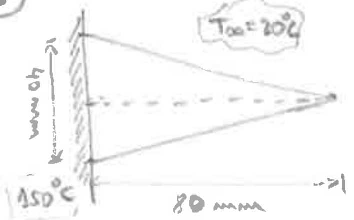
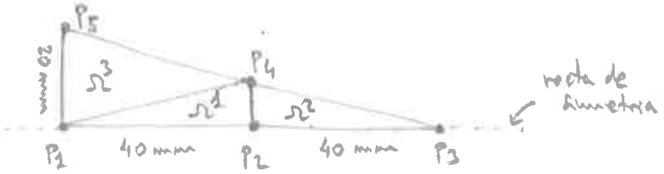
(8)



coef de convecció: $h = 30\ W/m^2{}^\circ C$

conductivitat: $k = 160\ W/m^\circ C$.



(a) Plantejament del problema i matrius de rigidesa elementals

$$\begin{cases} -k\,\Delta T = 0 & \text{a } \Omega \text{ (triangle de vèrtexs } P_1, P_3, P_5) \rightsquigarrow a_{11}=a_{22}=k,\ a_{12}=a_{21}=a_{00}=f=0. \\ T = 150 & \text{a } \overline{P_1P_5} \text{ (Dirichlet)} \\ k\dfrac{\partial T}{\partial n} + h(T - T_\infty) = 0 & \text{a } \overline{P_3P_5} \text{ (Robin)} \\ \dfrac{\partial T}{\partial n} = 0 & \text{a } \overline{P_1P_3} \text{ (Neumann)} \end{cases}$$

← per la simetria:

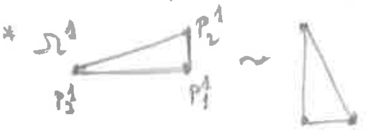
$T(x,y) = T(x,-y) \Rightarrow \dfrac{\partial T}{\partial y}(x,y) = -\dfrac{\partial T}{\partial y}(x,-y)$

$\Rightarrow$ a $y=0$, $0 = \dfrac{\partial T}{\partial y}(x,0) = -\dfrac{\partial T}{\partial n}(x,0)$

Nodes (en m): $P_1 = (0,0)$, $P_2 = (0.04, 0)$, $P_3 = (0.08, 0)$, $P_4 = (0.04, 0.01)$, $P_5 = (0, 0.02)$

Matrius de rigidesa locals:

* Ω^1



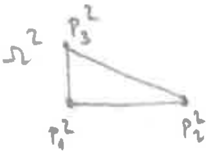
$a = 0.01$, $b = 0.04$

$\rightarrow \alpha = 4$, $\beta = 1/4$

$|A| = \dfrac{ab}{2} = 2\cdot 10^{-4}$

$$K^1 = \frac{k}{8}\begin{pmatrix} 17 & -16 & -1 \\ -16 & 16 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

* Ω^2



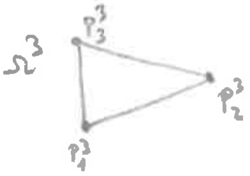
$a = 0.04$, $b = 0.01$

$\rightarrow \alpha = \frac{1}{4}$, $\beta = 4$

$|A| = 2\cdot 10^{-4}$

$$K^2 = \frac{k}{8}\begin{pmatrix} 17 & -1 & -16 \\ -1 & 1 & 0 \\ -16 & 0 & 16 \end{pmatrix}$$

* Ω^3



$x_2 - x_1 = 0.04$ $\quad y_1 - y_2 = -0.01$

$x_3 - x_2 = -0.04$ $\quad y_2 - y_3 = -0.01$

$x_1 - x_3 = 0$ $\quad y_3 - y_1 = 0.02$

$|A| = \dfrac{0.02\cdot 0.04}{2} = 4\cdot 10^{-4}$

$$K^3 = k\left(K^{3,11} + K^{3,22}\right) = \frac{k}{4|A|}\left[10^{-4}\begin{pmatrix} 1 & -2 & 1 \\ & 4 & -2 \\ \text{sim.} & & 1 \end{pmatrix} + 10^{-4}\begin{pmatrix} 16 & 0 & -16 \\ & 0 & 0 \\ \text{sim.} & & 16 \end{pmatrix}\right] = \frac{k}{16}\begin{pmatrix} 17 & -2 & -15 \\ -2 & 4 & -2 \\ -15 & -2 & 17 \end{pmatrix}$$

(b) Matriu de connectivitat:

$$C = \begin{pmatrix} 2 & 4 & 1 \\ 2 & 3 & 4 \\ 1 & 4 & 5 \end{pmatrix}$$

Matriu de rigidesa global: 5×5

- sumem K^1 a la 2a, 4a, 1a files/columnes:

$$\begin{array}{cc} \begin{pmatrix} 17 & -16 & -1 \\ -16 & 16 & 0 \\ -1 & 0 & 1 \end{pmatrix} & \begin{matrix} P_2 \\ P_4 \\ P_1 \end{matrix} \\ \begin{matrix} P_2 & P_4 & P_1 \end{matrix} & \end{array} \rightsquigarrow \begin{array}{cc} \begin{pmatrix} -1 & 0 & 1 \\ 17 & -16 & -1 \\ -16 & 16 & 0 \end{pmatrix} & \begin{matrix} P_1 \\ P_2 \\ P_4 \end{matrix} \\ \begin{matrix} P_2 & P_4 & P_1 \end{matrix} & \end{array} \rightsquigarrow \begin{array}{cc} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 17 & -16 \\ 0 & -16 & 16 \end{pmatrix} & \begin{matrix} P_1 \\ P_2 \\ P_4 \end{matrix} \\ \begin{matrix} P_1 & P_2 & P_4 \end{matrix} & \end{array}$$

- sumem K^2 a la 2a, 3a, 4a files/columnes
- sumem K^3 a la 1a, 4a, 5a files/columnes

} ← no s'han de permutar

Obtenim:

$$K = \frac{k}{16}\begin{pmatrix} 19 & -2 & 0 & -2 & -15 \\ -2 & 68 & -2 & -64 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ -2 & -64 & 0 & 68 & -2 \\ -15 & 0 & 0 & -2 & 17 \end{pmatrix}, \quad \underbrace{T = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{pmatrix}}_{\text{valors nodals}}, \quad \underbrace{Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix}}_{\text{termes de contorn}}$$

$\frac{k}{16} = 10$

Sistema global: $KT = Q$

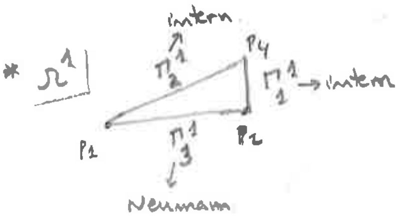
(c)

