini A és simètric respecte la recta $y=0$.

(g) $\iint_A \frac{y}{1+x^3}\,dx\,dy$, $A=\{(x,y)\in\mathbb{R}^2:\ 0\le x\le 1,\ 0\le y\le 1,\ x^2+y^2\ge 1\}$



$$I=\int_0^1\frac{dx}{1+x^3}\int_{\sqrt{1-x^2}}^{1}y\,dy=\int_0^1\frac{dx}{1+x^3}\left[\frac{y^2}{2}\right]_{y=\sqrt{1-x^2}}^{y=1}=\frac{1}{2}\int_0^1\frac{x^2}{1+x^3}=\frac{1}{6}\ln(1+x^3)\Big]_0^1=\underline{\underline{\frac{\ln 2}{6}}}$$

(h) $\iint_A x^2\sin xy\,dx\,dy$. $A=\{(x,y)\in\mathbb{R}^2:\ 0\le y\le 1,\ y\le x\le 1\}$



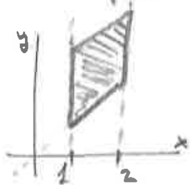
$$I=\int_0^1 x^2dx\int_0^x \sin xy\,dy=\int_0^1 x^2dx\left[-\frac{\cos xy}{x}\right]_{y=0}^{y=x}=\int_0^1 x(1-\cos x^2)\,dx=$$

$$=\frac{1}{2}(x^2-\sin x^2)\Big]_0^1=\underline{\underline{\frac{1}{2}(1-\sin 1)}}$$

(no té sentit si $x=0$, però això no afecta el resultat).

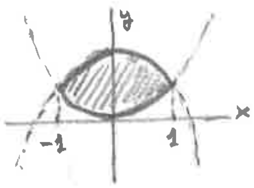
(10) Per a les integrals iterades següents escriviu les equacions de les corbes que limiten les regions d'integració i dibuixeu aquestes regions.

(a) $\int_1^2\left(\int_x^{x+3}f(x,y)\,dy\right)dx$



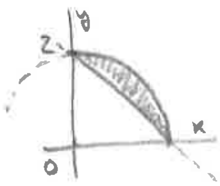
$x=1,\ x=2,\ y=x,\ y=x+3$

(b) $\int_{-1}^1\left(\int_{x^2}^{2-x^2}f(x,y)\,dy\right)dx$



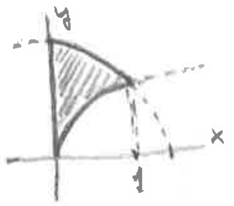
$y=x^2,\ y=2-x^2$

(c) $\int_0^2\left(\int_{2-y}^{\sqrt{4-y^2}}f(x,y)\,dx\right)dy$



$x+y=2,\ x^2+y^2=4$ (1r quadrant)

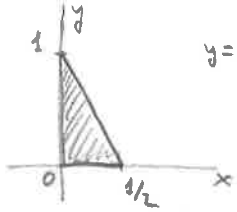
(d) $\int_0^1\left(\int_{\sqrt{x}}^{\sqrt{2-x^2}}f(x,y)\,dy\right)dx$



$x=0,\ x=y^2,\ x^2+y^2=2$ $(y\ge 0)$

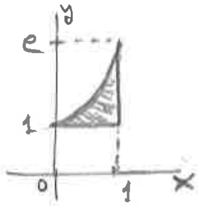
(11) Invertiu l'ordre d'integració en les integrals iterades següents.

(a) $$\int_0^{1/2}\left(\int_0^{1-2x} f(x,y)\,dy\right)dx = \int_0^1\left(\int_0^{(1-y)/2} f(x,y)\,dx\right)dy$$



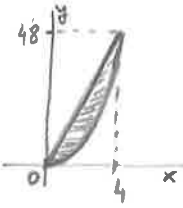
$y=1-2x \to x=\frac{1-y}{2}$

(b) $$\int_0^1\left(\int_1^{e^x} f(x,y)\,dy\right)dx = \int_1^e\left(\int_{\ln y}^1 f(x,y)\,dx\right)dy$$



$y=e^x \leadsto x=\ln y$

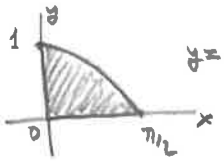
(c) $$\int_0^4\left(\int_{3x^2}^{12x} f(x,y)\,dy\right)dx = \int_0^{48}\left(\int_{y/12}^{\sqrt{y/3}} f(x,y)\,dx\right)dy$$



$y=3x^2 \longrightarrow x=\sqrt{y/3}$

$y=12x \longrightarrow x=y/12$

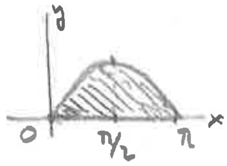
(d) $$\int_0^{\pi/2}\left(\int_0^{\cos x} f(x,y)\,dy\right)dx = \int_0^1\left(\int_0^{\arccos y} f(x,y)\,dx\right)dy$$



$y=\cos x \quad (0\le x\le \pi/2)$

$\longrightarrow x=\arccos y.$

(e) $$\int_0^{\pi}\left(\int_0^{\sin x} f(x,y)\,dy\right)dx = \int_0^1\left(\int_{\arcsin y}^{\pi-\arcsin y} f(x,y)\,dx\right)dy$$

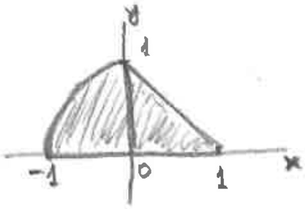


$y=\sin x$

$\longrightarrow \begin{cases} x=\arcsin y & \text{si } 0\le x\le \pi/2 \\ x=\pi-\arcsin y & \text{si } \frac{\pi}{2}\le x\le \pi \end{cases}$

(f) $$\int_0^1\left(\int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\,dx\right)dy =$$

$$= \int_{-1}^0\left(\int_0^{\sqrt{1-x^2}} f(x,y)\,dy\right)dx + \int_0^1\left(\int_0^{1-x} f(x,y)\,dy\right)dx$$



$x=-\sqrt{1-y^2} \quad (0\le y\le 1)$

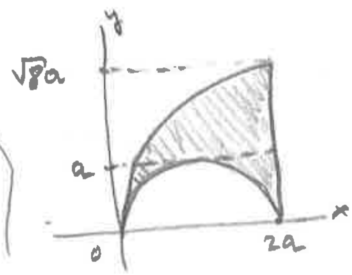
$\longrightarrow y=\sqrt{1-x^2}$

$x=1-y \longrightarrow y=1-x.$

(g) $$\int_0^{2a}\left(\int_{\sqrt{2ax-x^2}}^{\sqrt{4ax}} f(x,y)\,dy\right)dx =$$

$$= \int_0^a\left(\int_{y^2/4a}^{a-\sqrt{a^2-y^2}} f(x,y)\,dx + \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx\right)dy + \int_a^{\sqrt{8}a}\left(\int_{y^2/4a}^{2a} f(x,y)\,dx\right)dy$$

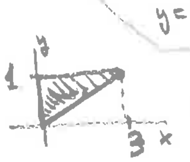
[a>0]



$y=\sqrt{4ax} \longrightarrow x=\frac{y^2}{4a}$

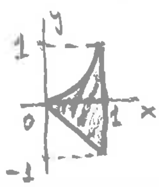
$y=\sqrt{2ax-x^2} \longrightarrow x=a\pm\sqrt{a^2-y^2}$

(h) $$\int_0^3\left(\int_{x/3}^1 f(x,y)\,dy\right)dx = \int_0^1\left(\int_0^{3y} f(x,y)\,dx\right)dy$$



$y=x/3 \to x=3y.$

(i) $$\int_0^1\left(\int_{-x}^{x^2} f(x,y)\,dy\right)dx = \int_{-1}^0\left(\int_{-y}^1 f(x,y)\,dx\right)dy + \int_0^1\left(\int_{\sqrt{y}}^1 f(x,y)\,dx\right)dy$$



$y=-x \to x=-y$

$y=x^2 \to x=\sqrt{y}.$

(13) Calculeu les següents integrals triples iterades.

(a) $$\int_1^2\int_0^1\int_0^{\pi/2} x^2y^3\sin z\,dz\,dy\,dx = \int_1^2 x^2\,dx\cdot\int_0^1 y^3\,dy\cdot\int_0^{\pi/2}\sin z\,dz =$$

$$= \left[\frac{x^3}{3}\right]_{x=1}^{x=2}\cdot\left[\frac{y^4}{4}\right]_{y=0}^{y=1}\cdot\left[-\cos z\right]_{z=0}^{z=\pi/2} = \frac{7}{3}\cdot\frac{1}{4}\cdot 1 = \underline{\underline{\frac{7}{12}}}$$

Obs | La integral es pot posar com un producte ja que tenim:
- funció: $f(x,y,z) = g(x)\cdot h(y)\cdot k(z)$
- domini: $R = [a,b]\times[c,d]\times[p,q]$.

(b) $$\int_0^1\int_0^x\int_0^{\sqrt{x^2+y^2}} z\,dz\,dy\,dx = \int_0^1\int_0^x\left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{x^2+y^2}}dy\,dx = \frac{1}{2}\int_0^1\int_0^x (x^2+y^2)\,dy\,dx =$$

$$= \frac{1}{2}\int_0^1\left[x^2y+\frac{y^3}{3}\right]_{y=0}^{y=x}dx = \frac{1}{2}\cdot\frac{4}{3}\int_0^1 x^3\,dx = \underline{\underline{\frac{1}{6}}}$$

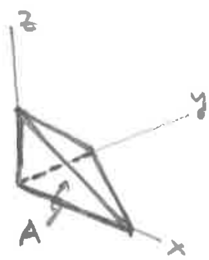
Nota: El domini és $A=\{(x,y,z):\ 0\le x\le 1,\ 0\le y\le x,\ 0\le z\le\sqrt{x^2+y^2}\}$

- $0\le x\le 1,\ 0\le y\le x$: [triangle $D\subset\mathbb{R}^2$ (la projecció de A sobre el pla xy)]
- $0\le z\le\sqrt{x^2+y^2}$: [entre 2 gràfiques sobre D: $z=0$ ← pla, $z=\sqrt{x^2+y^2}$ ← con]

(c) $$\int_0^3\int_0^{2x}\int_0^{\sqrt{xy}} z\,dz\,dy\,dx = \int_0^3\int_0^{2x}\left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{xy}}dy\,dx = \frac{1}{2}\int_0^3\int_0^{2x} xy\,dy\,dx = \frac{1}{2}\int_0^3\left[\frac{xy^2}{2}\right]_{y=0}^{y=2x} = \int_0^3 x^3\,dx = \underline{\underline{\frac{81}{4}}}$$

(14) | Per a les regions de $\mathbb{R}^3$ indicades escriviu la integral triple $\iiint_A f(x,y,z)\,dx\,dy\,dz$ en termes d'integrals iterades preses en diferents ordres.

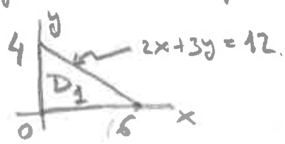
(a) A tetraedre limitat pels plans $x=0$, $y=0$, $z=0$, $2x+3y+4z=12$.



En total, tindrem $\underline{3!=6}$ maneres d'escriure la integral (tantes com permutacions de les variables x,y,z)

Escrivim el pla $2x+3y+4z=12$ com a gràfica, aïllant cada variable: $z=\dfrac{12-2x-3y}{4}$, $y=\dfrac{12-2x-4z}{3}$, $x=\dfrac{12-3y-4z}{2}$

- Projectant sobre el pla xy,



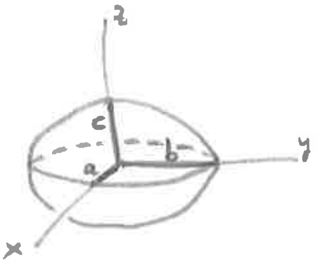
Sobre D_1, tenim la regió compresa entre les gràfiques $z=0$ i $z=\dfrac{12-2x-3y}{4}$

$$\int_A f = \iint_{D_1}dx\,dy\int_0^{(12-2x-3y)/4} f(x,y,z)\,dz = \int_0^6 dx\int_0^{(12-2x)/3}dy\int_0^{(12-2x-3y)/4} f(x,y,z)\,dz =$$

$$= \int_0^4 dy\int_0^{(12-3y)/2}dx\int_0^{(12-2x-3y)/4} f(x,y,z)\,dz.$$

- [Les altres projeccions (sobre el pla xz i sobre el pla yz) són anàlogues, i donen 4 maneres més d'escriure la integral.]

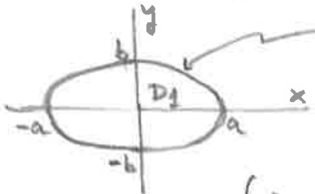
(b) A interior de l'el·lipsoide $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



Aïllant una variable, l'el·lipsoide ve donat per 2 gràfiques.

P. ex., $z=\pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ (i de manera semblant si aïllem y, o bé x)

- Projectant sobre el pla xy,



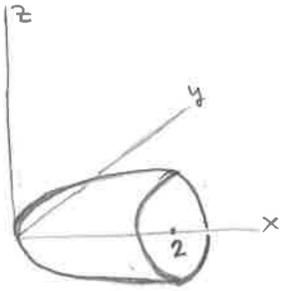
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=\pm b\sqrt{1-\frac{x^2}{a^2}}$, o bé $x=\pm a\sqrt{1-\frac{y^2}{b^2}}$

$$\int_A f=\iint_{D_1} dx\,dy\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} f(x,y,z)\,dz=\int_{-a}^{a}dx\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}dy\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} f(x,y,z)\,dz=$$

$$=\int_{-b}^{b}dy\int_{-a\sqrt{1-\frac{y^2}{b^2}}}^{a\sqrt{1-\frac{y^2}{b^2}}}dx\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} f(x,y,z)\,dz$$

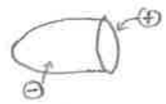
- Les altres projeccions són anàlogues.