Per la cond. de Dirichlet, $T_1 = T_5 = 150$

El vector Q prové de $Q^1 \to 2^a, 4^a, 1^a$ posicions
$Q^2 \to 2^a, 3^a, 4^a$ "
$Q^3 \to 1^a, 4^a, 5^a$ "

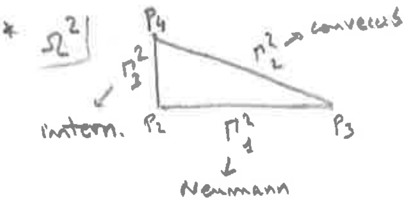
$$\longrightarrow Q = \begin{pmatrix} Q^1_3 + Q^3_1 \\ Q^1_1 + Q^2_1 \\ Q^2_2 \\ Q^1_2 + Q^2_3 + Q^3_2 \\ Q^3_3 \end{pmatrix}$$

* Ω^1



$$Q^1 = \underbrace{Q^1_{(1)}}_{\text{balanç}} + \underbrace{Q^1_{(2)}}_{\text{balanç}} + \underbrace{Q^1_{(3)}}_{= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \leftarrow Q^1_{i,3} = \int_{\Gamma^1_3} \underbrace{q^3_{n,i}}_{=0} \psi^1_i \, dl$$

* Ω^2



$$Q^2 = \underbrace{Q^2_{(1)}}_{= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} + Q^2_{(2)} + \underbrace{Q^2_{(3)}}_{\text{balanç amb } Q^1_{(1)}}$$

Relacionem $Q^2_{(2)}$ amb T_3, T_4:

$Q^2_{1,2} = 0$ ja que $\psi^2_1 = 0$ sobre Γ^2_2

$$Q^2_{2,2} = \int_{\Gamma^2_2} q^2_{n,2} \psi^2_2 \, dl = h\left[20 \int_{\Gamma^2_2} \psi^2_2 \, dl - T_3 \int_{\Gamma^2_2} (\psi^2_2)^2 \, dl - T_4 \int_{\Gamma^2_2} \psi^2_2 \psi^2_3 \, dl \right] =$$

$q_n = k \frac{\partial T}{\partial n} = -h(T - 20) = h(20 - T_3 \psi^2_2 - T_4 \psi^2_3)$
sobre Γ^2_2, $T = T_3 \psi^2_2 + T_4 \psi^2_3$

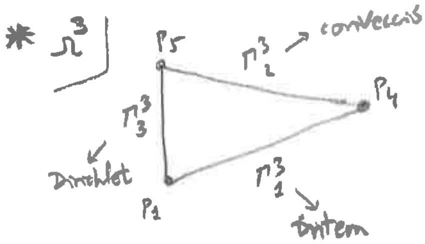
$L = \text{long}(\Gamma^2_2) = \sqrt{0.04^2 + 0.01^2} = \frac{\sqrt{17}}{100}$

$$= h \cdot \left[20 \cdot \frac{1!\,0!}{2!} L - T_3 \cdot \frac{2!\,0!}{3!} L - T_4 \frac{1!\,1!}{3!} L \right] = hL\left[10 - \frac{1}{3} T_3 - \frac{1}{6} T_4 \right]$$

$$Q^2_{3,2} = \int_{\Gamma^2_2} q^2_{n,2} \psi^2_3 \, dl = h\left[20 \int_{\Gamma^2_2} \psi^2_3 \, dl - T_3 \int_{\Gamma^2_2} \psi^2_2 \psi^2_3 \, dl - T_4 \int_{\Gamma^2_2} (\psi^2_3)^2 \, dl \right] = hL\left[10 - \frac{1}{6} T_3 - \frac{1}{3} T_4 \right]$$

(com abans)

$* \ \Omega^3$



$$Q^3 = \underbrace{Q^3_{(1)}}_{\text{balanç amb } Q^1_{(2)}} + Q^3_{(2)} + Q^3_{(3)}$$

$$Q^3_{(3)} = \begin{pmatrix} Q^3_{1,3} \\ Q^3_{2,3} \\ Q^3_{3,3} \end{pmatrix} \begin{matrix} \to \text{no conegut} \\ \to =0 \\ \to \text{no conegut.} \end{matrix}$$

$$Q^3_{i,3} = \int_{\Gamma^3_3} q^3_{n,3}\,\psi^3_i\,dl \quad (\psi^3_2 = 0 \text{ sobre } \Gamma^3_3)$$

Relacionem $Q^3_{(2)}$ amb T_4, T_5:

$Q^3_{1,2} = 0$ ja que $\psi^3_1 = 0$ sobre Γ^3_2

$$Q^3_{2,2} = \int_{\Gamma^3_2} q^3_{n,2}\,\psi^3_2\,dl = h\left[20\int_{\Gamma^3_2}\psi^3_2\,dl - T_4\int_{\Gamma^3_2}(\psi^3_2)^2\,dl - T_5\int_{\Gamma^3_2}\psi^3_2\,\psi^3_3\,dl\right] = hL\left[10 - \tfrac{1}{3}T_4 - \tfrac{1}{6}T_5\right]$$

$q_n = k\frac{\partial T}{\partial n} = -h(T-20) = h(20 - T_4\psi^3_2 - T_5\psi^3_3)$

sobre Γ^3_2, $T = T_4\psi^3_2 + T_5\psi^3_3$

com abans

$L = \text{long}(\Gamma^3_2) = \frac{\sqrt{17}}{100}$

$Q^3_{3,2}$ → no cal, ja que no coneixem $Q^3_{3,3}$

Obtenim:

$$Q = \begin{pmatrix} 0 + Q^3_{1,3} \\ 0 \\ Q^2_{2,2} \\ Q^2_{3,2} + Q^3_{2,2} + 0 \\ Q^3_{3,2} + Q^3_{3,3} \end{pmatrix} \begin{matrix} \to \text{no conegut} \\ \\ \to hL\left[10 - \frac{1}{3}T_3 - \frac{1}{6}T_4\right] \\ \to hL\left[20 - \frac{1}{6}T_3 - \frac{2}{3}T_4 - \frac{1}{6}T_5\right] \\ \to \text{no conegut} \end{matrix}$$