(c) A cos limitat per les superfícies $y^2+2z^2=4x$, $x=2$

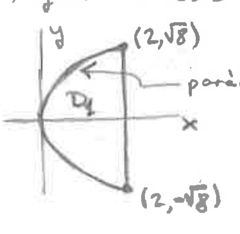


Posem com a gràfica la superfície $y^2+2z^2=4x$ (paraboloide el·líptic),

$z=\pm\sqrt{2x-\frac{y^2}{2}}$, $y=\pm\sqrt{4x-2z^2}$, $x=\frac{y^2}{4}+\frac{z^2}{2}$



- Projectant sobre el pla xy,

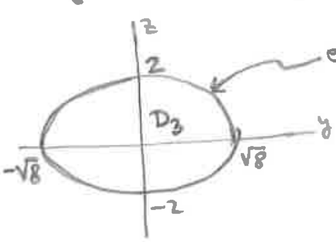


paràbola: $y^2=4x$ ↓ $\left[\begin{array}{l} y=\pm2\sqrt{x} \\ x=y^2/4\end{array}\right]$

$$\int_A f=\iint_{D_1}dx\,dy\int_{-\sqrt{2x-\frac{y^2}{2}}}^{\sqrt{2x-\frac{y^2}{2}}} f(x,y,z)\,dz=$$

$$=\int_0^2 dx\int_{-2\sqrt{x}}^{2\sqrt{x}}dy\int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z)\,dz=\int_{-\sqrt{8}}^{\sqrt{8}}dy\int_{y^2/4}^{2}dx\int_{-\sqrt{2x-y^2/2}}^{\sqrt{2x-y^2/2}} f(x,y,z)\,dz$$

- Projectant sobre el pla xz, és anàleg.
- Projectant sobre el pla yz,



el·lipse: $y^2+2z^2=8$ ↓ $\left[\begin{array}{l} z=\pm\sqrt{4-y^2/2} \\ y=\pm\sqrt{8-2z^2}\end{array}\right]$

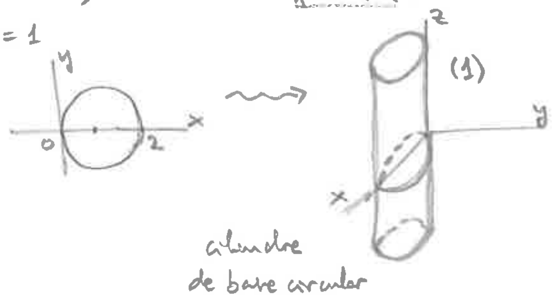
$$\int_A f=\iint_{D_3}dy\,dz\int_{\frac{y^2}{4}+\frac{z^2}{2}}^{2} f(x,y,z)\,dx=$$

$$=\int_{-\sqrt{8}}^{\sqrt{8}}dy\int_{-\sqrt{4-\frac{y^2}{2}}}^{\sqrt{4-y^2/2}}dz\int_{\frac{y^2}{4}+\frac{z^2}{2}}^{2} f(x,y,z)\,dx=\int_{-2}^{2}dz\int_{-\sqrt{8-2z^2}}^{\sqrt{8-2z^2}}dy\int_{\frac{y^2}{4}+\frac{z^2}{2}}^{2} f(x,y,z)\,dx$$

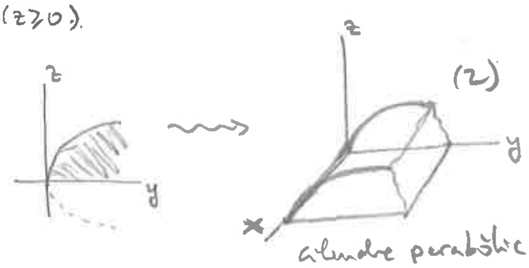
(15) Calculeu les integrals triples següents en les regions de $\mathbb{R}^3$ que s'indiquen.

(a) $\iiint_A xz\,dx\,dy\,dz$, A limitat pel cilindre de base circular $x^2+y^2-2x=0$ i la superfície $z^2=2y$ $(y,z\geq 0)$.

- $x^2+y^2-2x=0$, amb $z\in\mathbb{R}$ qualsevol
  $(x-1)^2+y^2=1$



- $z^2=2y$, amb $x\in\mathbb{R}$ qualsevol
  $(z\geq 0)$.



El domini A és la part de (2) que es troba dins de (1).

Projecció sobre el pla xy: $D=\{(x,y):\ x^2+y^2-2x\leq 0,\ y\geq 0\}$

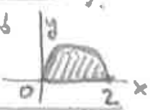
Llavors, A és la regió compresa entre les gràfiques $z=0$ i $z=\sqrt{2y}$, sobre la projecció D.

$$I=\iint_D dx\,dy\int_0^{\sqrt{2y}} xz\,dz=\int_0^2 x\,dx\int_0^{\sqrt{2x-x^2}}dy\int_0^{\sqrt{2y}}z\,dz=\int_0^2 x\,dx\int_0^{\sqrt{2x-x^2}}dy\left[\frac{z^2}{2}\right]_{z=0}^{z=\sqrt{2y}}=$$

$$=\int_0^2 x\,dx\int_0^{\sqrt{2x-x^2}}y\,dy=\int_0^2 x\,dx\left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{2x-x^2}}=\frac{1}{2}\int_0^2 x(2x-x^2)\,dx=\frac{2}{3}$$

(b) $\iiint_A zy\sqrt{x^2+y^2}\,dx\,dy\,dz$, $A=\{(x,y,z)\in\mathbb{R}^3:\ 0\leq z\leq x^2+y^2,\ 0\leq y\leq\sqrt{2x-x^2}\}$

entre 2 gràfiques — projecció



$$I=\int_0^2 dx\int_0^{\sqrt{2x-x^2}}dy\int_0^{x^2+y^2}zy\sqrt{x^2+y^2}\,dz=\int_0^2 dx\int_0^{\sqrt{2x-x^2}}y\sqrt{x^2+y^2}\,dy\left[\frac{z^2}{2}\right]_{z=0}^{z=x^2+y^2}=$$

$$=\frac{1}{2}\int_0^2 dx\int_0^{\sqrt{2x-x^2}}y(x^2+y^2)^{5/2}dy=\frac{1}{4}\int_0^2 dx\left[\frac{(x^2+y^2)^{7/2}}{7/2}\right]_{y=0}^{y=\sqrt{2x-x^2}}=\frac{1}{14}\int_0^2\left((2x)^{7/2}-(x^2)^{7/2}\right)dx=$$

$$=\frac{1}{14}\left[\frac{1}{2}\cdot\frac{(2x)^{9/2}}{9/2}-\frac{x^8}{8}\right]_0^2=\frac{1}{14}\left(\frac{2^9}{9}-\frac{2^8}{8}\right)=\frac{16}{9}$$

(c) $\iiint_A dx\,dy\,dz$, $A=\{(x,y,z)\in\mathbb{R}^3:\ 1\leq x\leq 3,\ 1\leq y\leq 3,\ 0\leq z\leq xy\}$

$$I=\int_1^3 dx\int_1^3 dy\int_0^{xy}dz=\int_1^3 dx\int_1^3 xy\,dy=\int_1^3 x\,dx\cdot\int_1^3 y\,dy=4^2=\underline{16}$$

(16) Usem coordenades polars per calcular les següents integrals dobles.

(a) $\iint_A (x^2+y^2)\,dx\,dy$, $\quad A=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 4\}$

Canvi a coordenades polars: $\left\{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right. \quad \begin{array}{l} r\ge 0,\ 0\le\theta\le 2\pi \\ \text{jacobià} = r \end{array}$

Nou domini: $A^* = \{(r,\theta) : 0\le r\le 2,\ 0\le\theta\le 2\pi\}$

[Nota. En realitat, caldria dir que $(x,y)=T(r,\theta)=(r\cos\theta, r\sin\theta)$ és un canvi de variables $T: B^*\longrightarrow B$, entre els conjunts oberts
$B^*=\{(r,\theta): 0<r<2,\ 0<\theta<2\pi\}$, $\quad B=\{(x,y): x^2+y^2<4\}-\{(x,0): x\ge 0\}$,
però com que les parts que caldria excloure tenen àrea zero, això no afecta el valor de la integral.]