$hL = \frac{3\sqrt{17}}{10}$

Usant els valors coneguts de T_1 i T_5, i restringint-nos a la 2a, 3a i 4a equacions, obtenim el <u>sistema reduït</u>:

$$10\begin{pmatrix} 68 & -2 & -64 \\ -2 & 2 & 0 \\ -64 & 0 & 68 \end{pmatrix}\begin{pmatrix} T_2 \\ T_3 \\ T_4 \end{pmatrix} + \begin{pmatrix} -20 \\ 0 \\ -20 \end{pmatrix}150 + \begin{pmatrix} 0 \\ 0 \\ -20 \end{pmatrix}150 = \frac{3\sqrt{17}}{10}\cdot\left[\begin{pmatrix} 0 \\ 10 \\ -5 \end{pmatrix} - \begin{pmatrix} 0 \\ 1/3 \\ 1/6 \end{pmatrix}T_3 - \begin{pmatrix} 0 \\ 1/6 \\ 2/3 \end{pmatrix}T_4\right]$$

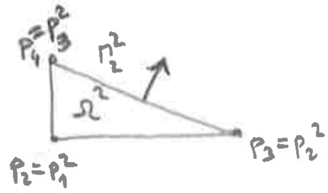
$$\tilde{K}\begin{pmatrix} T_2 \\ T_3 \\ T_4 \end{pmatrix} = \tilde{Q}, \quad \text{amb } \tilde{K} = 10\begin{pmatrix} 68 & -2 & -64 \\ -2 & 2 & 0 \\ -64 & 0 & 68 \end{pmatrix} + \frac{\sqrt{17}}{10}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 1/2 & 2 \end{pmatrix}, \quad \tilde{Q} = 150\begin{pmatrix} 20 \\ 0 \\ 40 \end{pmatrix} + \sqrt{17}\begin{pmatrix} 0 \\ 3 \\ -3/2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} T_2 = 146.127\,°C \\ T_3 = 142.305\,°C \\ T_4 = 146.125\,°C \end{cases}$$

El flux de calor en cada node es determina a partir dels fluxos a través dels costats dels elements al qual pertany.

P.ex., per al costat Γ_2^2 de l'element Ω^2,

* la densitat de flux ve donada per la variable secundària (si és >0, vol dir flux d'entrada)



$$q^2_{n,2} = k\,\frac{\partial T}{\partial n} \qquad [W/m^2]$$

* integrant, tenim el flux total a través d'aquest costat:

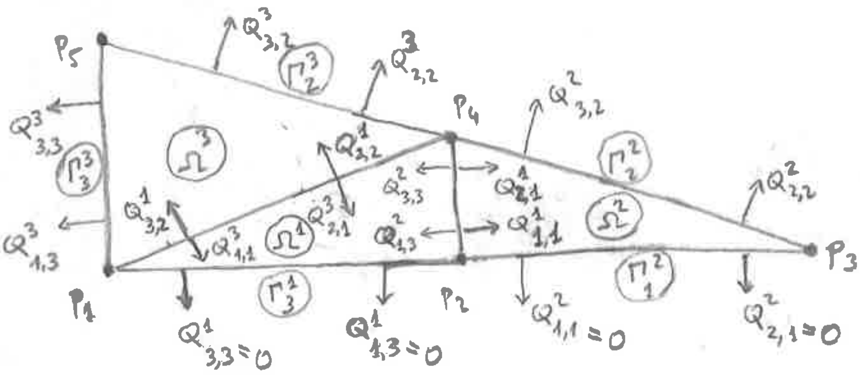
$$\text{flux}(\Gamma_2^2) = \int_{\Gamma_2^2} q^2_{n,2}\, dl = Q^2_{2,2} + Q^2_{3,2}$$

$$\left[\begin{array}{l} \psi_1^2+\psi_2^2+\psi_3^2=1 \\ \psi_1^2=0 \text{ sobre } \Gamma_2^2 \end{array}\right]$$

[W/m; multiplicant pel gruix obtindríem el flux en W]

Per obtenir els fluxos en els nodes, repartim el flux a través de Γ_2^2 entre els seus nodes: $Q^2_{2,2} = \int_{\Gamma_2^2} q^2_{n,2}\,\psi_2^2$ per a P_3, $\quad Q^2_{3,2} = \int_{\Gamma_3^2} q^3_{n,2}\,\psi_3^2$ per a P_4

En cada node, cal tenir en compte tots els costats que hi incideixen.



Després de fer el balanç, els fluxos en els nodes són:

$$\left\{\begin{array}{llll}
(P_1) & Q^3_{1,3} & = Q_1 = 154.955 & W/m \\
(P_2) & 0 & = Q_2 = 0 & \\
(P_3) & Q^2_{2,2} & = Q_3 = -76.4292 & W/m \\
(P_4) & Q^2_{3,2}+Q^3_{2,2} & = Q_4 = -156.020 & W/m \\
(P_5) & Q^3_{3,2}+Q^3_{3,3} & = Q_5 = 77.4937 & W/m
\end{array}\right.$$

[ho calculem fent $Q=K\cdot T$]

Els fluxos a través dels costats:

$$\left[\begin{array}{l}
\text{flux}(\Gamma_3^1)=0 \\
\text{flux}(\Gamma_1^2)=0 \\
\text{flux}(\Gamma_2^2) = Q^2_{2,2}+Q^2_{3,2} = hL\left[20-\frac{1}{2}T_3-\frac{1}{2}T_4\right] = -153.846 \text{ W/m} \\
\text{flux}(\Gamma_2^3) = Q^3_{2,2}+Q^3_{3,2} = hL\left[20-\frac{1}{2}T_4-\frac{1}{2}T_5\right] = -158.405 \text{ W/m} \\
\text{flux}(\Gamma_3^3) = Q^3_{3,3}+Q^3_{1,3} = Q_5 - \underbrace{Q^3_{3,2}}_{hL\left[10-\frac{1}{6}T_4-\frac{1}{3}T_5\right]} + Q_1 = 312.051 \text{ W/m}
\end{array}\right.$$

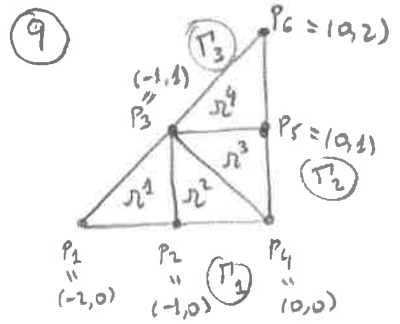
Notem que

$\sum Q_i = 0$

$\sum \text{flux}(\Gamma_j^k)=0$

ja que a l'EDP no hi ha terme de convecció a l'interior del domini.

9



$$-\Delta u = 5$$

$$\left.\begin{array}{ll} \frac{\partial u}{\partial n}(x,0) = -2x & \text{a } \Gamma_1 \\ \frac{\partial u}{\partial n}(0,y) = 2y & \text{a } \Gamma_2 \end{array}\right\} \text{Neumann}$$

$$u(x,y) = -\tfrac{1}{2}y^2 - 2y - 2 \quad \text{a } \Gamma_3 \leftarrow \text{Dirichlet}$$

(a) Matriu de connectivitat:

$$B = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 4 & 3 \\ 5 & 3 & 4 \\ 5 & 6 & 3 \end{pmatrix}$$

[comencem cada triangle per l'angle recte i el recorrem en sentit antihorari.