Fent el canvi, obtenim: $I = \iint_{A^*} r^2 \underbrace{r}_{\text{jacobià }(>0)}\,dr\,d\theta = \underbrace{\int_0^{2\pi} d\theta}_{=2\pi}\cdot\int_0^2 r^3\,dr = 2\pi\cdot 4 = \underline{\underline{8\pi}}$

(b) $\iint_A \cos(x^2+y^2)\,dx\,dy$, $\quad A=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le \pi/2\}$

Nou domini: $A^*=\{(r,\theta): 0\le r\le\sqrt{\pi/2},\ 0\le\theta\le 2\pi\}$

$I=\iint_{A^*}\cos(r^2)\underbrace{r}_{\text{jacobià}}\,dr\,d\theta = \underbrace{\int_0^{2\pi}d\theta}_{=2\pi}\cdot\int_0^{\sqrt{\pi/2}}\cos(r^2)\,r\,dr = 2\pi\cdot\frac{1}{2}\sin(r^2)\Big]_{r=0}^{r=\sqrt{\pi/2}} = \underline{\underline{\pi}}$

(c) $\iint_A \frac{(x+y)^2}{x^2+y^2+2}\,dx\,dy$, $\quad A=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1\}$

$A^*=\{(r,\theta): 0\le r\le 1,\ 0\le\theta\le 2\pi\}$

$I=\iint_{A^*}\frac{(r\cos\theta+r\sin\theta)^2}{r^2+2}\underbrace{r}_{\text{jacobià}}\,dr\,d\theta = \int_0^{2\pi}(1+2\cos\theta\sin\theta)\,d\theta\cdot\int_0^1\frac{r^3}{r^2+2}\,dr = \boxed{2\pi\left(\frac{1}{2}-\ln\frac{3}{2}\right)}$

$\int_0^{2\pi}(1+2\cos\theta\sin\theta)\,d\theta = \theta+\sin^2\theta\Big]_0^{2\pi}=2\pi$; $\quad \int_0^1\frac{r^3}{r^2+2}\,dr = \int_0^1\left(r-\frac{2r}{r^2+2}\right)dr = \frac{1}{2}-\ln\frac{3}{2}$

(d) $\iint_A \frac{dx\,dy}{(1+x^2+y^2)^2\sqrt{x^2+y^2}}$, $\quad A=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le R^2\}$

[Ind. Usem propietats elementals de sin i cos per veure que
$\sin(\operatorname{arctg} R)=\frac{R}{\sqrt{1+R^2}}$, $\cos(\operatorname{arctg} R)=\frac{1}{\sqrt{1+R^2}}$]

$I=\int_0^{2\pi}d\theta\cdot\int_0^R\frac{dr}{(1+r^2)^2} \underset{r=\operatorname{tg} u}{=} 2\pi\int_0^{\operatorname{arctg}R}\frac{du}{1+\operatorname{tg}^2 u} =$

$= 2\pi\left[\frac{u}{2}+\frac{\sin 2u}{4}\right]_0^{\operatorname{arctg}R} = \pi\left(\operatorname{arctg}R+\frac{\sin(2\operatorname{arctg}R)}{2}\right) = \boxed{\pi\left(\operatorname{arctg}R+\frac{R}{1+R^2}\right)}$

($\frac{1}{1+\operatorname{tg}^2 u}=\cos^2 u=\frac{1+\cos 2u}{2}$)

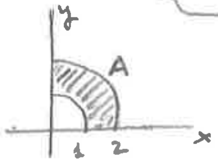
hem usat: si $v=\operatorname{arctg}R$ $\ (0<v<\pi/2)$,
$1+R^2=1+\operatorname{tg}^2 v=\frac{1}{\cos^2 v}$
$\Rightarrow \cos v=\frac{1}{\sqrt{1+R^2}}$, $\sin v=\frac{R}{\sqrt{1+R^2}}$ (>0)

(e) $\iint_A \sqrt{x^2+y^2-9}\, dx\, dy$, $\quad A = \{ (x,y) \in \mathbb{R}^2 : 9 \le x^2+y^2 \le 25 \}$.

$A^* = \{ (r,\theta) : 3 \le r \le 5, \ 0 \le \theta \le 2\pi \}$

$$I = 2\pi \int_3^5 \sqrt{r^2-9} \cdot \underbrace{r}_{\text{jacobià}} dr = \pi \left[\frac{(r^2-9)^{3/2}}{3/2} \right]_{r=3}^{r=5} = \frac{2\pi}{3}(16^{3/2} - 0) = \frac{128\pi}{3}$$

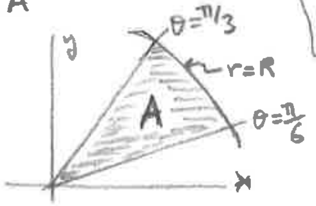
(f) $\iint_A xy\, dx\, dy$, A intersecció amb el primer quadrant de la corona circular de centre (0,0) i radi interior 1 i radi exterior 2.



$\rightsquigarrow \quad A^* = \{ (r,\theta) : 1 \le r \le 2, \ 0 \le \theta \le \frac{\pi}{2} \}$

$$I = \iint_{A^*} r\cos\theta \cdot r\sin\theta \underbrace{r}_{\text{jacobià}} dr\, d\theta = \int_0^{\pi/2} \cos\theta \cdot \sin\theta\, d\theta \cdot \int_1^2 r^3 dr = \left[\frac{\sin^2\theta}{2} \right]_{\theta=0}^{\theta=\pi/2} \cdot \left[\frac{r^4}{4} \right]_{r=1}^{r=2} = \frac{1}{2} \cdot \frac{15}{4} = \frac{15}{8}$$

(g) $\iint_A x(x^2+y^2)\, dx\, dy$, A sector circular de centre (0,0) i radi R formant angles entre $\pi/3$ i $\pi/6$ amb l'eix x positiu.



$\rightsquigarrow \quad A^* = \{ (r,\theta) : 0 \le r \le R, \ \frac{\pi}{6} \le \theta \le \frac{\pi}{3} \}$

$$I = \iint_{A^*} r\cos\theta\, r^2 \underbrace{r}_{\text{jacobià}} dr\, d\theta = \int_{\pi/6}^{\pi/3} \cos\theta\, d\theta \int_0^R r^4 dr = [\sin\theta]_{\theta=\pi/6}^{\theta=\pi/3} \left[\frac{r^5}{5} \right]_{r=0}^{r=R} = \left(\frac{\sqrt{3}}{2} - \frac{1}{2} \right) \frac{R^5}{5} = \frac{\sqrt{3}-1}{10} \cdot R^5$$

(17) Calculeu les àrees dels dominis $A \subset \mathbb{R}^2$ definits en coordenades polars, $x = r\cos\theta$, $y = r\sin\theta$, que s'indiquen tot seguit.