(b)

$$\underset{\alpha=\beta=1}{K^k = \frac{1}{2}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}}, \quad \underset{f=5,\ |A|=1/2}{F^k = \frac{5}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}, \quad k = 1,2,3,4.$$

(c) Per a cada element, permutem les matrius locals i les sumem a la matriu global.

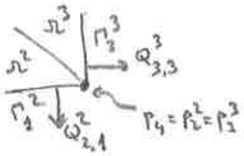
$$\Omega^1: \begin{array}{c} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{matrix} P_2 \\ P_3 \\ P_1 \end{matrix} \\ P_2\ P_3\ P_1 \end{array} \longrightarrow \begin{array}{c} \begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \\ P_1\ P_2\ P_3 \end{array} \rightsquigarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}; \quad \Omega^2: \begin{array}{c} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{matrix} P_2 \\ P_4 \\ P_3 \end{matrix} \\ P_2\ P_4\ P_3 \end{array} \longrightarrow \begin{array}{c} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \\ P_2\ P_4\ P_3 \end{array} \rightarrow \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\Omega^3: \begin{array}{c} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{matrix} P_5 \\ P_3 \\ P_4 \end{matrix} \\ P_5\ P_3\ P_4 \end{array} \longrightarrow \begin{array}{c} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix} \\ P_5\ P_3\ P_4 \end{array} \rightsquigarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}; \quad \Omega^4: \text{ com } \Omega^1.$$

Obtenim:

$$K = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & & & \\ -1 & 4 & -2 & -1 & & \\ 0 & -2 & 4 & 0 & -2 & 0 \\ & -1 & 0 & 2 & -1 & \\ & & -2 & -1 & 4 & -1 \\ & & 0 & & -1 & 1 \end{pmatrix}, \text{ i de manera semblant: } F = \frac{5}{6}\begin{pmatrix} 1 \\ 2 \\ 4 \\ 2 \\ 2 \\ 1 \end{pmatrix}$$

(d)



$$Q_4 = Q_{2,1}^2 + Q_{3,3}^3 = \frac{2}{3}$$

$$\left[\begin{array}{l} Q_{2,1}^2 = \int_{\Gamma_1^2} (-2x)\,\varphi_2^2\,dl = \int_0^1 2(1-t)t\,dt = \frac{1}{3} \quad \left(\begin{array}{l}\text{parametritzant:} \\ \gamma(t) = (-1+t,0),\ 0 \le t \le 1\end{array}\right) \\ Q_{3,3}^3 = \int_{\Gamma_2^3} 2y\,\varphi_3^3\,dl = \int_0^1 2t(1-t)\,dt = \frac{1}{3} \quad \left(\begin{array}{l}\text{parametritzant:} \\ \delta(t) = (0,t),\ 0 \le t \le 1\end{array}\right). \end{array}\right.$$

Anàlogament trobarem $Q_2 = Q_{1,3}^1 + Q_{1,1}^2 = \frac{4}{3} + \frac{2}{3} = 2$, $Q_5 = Q_{1,3}^3 + Q_{1,1}^4 = \frac{2}{3} + \frac{4}{3} = 2$.

(e) Sistema global: $KU = F + Q$, amb $U_1 = -2$, $U_3 = -9/2$, $U_6 = -8$ (usant cond. Dirichlet)
$Q_2 = 2$, $Q_4 = 2/3$, $Q_5 = 2$

Prenent la $2^a, 4^a, 5^a$ eqs. tenim el sistema reduït:

$$\frac{1}{2}\begin{pmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 4 \end{pmatrix}\begin{pmatrix} U_2 \\ U_4 \\ U_5 \end{pmatrix} = \frac{5}{6}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2/3 \\ 2 \end{pmatrix} - \frac{1}{2}\left[\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\cdot(-2) + \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix}\cdot(-9/2) + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}(-8)\right] = \frac{1}{6}\begin{pmatrix} -11 \\ 14 \\ -29 \end{pmatrix}$$

$$\Rightarrow \underline{U_2 = -\frac{25}{36},\ U_4 = \frac{8}{9},\ U_5 = -\frac{79}{36}}$$

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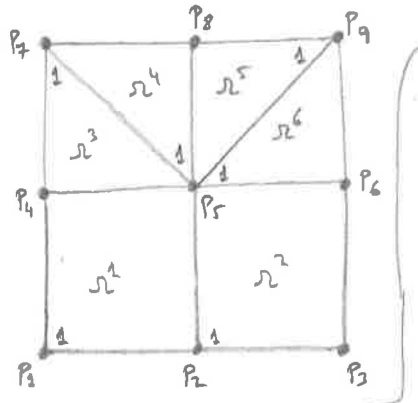
$\Omega=[-1,1]\times[-1,1] \longrightarrow$ EDP : $-\Delta u = f$.



(a) Matrius elementals :

$$K^1=K^2=\frac{1}{6}\begin{pmatrix}4&-1&-2&-1\\ &4&-1&-2\\ \text{sim.}& &4&-1\\ & & &4\end{pmatrix} \leftarrow \text{rectangles amb } \alpha=\beta=1.$$

$$K^3=K^4=K^5=K^6=\frac{1}{2}\begin{pmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{pmatrix} \leftarrow$$ triangles rectangles isòsceles, amb el node 2 a l'angle recte. (probl. 5)

(b) Quins elements de les files 6 i 7 de la matriu acoblada són zero?

Matriu de connectivitat : $C=\begin{pmatrix}1&2&5&4\\2&3&6&5\\7&4&5&\\5&8&7&\\9&8&5&\\5&6&9&\end{pmatrix}$ $\begin{matrix}\}\text{ rectangles}\\ \}\text{ triangles.}\end{matrix}$

Fila 6 : $K_{61}=K_{64}=K_{67}=K_{68}=0.$

$K_{62}=K^2_{31}=-\frac{1}{3}$, $K_{63}=K^2_{32}=-\frac{1}{6}$, $K_{65}=K^2_{34}+K^6_{21}=-\frac{2}{3}$, $K_{66}=K^2_{33}+K^6_{22}=\frac{5}{3}$, $K_{69}=K^6_{23}=-\frac{1}{2}$.

Fila 7 : $K_{71}=K_{72}=K_{73}=K_{76}=K_{79}=0.$

$K_{74}=K^3_{12}=-\frac{1}{2}$, $K_{75}=K^3_{13}+K^4_{31}=0$, $K_{77}=K^3_{11}+K^4_{33}=1$, $K_{78}=K^4_{32}=-\frac{1}{2}$.

(c)

$$f(x,y)=\begin{cases} y-x & \text{si } (x,y)\in\Omega^5\\ x-y & \text{si } (x,y)\in\Omega^6\\ 0 & \text{altrament.}\end{cases} \qquad \underset{\uparrow\atop \text{(cond. de Dirichlet)}}{u=0} \text{ a tot } \partial\Omega$$

$\rightsquigarrow u(0,0),\ u(\frac{1}{2},1/4)$?

Per la cond de contorn, $U_1=\dots=U_4=U_6=\dots=U_9=0$.

A més, $Q_5=0$ ja que es tracta d'un node intern (balanç = 0).

En el sistema global, tenim :

$$\begin{pmatrix} K\\ (9\times 9)\end{pmatrix}\begin{pmatrix}0\\ \vdots\\ 0\\ U_5\\ 0\\ \vdots\\ 0\end{pmatrix}=\begin{pmatrix}F_1\\ \vdots\\ F_5\\ \vdots\\ F_9\end{pmatrix}+\begin{pmatrix}Q_1\\ \vdots\\ Q_4\\ 0\\ Q_5\\ \vdots\\ Q_9\end{pmatrix}$$

Prenent la 5ª eq., tenim el "sistema reduït" (1×1) : $\boxed{K_{55}U_5=F_5}$

Tenim :

$$K_{55}=K^1_{33}+K^2_{44}+K^3_{33}+K^4_{11}+K^5_{33}+K^6_{11}=\frac{1}{6}(4+4)+\frac{1}{2}(1+1+1+1)=\frac{10}{3}$$

$$F_5=F^5_3+F^6_1=\int_{\Omega^5} f\cdot\psi^5_3\,dx\,dy+\int_{\Omega^6} f\cdot\psi^6_1\,dx\,dy \quad (f=0 \text{ fora de } \Omega^5 \text{ i } \Omega^6)$$

$\Omega^5 \rightarrow$ com que f és un polinomi de grau 1 sobre Ω^5, és una combinació lineal de les funcions interpoladores en aquest element:

$$f = f_1^5\varphi_1^5 + f_2^5\varphi_2^5 + f_3^5\varphi_3^5 \quad \text{sobre } \Omega^5, \quad \text{essent } f_j^5 = f(p_j^5),\ j=1,2,3.$$

Tenim: $\left.\begin{array}{l} f_1^5 = f(p_9) = f(1,1) = 0 \\ f_2^5 = f(p_8) = f(0,1) = 1 \\ f_3^5 = f(p_5) = f(0,0) = 0 \end{array}\right\} \Rightarrow f = \varphi_2^5$

Llavors, $$F_3^5 = \int_{\Omega^5} (\varphi_2^5)^1 (\varphi_3^5)^1 dx\,dy = \frac{1!\,1!\,0!}{4!}\, 2|A_5| = \frac{1}{24}$$

$\Omega^6 \rightarrow$ anàlogament, $f = f_1^6\varphi_1^6 + f_2^6\varphi_2^6 + f_3^6\varphi_3^6$ sobre Ω^6.

$\left.\begin{array}{l} f_1^6 = f(p_5) = f(0,0) = 0 \\ f_2^6 = f(p_6) = f(1,0) = 1 \\ f_3^6 = f(p_9) = f(1,1) = 0 \end{array}\right\} \rightarrow f = \varphi_2^6$

$$F_1^6 = \int_{\Omega^6} (\varphi_2^6)^1 (\varphi_1^6)^1 dx\,dy = \frac{1!\,1!\,0!}{4!}\, 2|A_6| = \frac{1}{24}$$

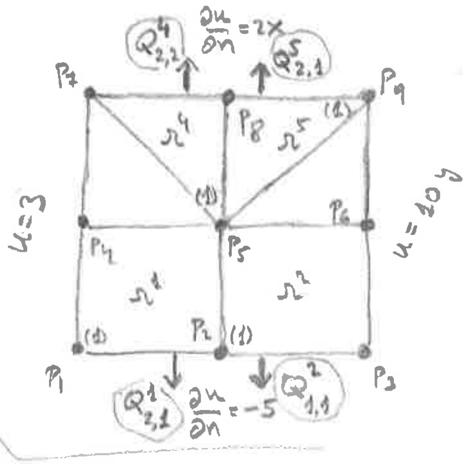
$\Rightarrow F_5 = F_3^5 + F_1^6 = \frac{1}{12}$, i obtenim $\boxed{u(0,0) \simeq U_5 = \frac{F_5}{K_{55}} = \frac{1}{40}}$

Finalment, $(\frac{1}{2},\frac{1}{4}) \in \Omega^6 \longrightarrow \boxed{u(1/2,1/4) \simeq U_5\varphi_1^6 + \underbrace{U_6}_{=0}\varphi_2^6 + \underbrace{U_9}_{=0}\varphi_3^6 = \frac{1}{80}}$

Per a Ω^6, $$M = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccc} p_5 & p_6 & p_9 \end{array} \end{array} \longrightarrow M^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow \varphi_1^6 = 1-x$$
$(2A_6 = 1)$

(d) CC: $\left\{\begin{array}{l} u(-1,y) = 3 \\ u(1,y) = 10y \end{array}\right]$ ← Dirichlet

$\left.\begin{array}{l} \frac{\partial u}{\partial y}(x,-1) = 5 \\ \frac{\partial u}{\partial y}(x,1) = 2x \end{array}\right]$ ← Neumann $\quad \frac{\partial u}{\partial n}(x,-1) = -5, \quad \frac{\partial u}{\partial n}(x,1) = 2x$



Dirichlet: $\left[\begin{array}{l} U_1 = U_4 = U_7 = 3 \\ U_3 = -10,\ U_6 = 0,\ U_9 = 10. \end{array}\right.$

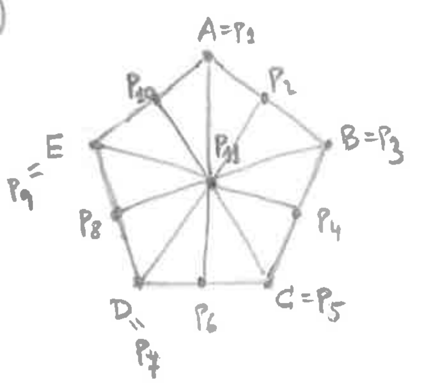
Neumann: $\left[\begin{array}{l} Q_2 = Q^1_{2,1} + Q^2_{1,1} = \frac{-5\cdot 1}{2} + \frac{-5\cdot 1}{2} = -5 \quad (q_n = -5 \text{ const, long.} = 1). \\ Q_8 = Q^4_{2,2} + Q^5_{2,1} = 0 \end{array}\right.$ ← per simetria, obté:

$$Q^4_{2,2} = \int_{\Gamma^4_2} 2x\cdot\varphi^4_2\, dl = \int_{-1}^{0} 2x(x+1)\,dx = -\frac{1}{3}$$
$$Q^5_{2,1} = \int_{\Gamma^5_1} 2x\cdot\varphi^5_1\, dl = \int_{0}^{1} 2x(1-x)\,dx = \frac{1}{3}.$$

A més, tindrem $Q_5 = 0$ (intern)

→ sistema reduït: 3×3.