(a) A figura definida per $a\cos\theta \le r \le a(1+\cos\theta)$ $(a>0)$

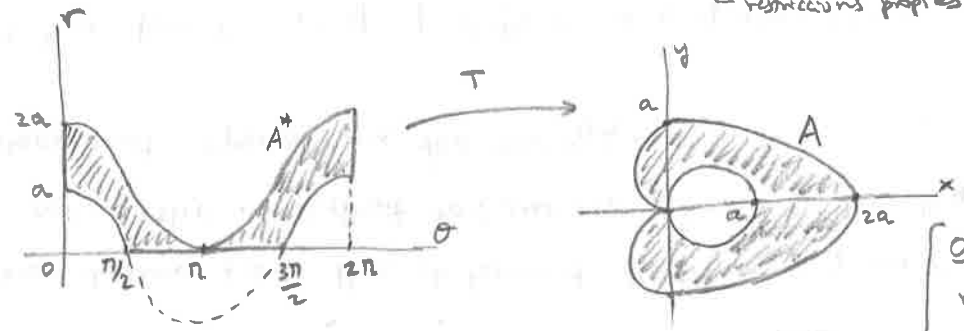
<u>Ind.</u> Observar que l'expressió té sentit quan $\cos\theta \ge 0$. Dibuixar les gràfiques de $a\cos\theta$ i $a(1+\cos\theta)$ pot ajudar a veure els valors de r admissibles.

[Obs. $\boxed{\text{àrea}(A) = \iint_A dx\,dy = \iint_{A^*} r\,dr\,d\theta}$, on A^* és el nou domini obtingut en fer el canvi a polars.]

En el nostre cas, el nou domini A^* ens ve donat directament per les desigualtats:

$$A^* = \{(r,\theta) :\ a\cos\theta \le r \le a(1+\cos\theta),\ \underbrace{r \ge 0,\ 0 \le \theta \le 2\pi}\}$$

↑ restriccions pròpies de les polars.



[Obs: $r = a(1+\cos\theta)$ cardioide; $r = a\cos\theta$ circumferència]

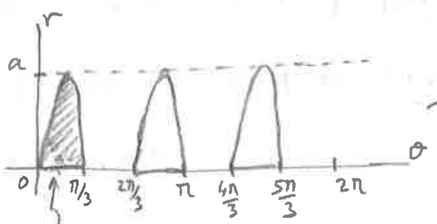
$$\text{àrea}(A) = \iint_{A^*} r\,dr\,d\theta = 2\left(\int_0^{\pi/2} d\theta \int_{a\cos\theta}^{a(1+\cos\theta)} r\,dr + \int_{\pi/2}^{\pi} d\theta \int_0^{a(1+\cos\theta)} r\,dr\right) =$$

(per simetria fem els càlculs només per al 1r i 2n quadrants)

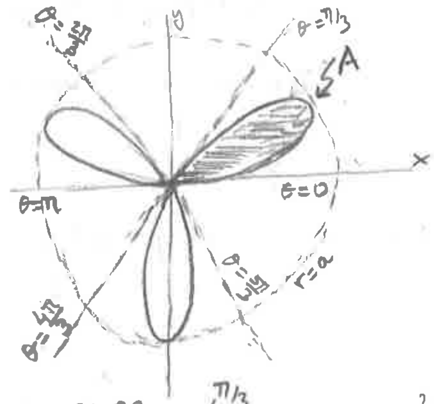
$$= 2\left(\int_0^{\pi/2} d\theta \left[\frac{r^2}{2}\right]_{r=a\cos\theta}^{r=a(1+\cos\theta)} + \int_{\pi/2}^{\pi} d\theta \left[\frac{r^2}{2}\right]_{r=0}^{r=a(1+\cos\theta)}\right) = a^2\left(\int_0^{\pi/2} (1+2\cos\theta)\,d\theta + \int_{\pi/2}^{\pi} (1+2\cos\theta+\underbrace{\cos^2\theta}_{=\frac{1+\cos 2\theta}{2}})\,d\theta\right) =$$

$$= a^2\left(\left[\theta + 2\sin\theta\right]_0^{\pi} + \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}\right) = a^2\left(\pi + \frac{\pi}{4}\right) = \frac{5}{4}\pi a^2$$

(b) A regió limitada per un pètal de la rosa definit per $r = a\sin 3\theta$ $(0 \le \theta \le \frac{\pi}{3},\ a>0)$

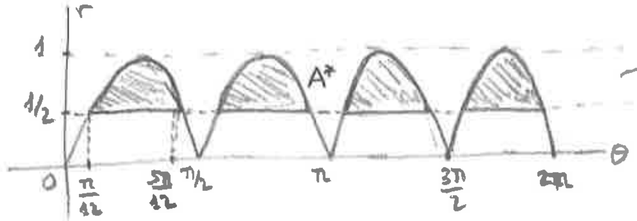


$A^* = \{(r,\theta) :\ 0 \le \theta \le \frac{\pi}{3},\ 0 \le r \le a\sin 3\theta\}$

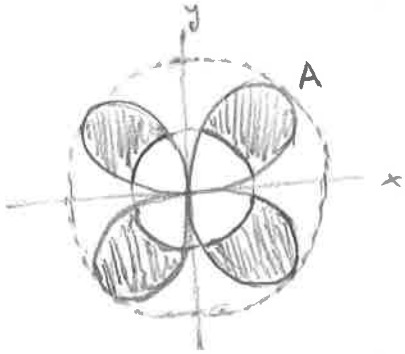


$$\text{àrea}(A) = \int_{A^*} r\,dr\,d\theta = \int_0^{\pi/3} d\theta \int_0^{a\sin 3\theta} r\,dr = \int_0^{\pi/3} d\theta \left[\frac{r^2}{2}\right]_{r=0}^{r=a\sin 3\theta} = \frac{a^2}{2}\int_0^{\pi/3} \underbrace{\sin^2 3\theta}_{=\frac{1-\cos 6\theta}{2}}\,d\theta = \frac{a^2}{2}\left[\frac{\theta}{2} - \frac{\sin 6\theta}{12}\right]_0^{\pi/3} = \frac{\pi a^2}{12}$$

(c) A regió definida per $\frac{1}{2} \le r \le |\sin 2\theta|$.
Ind. Cal $|\sin 2\theta| \ge 1/2$ perquè l'expressió tingui sentit



T →



- Per simetria, ens restringim al 1r quadrant.
- Per trobar els extrems d'integració de θ, resolem: $\sin 2\theta = \frac{1}{2}$ $(0 \le \theta \le \frac{\pi}{2}) \longrightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$

$$\text{àrea}(A) = \iint_{A^*} r\,dr\,d\theta = 4\int_{\pi/12}^{5\pi/12} d\theta \int_{1/2}^{|\sin 2\theta|} r\,dr = 4\int_{\pi/12}^{5\pi/12} d\theta \left[\frac{r^2}{2}\right]_{r=\frac{1}{2}}^{r=|\sin 2\theta|} = 2\int_{\pi/12}^{5\pi/12}\left(\sin^2 2\theta - \frac{1}{4}\right)d\theta =$$

$$= \frac{1}{2}\int_{\pi/12}^{5\pi/12}(1 - 2\cos 4\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{\sin 4\theta}{2}\right]_{\pi/12}^{5\pi/12} = \frac{\pi}{6} - \frac{\sin\frac{5\pi}{3}}{4} + \frac{\sin\frac{\pi}{3}}{4} = \underline{\frac{\pi}{6} + \frac{\sqrt{3}}{4}}$$

$$\left(\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3}\right)$$

(d) Anàlogament, calculeu la integral doble $\iint_A \arcsin(x^2+y^2)\,dx\,dy$, on A és la regió limitada per la corba $r = \sqrt{\sin\theta}$ $(0 \le \theta \le \pi/2)$.