14



u: variables primàries

q: variables secundàries.

$\overline{AB} \rightarrow q=4$ constant

$\overline{BC} \rightarrow q$ lineal de 4 a 0

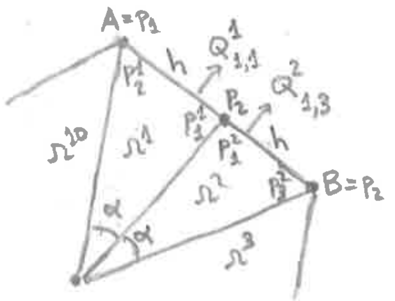
$\overline{CD} \rightarrow u=10$ constant.

$\overline{DE} \rightarrow q=-5(u-1)$

$\overline{EA} \rightarrow u$ lineal de 0 a 5.

Condicions de contorn als nodes centrals dels costats: $P_2, P_4, P_6, P_8, P_{10}$.

Enumerem els elements, i els nodes dins de cada element



Cada element és un triangle rectangle amb hipotenusa = 1, $\alpha = \frac{2\pi}{10} = \frac{\pi}{5}$ $\Big\} \longrightarrow h = \sin\frac{\pi}{5} = 0.587785$

$\underline{P_2}$: $Q_2 = Q^1_{1,1} + Q^2_{1,3} = \frac{qh}{2} + \frac{qh}{2} = 4h$

$\underline{P_4}$: $Q_4 = Q^3_{1,1} + Q^4_{1,3} = 2h$

$$Q^3_{1,1} = \int_{\Gamma^3_1} q\, \psi^3_1\, dl = \int_{\Gamma^3_1} (4\psi^3_2 + 2\psi^3_1)\psi^3_1\, dl = 4\cdot\frac{1!1!}{3!}h + 2\cdot\frac{2!0!}{3!}h = \frac{4h}{3}$$

q lineal de 4 a 2

$$Q^4_{1,3} = \int_{\Gamma^4_3} q\, \psi^4_1\, dl = \int_{\Gamma^4_3} (2\psi^4_1 + 0\,\psi^4_3)\psi^4_1\, dl = 2\cdot\frac{2!0!}{3!}h = \frac{2h}{3}$$

q lineal de 2 a 0

$\underline{P_6}$: $U_6 = 10$.

$U_7 = 10,\ U_9 = 0$

$\underline{P_8}$: $Q_8 = Q^7_{1,1} + Q^8_{1,3} = 5h\left[1 - \frac{1}{6}U_7 - \frac{2}{3}U_8 - \frac{1}{6}U_9\right] = -\frac{10}{3}h\left[1 + U_8\right]$.

$$Q^7_{1,1} = \int_{\Gamma^7_1} q\, \psi^7_1\, dl = 5\int_{\Gamma^7_1} (1-u)\psi^7_1\, dl = 5\int_{\Gamma^7_1} (1 - U_7\psi^7_2 - U_8\psi^7_1)\psi^7_1\, dl =$$

$$= 5\left[\frac{1!0!}{2!}h - U_7\frac{1!1!}{3!}h - U_8\frac{2!0!}{3!}h\right] = 5h\left[\frac{1}{2} - \frac{1}{6}U_7 - \frac{1}{3}U_8\right]$$

$$Q^8_{1,3} = \int_{\Gamma^8_3} q\, \psi^8_1\, dl = 5\int_{\Gamma^8_3} (1-u)\psi^8_1\, dl = 5\int_{\Gamma^8_3} (1 - U_8\psi^8_1 - U_9\psi^8_3)\psi^8_1\, dl =$$

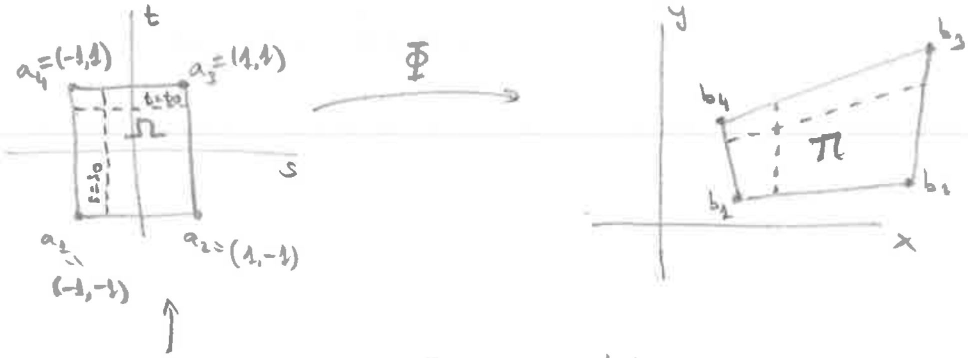
$$= 5\left[\frac{1!0!}{2!}h - U_8\frac{2!0!}{3!}h - U_9\frac{1!1!}{3!}h\right] = 5h\left[\frac{1}{2} - \frac{1}{3}U_8 - \frac{1}{6}U_9\right]$$

$\underline{P_{10}}$: $U_{10} = 2.5$

16

R: element finit quadrilàter isoparamètric.

→ les rectes $s=$ const. i $t=$ const. de l'element de referència es transformen en rectes de R.



quadrilàter (convex)

Funcions de forma per a Ω:

$$\begin{cases} \psi_1(s,t) = \frac{1}{4}(s-1)(t-1) \\ \psi_2(s,t) = -\frac{1}{4}(s+1)(t-1) \\ \psi_3(s,t) = \frac{1}{4}(s+1)(t+1) \\ \psi_4(s,t) = -\frac{1}{4}(s-1)(t+1) \end{cases}$$

Llavors, $\Phi(s,t) = b_1\psi_1(s,t) + b_2\psi_2(s,t) + b_3\psi_3(s,t) + b_4\psi_4(s,t) = (d_1+d_2s+d_3t+d_4st,\ e_1+e_2s+e_3t+e_4st)$ (canvi no lineal)

Per a $s=s_0$, $\Phi(s_0,t) = \big((d_1+d_2s_0)+(d_3+d_4s_0)t,\ (e_1+e_2s_0)+(e_3+e_4s_0)t\big)$, recta parametritzada per $t\in[-1,1]$

Per a $t=t_0$, $\Phi(s,t_0) \longrightarrow$ recta parametritzada per $s\in[-1,1]$

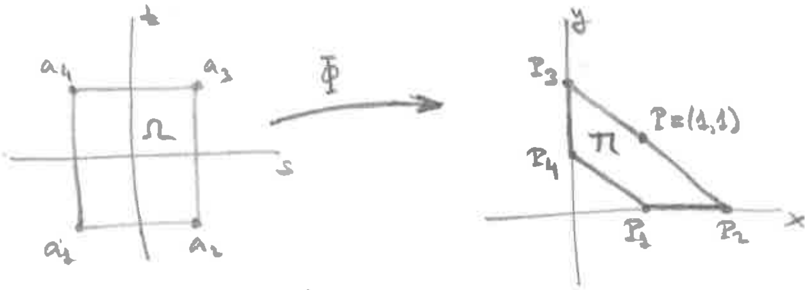
En canvi, en general rectes oblíqües es transformen en arcs de paràbola:

p.ex., $t=s \rightsquigarrow \Phi(s,s) = \big(d_1+(d_2+d_3)s+d_4s^2,\ e_1+(e_2+e_3)s+e_4s^2\big)$.

17

Quadrilàter bilineal isoparamètric: $P_1=(1,0)$, $P_2=(2,0)$, $P_3=(0,2)$, $P_4=(0,1)$.

→ valors nodals: $u_1=2.3$, $u_2=1.8$, $u_3=1.9$, $u_4=2.5$ $\rightsquigarrow u(1,1)\simeq ?$



$\psi_1,\dots,\psi_4$ com al probl. 16

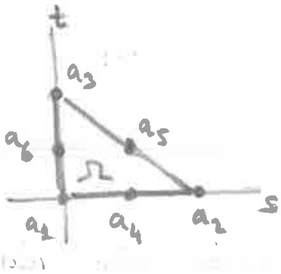
$$\hookrightarrow \Phi(s,t) = \begin{pmatrix}1\\0\end{pmatrix}\psi_1 + \begin{pmatrix}2\\0\end{pmatrix}\psi_2 + \begin{pmatrix}0\\2\end{pmatrix}\psi_3 + \begin{pmatrix}0\\1\end{pmatrix}\psi_4 = \begin{pmatrix}\psi_1+2\psi_2\\ 2\psi_3+\psi_4\end{pmatrix} = \frac{1}{4}\begin{pmatrix}-(s+3)(t-1)\\ (s+3)(t+1)\end{pmatrix}$$

Resolent $\Phi(s,t) = \begin{pmatrix}1\\1\end{pmatrix} \longrightarrow s=1, t=0 \Rightarrow \Phi^{-1}(1,1)=(1,0)$

Llavors, $u(1,1) \simeq u_1\varphi_1(1,1) + u_2\varphi_2(1,1) + u_3\varphi_3(1,1) + u_4\varphi_4(1,1) =$

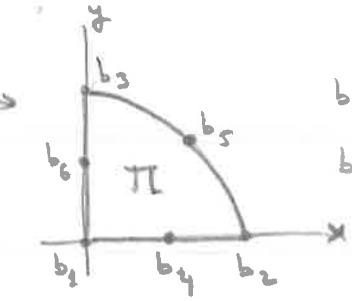
$$= u_1\underbrace{\psi_1(1,0)}_{=0} + u_2\underbrace{\psi_2(1,0)}_{=1/2} + u_3\underbrace{\psi_3(1,0)}_{=1/2} + u_4\underbrace{\psi_4(1,0)}_{=0} = \underline{\underline{1.85}}$$

(18)



$a_1=(0,0)$
$a_2=(1,0)$
$a_3=(0,1)$
$a_4=(1/2,0)$
$a_5=(1/2,1/2)$
$a_6=(0,1/2)$

$\xrightarrow{\Phi}$



$b_i=a_i$, $i=1,2,3,4,6$

$b_5=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$

element finit triangular quadràtic isoparamètric.

(a) Polinomis $\psi_i(s,t)$, $i=1,\dots,6$.

Escrivim $a_i=(s_i,t_i)$, $i=1,\dots,6$

$$\psi_j(s,t)=\alpha_j+\beta_j s+\gamma_j t+\delta_j s^2+\varepsilon_j st+\eta_j t^2,\quad j=1,\dots,6 \quad \text{(polinomis de grau 2)}.$$

Considerant la matriu

$$M=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ s_1 & s_2 & \cdots & & & s_6\\ t_1 & t_2 & \cdots & & & t_6\\ s_1^2 & s_2^2 & \cdots & & & s_6^2\\ s_1t_1 & s_2t_2 & \cdots & & & s_6t_6\\ t_1^2 & t_2^2 & \cdots & & & t_6^2\end{pmatrix}=\begin{pmatrix} 1&1&1&1&1&1\\ 0&1&0&1/2&1/2&0\\ 0&0&1&0&1/2&1/2\\ 0&1&0&1/4&1/4&0\\ 0&0&0&0&1/4&0\\ 0&0&1&0&1/4&1/4\end{pmatrix},$$

la condició $\psi_j(a_i)=\delta_{ij}\ \forall i,j$ ens diu:

$$\begin{pmatrix}\alpha_1 & \cdots & \eta_1\\ \vdots & & \vdots\\ \alpha_6 & \cdots & \eta_6\end{pmatrix}=M^{-1}=\begin{pmatrix}1&-3&-3&2&4&2\\ 0&-1&0&2&0&0\\ 0&0&-1&0&0&2\\ 0&4&0&-4&-4&0\\ 0&0&0&0&4&0\\ 0&0&4&0&-4&-4\end{pmatrix}$$

$\Rightarrow$ tenim $\psi_1,\dots,\psi_6$ (files de M^{-1})

Una altra possibilitat és considerar en primer lloc les funcions de forma del triangle lineal de nodes a_1,a_2,a_3:
$$\begin{cases}\gamma_1(s,t)=1-s-t\\ \gamma_2(s,t)=s\\ \gamma_3(s,t)=t\end{cases}$$

Demanem:

- ψ_1 s'anul·la sobre les rectes $\gamma_1=0$ i $\gamma_1=1/2$, i el seu valor a a_1 és 1: $\psi_1=\dfrac{\gamma_1(\gamma_1-1/2)}{1(1-1/2)}=\gamma_1(2\gamma_1-1)=(1-s-t)(1-2s-2t)$

$\longrightarrow$ anàlogament, $\begin{cases}\psi_2=\gamma_2(2\gamma_2-1)=s(2s-1)\\ \psi_3=\gamma_3(2\gamma_3-1)=t(2t-1)\end{cases}$