$A^* = \{(r,\theta) : 0 \le \theta \le \pi/2,\ 0 \le r \le \sqrt{\sin\theta}\}$

$$I = \iint_{A^*} \arcsin(r^2)\,\underset{\text{jacobià}}{r}\,dr\,d\theta = \int_0^{\pi/2} d\theta \int_0^{\sqrt{\sin\theta}} \arcsin(r^2)\,r\,dr \underset{\text{canvi } s=r^2}{=} \frac{1}{2}\int_0^{\pi/2} d\theta \int_0^{\sin\theta} \arcsin s\,ds \underset{\text{parts}}{=}$$

parts: $u = \arcsin s \rightarrow du = \frac{ds}{\sqrt{1-s^2}}$; $dv = ds \rightarrow v = s$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta \left(\left[s \arcsin s\right]_{s=0}^{s=\sin\theta} - \int_0^{\sin\theta} \frac{s}{\sqrt{1-s^2}}\,ds\right) =$$

$$= \frac{1}{2}\int_0^{\pi/2} d\theta \left(\theta\sin\theta + \left[\sqrt{1-s^2}\right]_{s=0}^{s=\sin\theta}\right) = \frac{1}{2}\int_0^{\pi/2} (\underset{\text{parts}}{\theta\sin\theta} + \cos\theta + 1)\,d\theta = \frac{1}{2}\left(1 + 1 - \frac{\pi}{2}\right) = \underline{1 - \frac{\pi}{4}}.$$

(18) Calculeu les integrals dobles següents mitjançant el canvi de variables que s'indica en cada cas.

(a) $\displaystyle\iint_D xy\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 : 6\le 2y-x\le 12,\ 0\le x\le 4\}$,

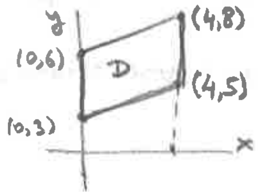
fent $x=4u$ i $y=2u+3v$

Tenim $(x,y)=T(u,v)=(4u,2u+3v)$, $\quad JT(u,v)=\begin{vmatrix}4 & 0\\ 2 & 3\end{vmatrix}=12.$

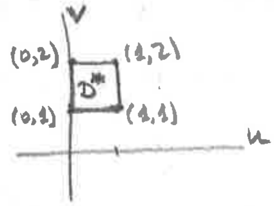
Nou domini: $\begin{cases} 6\le 2(2u+3v)-4u\le 12 & \longrightarrow\ 1\le v\le 2\\ 0\le 4u\le 4 & \longrightarrow\ 0\le u\le 1\end{cases}$

$D^*=\{(u,v):\ 0\le u\le 1,\ 1\le v\le 2\}$

Obs: També podem usar que el canvi T és lineal $\Rightarrow$ transforma segments en segments. Com que D és un polígon, només cal transformar els vèrtexs.



$\begin{cases} u = x/4 \\ v=\dfrac{2y-x}{6}\end{cases} \Longrightarrow$



Aplicant el canvi de variables,

$$I=\iint_D f(x,y)\,dx\,dy=\iint_{D^*} f(T(u,v))\cdot|JT(u,v)|\,du\,dv=\iint_{D^*}4u\cdot(2u+3v)\cdot|12|\,du\,dv=$$

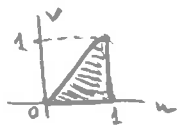
$$=48\int_0^1 du\int_1^2(2u^2+3uv)\,dv=48\int_0^1 du\left[2u^2v+\frac{3uv^2}{2}\right]_{v=1}^{v=2}=48\int_0^1\left(2u^2+\frac{9u}{2}\right)du=48\left(\frac{2}{3}+\frac{9}{4}\right)=\underline{\underline{140}}$$

(b) $\displaystyle\iint_D \frac{1}{(1+x+y)^5}\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2:\ x\ge 0,\ y\ge 0,\ x+y\le 1\}$, fent $u=x+y$ i $v=y$.

$\begin{cases}u=x+y\\ v=y.\end{cases}$ $\longrightarrow$ aïllem variables antigues en funció de les noves: $\begin{cases}x=u-v\\ y=v\end{cases}$, $\quad (x,y)=T(u,v)=(u-v,v)$

$JT(u,v)=\begin{vmatrix}1 & -1\\ 0 & 1\end{vmatrix}=1.$

Nou domini:
$x\ge 0 \longrightarrow u\ge v$
$y\ge 0 \longrightarrow v\ge 0$
$x+y\le 1 \longrightarrow u\le 1$
$\longrightarrow D^*=\{(u,v):\ v\ge 0,\ v\le u\le 1\}$



$$I=\iint_{D^*}\frac{1}{(1+u)^5}\,\boxed{1}\,du\,dv=\int_0^1 dv\int_v^1\frac{du}{(1+u)^5}=\int_0^1 dv\left[\frac{(1+u)^{-4}}{-4}\right]_{u=v}^{u=1}=\frac{1}{4}\int_0^1\left((1+v)^{-4}-2^{-4}\right)dv=$$

($\uparrow$ jacobià)

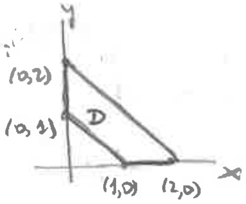
$$=\frac{1}{4}\left(\left[\frac{(1+v)^{-3}}{-3}\right]_0^1-\frac{1}{16}\right)=\frac{1}{4}\left(\frac{1}{3}(1-2^{-3})-\frac{1}{16}\right)=\frac{1}{4}\left(\frac{1}{3}\cdot\frac{7}{8}-\frac{1}{16}\right)=\underline{\underline{\frac{11}{192}}}$$

(c) $\displaystyle\iint_D \frac{dx\,dy}{(x+y)^{n+1}}$, $D=\{(x,y)\in\mathbb{R}^2 : 1\le x+y\le 2,\ x\ge 0,\ y\ge 0\}$, fent $u=x+y$ i $v=x$

$\begin{cases} u=x+y \\ v=x \end{cases} \longrightarrow$ aïllant, $\begin{cases} x=v \\ y=u-v \end{cases}$

$(x,y)=T(u,v)$, $\quad JT(u,v)=\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$

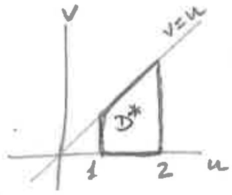
Domini:



Nou domini:

$D^*=\{(u,v) : 1\le u\le 2,\ v\ge 0,\ u\ge v\}$

(com que el canvi és lineal, també podem transformar els vèrtexs)



Calculem:

$$I=\iint_{D^*} \frac{1}{u^{n+1}}\cdot\underbrace{|-1|}_{\text{jacobià}}\,du\,dv = \int_1^2 \frac{du}{u^{n+1}}\int_0^u dv = \int_1^2 \frac{du}{u^n}$$

$\longrightarrow$ si $n\neq 1$, $\quad I=\left.\frac{u^{-n+1}}{-n+1}\right]_1^2 = \frac{1}{n-1}\left(1-\frac{1}{2^{n-1}}\right)$

si $n=1$, $\quad I=\ln u\Big]_1^2 = \ln 2$

(d) $\displaystyle\iint_D \left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{3/2} dx\,dy$, $D=\left\{(x,y)\in\mathbb{R}^2 : \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\right\}$, fent $x=a r\cos\theta$ i $y=b r\sin\theta$.

$\begin{cases} x=ar\cos\theta \\ y=br\sin\theta \end{cases}$, coordenades polars "adaptades", jacobià $=abr\ (>0)$

Notem que $\frac{x^2}{a^2}+\frac{y^2}{b^2}=(r\cos\theta)^2+(r\sin\theta)^2=r^2 \Rightarrow$ l'el·lipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ve donada per $r=1$.

Nou domini: $D^*=\{(r,\theta) : 0\le r\le 1,\ 0\le\theta\le 2\pi\}$

Calculem:

$$I=\iint_{D^*}(1-r^2)^{3/2}\underbrace{abr}_{\text{jacobià}}\,dr\,d\theta = ab\int_0^{2\pi}d\theta\cdot\int_0^1 (1-r^2)^{3/2}r\,dr =$$

$$= 2\pi ab\cdot\left(-\frac{1}{2}\right)\left[\frac{(1-r^2)^{5/2}}{5/2}\right]_{r=0}^{r=1} = \frac{2}{5}\pi ab.$$

(e) $\displaystyle\iint_D \operatorname{arctg}\left(x^2+\frac{y^2}{2}\right)dx\,dy$, $D=\left\{(x,y)\in\mathbb{R}^2 : x^2+\frac{y^2}{2}\le 1,\ x\ge 0,\ y\ge 0\right\}$,

fent $x=r\cos\theta$ i $y=\sqrt{2}\,r\sin\theta$.

$\begin{cases} x=r\cos\theta \\ y=\sqrt{2}r\sin\theta \end{cases}$, coord. polars "adaptades", jacobià $=\sqrt{2}\,r$

Domini:

Nou domini:

$D^*=\{(r,\theta) : 0\le r\le 1,\ 0\le\theta\le\frac{\pi}{2}\}$

canvi: $s=r^2$, $ds=2r\,dr$

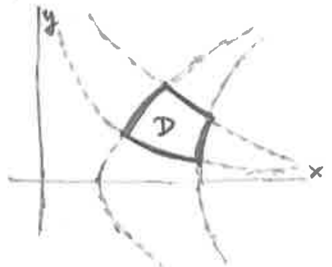
Calculem:

$$I=\iint_{D^*}\operatorname{arctg}(r^2)\cdot\sqrt{2}\,r\cdot dr\,d\theta = \sqrt{2}\int_0^{\pi/2}d\theta\cdot\int_0^1 \operatorname{arctg}(r^2)\cdot r\,dr = \sqrt{2}\cdot\frac{\pi}{2}\cdot\frac{1}{2}\int_0^1 \operatorname{arctg} s\,ds =$$

$$= \frac{\pi\sqrt{2}}{4}\left(\Big[s\operatorname{arctg}s\Big]_0^1 - \int_0^1 \frac{s}{1+s^2}ds\right) = \frac{\pi\sqrt{2}}{4}\left(\frac{\pi}{4}-\frac{1}{2}\Big[\ln(1+s^2)\Big]_0^1\right) = \frac{\pi\sqrt{2}}{8}\left(\frac{\pi}{2}-\ln 2\right)$$

parts: $u=\operatorname{arctg}s$, $dv=ds$

(f) $\iint_D (x^2+y^2)\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2:\ 1\le x^2-y^2\le 9,\ 2\le xy\le 4,\ x\ge 0, y\ge 0\}$,
fent $u=x^2-y^2$ i $v=2xy$.



$\begin{cases} u=x^2-y^2 \\ v=2xy \end{cases}$, tenim les variables noves en funció de les antigues $\to$ és el canvi invers $(u,v)=T^{-1}(x,y)$.

Aïllant, tindríem $(x,y)=T(u,v)$, però també podem aplicar el canvi de variables usant $(u,v)=T^{-1}(x,y)$.

Nou domini: $D^*=\{(u,v):\ 1\le u\le 9,\ 4\le v\le 8\}$

Jacobià: $JT^{-1}(x,y)=\begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2)$

$\to$ pel teo. f. inversa, $JT(u,v)=\dfrac{1}{JT^{-1}(x,y)}=\dfrac{1}{4(x^2+y^2)}$,

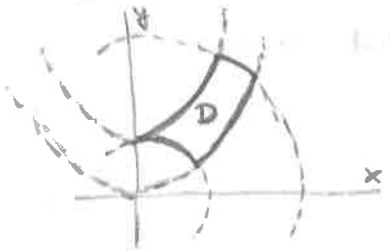
on $x=x(u,v)$, $y=y(u,v)$ són els components de $(x,y)=T(u,v)$, que no hem escrit explícitament.

Aplicant el canvi, $I=\iint_{D^*}\underbrace{(x^2+y^2)\left|\dfrac{1}{4(x^2+y^2)}\right|}\,du\,dv = \dfrac{1}{4}\iint_{D^*} du\,dv = \dfrac{1}{4}\,\text{àrea}(D^*) = \dfrac{1}{4}\cdot 32 = \underline{8}$

(entenent $\begin{cases} x=x(u,v) \\ y=y(u,v) \end{cases}$)

Nota: Es comprova que $T: D^*\longrightarrow D$ és bijectiva, i que $T(u,v)=\left(\underbrace{\sqrt{\dfrac{u+\sqrt{u^2+v^2}}{2}}}_{x(u,v)},\ \underbrace{\sqrt{\dfrac{\sqrt{u^2+v^2}-u}{2}}}_{y(u,v)}\right)$ (probl. 2.42) $\leftarrow$ (expressions complicades que no hem usat)

(g) $\iint_D \dfrac{x+2xy}{x^2+y^2}\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2:\ x^2\le y\le x^2+1,\ 1\le x^2+y^2\le e^2,\ x\ge 0\}$,
fent $u=x^2+y^2$ i $v=y-x^2$.



$\begin{cases} u=x^2+y^2 \\ v=y-x^2 \end{cases}$, tenim $(u,v)=T^{-1}(x,y)$

Nou domini: $D^*=\{(u,v):\ 0\le v\le 1,\ 1\le u\le e^2\}$

Jacobià: $JT^{-1}(x,y)=\begin{vmatrix} 2x & 2y \\ -2x & 1 \end{vmatrix} = 2x+4xy \leadsto JT(u,v)=\dfrac{1}{2x+4xy}$

Per tant, $I=\iint_{D^*}\dfrac{x+2xy}{u}\left|\dfrac{1}{2x+4xy}\right|du\,dv = \dfrac{1}{2}\iint_{D^*}\dfrac{du\,dv}{u} = \dfrac{1}{2}\int_1^{e^2}\dfrac{du}{u}\cdot\int_0^1 dv =$

(amb $x=x(u,v)$, $y=y(u,v)$)

$= \dfrac{1}{2}\Big[\ln u\Big]_{u=1}^{u=e^2}\cdot 1 = \dfrac{1}{2}(\ln(e^2)-\ln 1) = \underline{1}$

(19) Usem coordenades cilíndriques per calcular les següents integrals triples

(a) $\iiint_B \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$, $\quad B=\{(x,y,z)\in\mathbb{R}^3 : \sqrt{x^2+y^2}\le z\le 4\}$

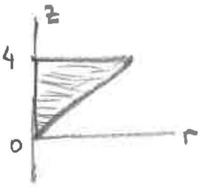
[Obs. B és sòlid de revolució (resp. l'eix z), ja que les coordenades x, y només apareixen com a x^2+y^2.]