- ψ_4 s'anul·la sobre les rectes $\gamma_1=0$ i $\gamma_2=0$, i el seu valor a a_4 és 1: $\psi_4=\dfrac{\gamma_1\gamma_2}{1/2\cdot 1/2}=4\gamma_1\gamma_2=4s(1-s-t)$

$\longrightarrow$ anàlogament, $\begin{cases}\psi_5=4\gamma_2\gamma_3=4st\\ \psi_6=4\gamma_1\gamma_3=4t(1-s-t)\end{cases}$

(b) $\Phi(s,t)=b_1\psi_1+\dots+b_6\psi_6=\begin{pmatrix}1\\0\end{pmatrix}s(2s-1)+\begin{pmatrix}0\\1\end{pmatrix}t(2t-1)+\begin{pmatrix}1/2\\0\end{pmatrix}4s(1-s-t)+\begin{pmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix}4st+\begin{pmatrix}0\\1/2\end{pmatrix}4t(1-s-t)=$
$=\begin{pmatrix}s-2st+2\sqrt{2}\,st\\ t-2st+2\sqrt{2}\,st\end{pmatrix}$

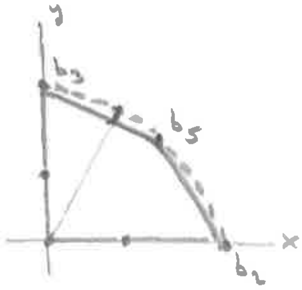
$\Rightarrow \Phi(s,t)=(s+kst,\ t+kst)$,

amb $\underline{k=2(\sqrt{2}-1)}$

Nota. La imatge del costat $\overline{a_2 a_3}$ és una paràbola, parametritzada per

$$\Phi(s,1-s) = \left((1+k)s - ks^2,\ 1-(1-k)s - ks^2\right)$$

→ és una bona aprox. de la circumferència de radi 1 (sobre un radi, dist. ≤ 0.011)



(c) Restricció de φ_1 sobre els costats $\overline{b_1 b_2}$ i $\overline{b_3 b_1}$.

$\Phi(s,0) = (s,0) \Rightarrow \Phi^{-1}(x,0) = (x,0) \Rightarrow \varphi_1(x,0) = \psi_1(x,0) = (1-x)(1-2x) = 1-3x+2x^2$

$\Phi(0,t) = (0,t) \Rightarrow \Phi^{-1}(0,y) = (0,y) \Rightarrow \varphi_1(0,y) = \psi_1(0,y) = (1-y)(1-2y) = 1-3y+2y^2$

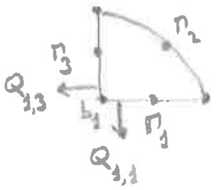
(en general, els $\varphi_j(x,y)$ no seran polinomis)

(*) Calculem també F_1, si f = const. sobre Π.

$F_1 = \displaystyle\int_\Pi f\cdot\varphi_1\,dx\,dy \underset{\text{canvi } \Phi}{=} f\cdot\int_\Omega \psi_1\cdot|J\Phi|\,ds\,dt$,

etc. (usant que $J\Phi = 1+k(s+t) > 0$

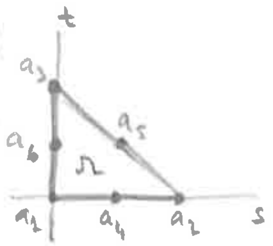
(d) Calculem Q_1 si la variable secundària és $q_n = q_0$ const. sobre $\partial\Pi$.



$$Q = Q_{1,1} + Q_{1,3} = \int_{\Pi_1} q_0\varphi_1\,dl + \int_{\Pi_3} q_0\varphi_1\,dl = \int_0^1 q_0(1-3x+2x^2)\,dx + \int_0^1 q_0(1-3y+2y^2)\,dy =$$

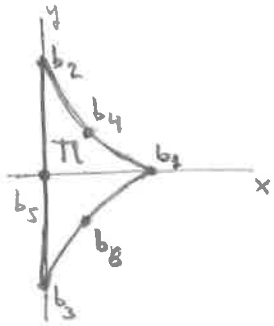
$$= q_0\frac{1}{6} + q_0\frac{1}{6} = \underline{\frac{q_0}{3}}$$

(19)



$a_1 = (0,0)$
$a_2 = (1,0)$
$a_3 = (0,1)$
$a_4 = (1/2,0)$
$a_5 = (1/2,1/2)$
$a_6 = (0,1/2)$

$\xrightarrow{\Phi}$



$b_1 = (1,0)$
$b_2 = (0,1)$
$b_3 = (0,-1)$
$b_4 = (1/2,1/3)$
$b_5 = (0,0)$
$b_6 = (1/2,-1/3)$

(a) Polinomis $\psi_i(s,t)$, $i=1,\dots,6$. ⤳ prob. 18a

(b)

$$\Phi(s,t) = b_1\psi_1 + \dots + b_6\psi_6 = \binom{1}{0}(1-s-t)(1-2s-2t) + \binom{0}{1}s(2s-1) + \binom{0}{-1}t(2t-1)$$

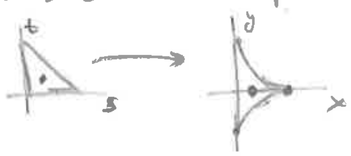
$$+ \binom{1/2}{1/3}4s(1-s-t) + \binom{1/2}{-1/3}4t(1-s-t) =$$

$$= \left(1-s-t,\ \tfrac{1}{3}(s-t)(1+2s+2t)\right)$$

(c) Valors nodals sobre Π: $u_1 = 10$, $u_2 = 0$, $u_3 = 0$, $u_4 = 5$, $u_5 = 0$, $u_6 = 5$

→ trobem $u(1/2,0)$

Trobem $\Phi^{-1}(1/2,0)$: $\begin{cases} 1-s-t = 1/2 \\ \frac{1}{3}(s-t)(1+2s+2t) = 0 \end{cases}$ → $s+t = \frac{1}{2}$, $s-t = 0$ → $s = t = 1/4$



$$u(1/2,0) \cong u_1\varphi_1(1/2,0) + \dots + u_6\varphi_6(1/2,0) =$$

$$= u_1\psi_1(1/4,1/4) + \dots + u_6\psi_6(1/4,1/4) =$$

$$= 10\cdot 0 + 0\cdot\left(-\tfrac{1}{8}\right) + 0\cdot\left(-\tfrac{1}{8}\right) + 5\cdot\tfrac{1}{2} + 0\cdot\tfrac{1}{4} + 5\cdot\tfrac{1}{2} = \underline{5}$$