Usem coordenades cilíndriques:

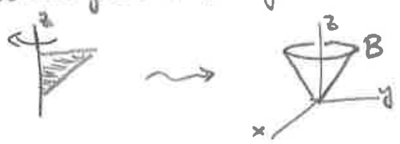
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad r\ge 0,\ 0\le\theta\le 2\pi,\ z\in\mathbb{R} \qquad \text{jacobià} = r.$$

Nou domini: $B^* = \{(r,\theta,z) : r\le z\le 4,\ \underbrace{r\ge 0,\ 0\le\theta\le 2\pi}\}$

restriccions pròpies de les cilíndriques.



[Obs: afegim $0\le\theta\le 2\pi$ correspon a obtenir el sòlid de revolució generat en girar el triangle resp. l'eix z.]



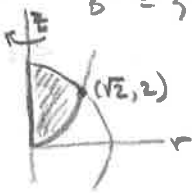
Apliquem el canvi:

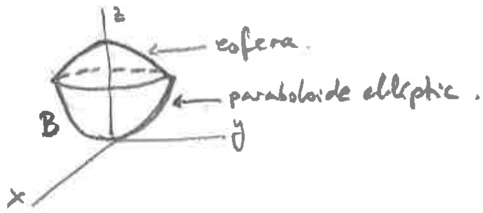
$$I = \iiint_{B^*} \sqrt{r^2+z^2}\,\underset{\substack{\uparrow\\ \text{jacobià}}}{r}\,dr\,d\theta\,dz = \underbrace{\int_0^{2\pi} d\theta}_{2\pi}\cdot\int_0^4 dz\int_0^z \sqrt{r^2+z^2}\cdot r\,dr = \pi\int_0^4 dz\left[\frac{(r^2+z^2)^{3/2}}{3/2}\right]_{r=0}^{r=z} =$$

$$= \frac{2\pi}{3}\int_0^4 \left((2z^2)^{3/2}-(z^2)^{3/2}\right)dz = \frac{2\pi}{3}\cdot(2^{3/2}-1)\int_0^4 z^3\,dz = \frac{2\pi}{3}(2\sqrt{2}-1)\left[\frac{z^4}{4}\right]_0^4 = \frac{128\pi}{3}(2\sqrt{2}-1)$$

(e) $\iiint_B z\,dx\,dy\,dz$, $\quad B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 6,\ x^2+y^2\le z,\ z\ge 0\}$

Nou domini: $B^* = \{(r,\theta,z) : r^2+z^2\le 6,\ r^2\le z,\ z\ge 0,\ r\ge 0,\ 0\le\theta\le 2\pi\}$





Calculem:

$$I = \iiint_{B^*} z\,\underset{\substack{\uparrow\\ \text{jacobià}}}{r}\,dr\,d\theta\,dz = 2\pi\int_0^{\sqrt{2}} r\,dr\int_{r^2}^{\sqrt{6-r^2}} z\,dz = 2\pi\int_0^{\sqrt{2}} r\,dr\left[\frac{z^2}{2}\right]_{z=r^2}^{z=\sqrt{6-r^2}} =$$

$$= \pi\int_0^{\sqrt{2}} r(6-r^2-r^4)\,dr = \pi\cdot\left[3r^2-\frac{r^4}{4}-\frac{r^6}{6}\right]_0^{\sqrt{2}} = \frac{11\pi}{3}.$$

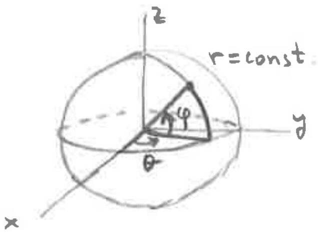
(20) Usem coordenades esfèriques per calcular les següents integrals triples.

(b) $\iiint_B z(x^2+y^2)\,dx\,dy\,dz$, $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le a^2,\ z\ge 0\}$

Tenim una semiesfera

Usem coordenades esfèriques:

$$\begin{cases} x = r\cos\varphi\cos\theta \\ y = r\cos\varphi\cdot\sin\theta \\ z = r\sin\varphi \end{cases} \qquad r\ge 0,\ 0\le\theta\le 2\pi,\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}, \quad \text{jacobià} = r^2\cos\varphi$$



Nou domini: $B^* = \{(r,\theta,\varphi) : r^2\le a^2,\ r\sin\varphi\ge 0,\ \underbrace{r\ge 0,\ 0\le\theta\le 2\pi,\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}}_{\text{restriccions pròpies de les esfèriques}}\} =$

$= \{(r,\theta,\varphi) : 0\le r\le a,\ 0\le\theta\le 2\pi,\ 0\le\varphi\le\frac{\pi}{2}\}$

Apliquem el canvi:

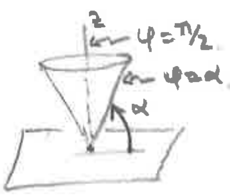
$$I = \iiint_{B^*} r\sin\varphi\cdot r^2\cos^2\varphi\cdot \underbrace{r^2\cos\varphi}_{\text{jacobià}}\,dr\,d\theta\,d\varphi = \int_0^{2\pi} d\theta\cdot\int_0^a r^5\,dr\cdot\int_0^{\pi/2}\sin\varphi\cdot\cos^3\varphi\,d\varphi =$$

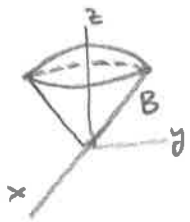
$$= 2\pi\left[\frac{r^6}{6}\right]_{r=0}^{r=a}\cdot\left[-\frac{\cos^4\varphi}{4}\right]_{\varphi=0}^{\varphi=\pi/2} = 2\pi\cdot\frac{a^6}{6}\cdot\frac{1}{4} = \frac{\pi a^6}{12}$$

(21) Calculeu els volums dels dominis $B\subset\mathbb{R}^3$ definits en coordenades esfèriques, $x=r\cos\varphi\cos\theta$, $y=r\cos\varphi\cdot\sin\theta$, $z=r\sin\varphi$ $(0<\theta<2\pi,\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2})$, que s'indiquen tot seguit

(a) B domini tallat sobre la bola $r\le a$ pel con $\alpha\le\varphi\le\frac{\pi}{2}$ $(a>0,\ 0<\alpha<\pi/2)$

Obs. L'equació $\varphi=\alpha$ ens dóna un con (meitat superior)
Les desigualtats $\alpha\le\varphi\le\frac{\pi}{2}$ ens donen el con sòlid.





Nou domini: $B^* = \{(r,\theta,\varphi) : 0\le r\le a,\ 0\le\theta\le \pi/2,\ \alpha\le\varphi\le\frac{\pi}{2}\}$

Calculem:

$$\text{vol}(B) = \iiint_B dx\,dy\,dz = \iiint_{B^*} \underset{\uparrow\ \text{jacobià}}{r^2\cos\varphi}\,dr\,d\theta\,d\varphi = 2\pi\int_0^a r^2\,dr\cdot\int_\alpha^{\pi/2}\cos\varphi\,d\varphi = \frac{2\pi}{3}a^3(1-\sin\alpha)$$