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The epicycle model

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Abstract

Las soluciones de las ecuaciones del movimiento para una partícula de masa puntual bajo un campo de fuerzas conservativo estan generalmente ligadas a un conjunto básico de integrales del movimiento, que dependen del conocimiento que se tiene de la función potencial y de las simetrías y propiedades espaciales y temporales satisfechas por el sistema dinámico. La aproximación epicíclica es un caso particular de integración de las ecuaciones del movimiento bajo un conjunto mínimo de hipótesis, como son plano de simetría y simetría axial, que permite obtener soluciones para órbitas casi circulares en el espacio tridimensional. Bajo esta proximación, cualquier órbita proyectada sobre el plano de simetría describe una elipse con origen en un centro guía o epicentro, en movimiento circular uniforme alrededor del centro del sistema. La aproximación es fácil de modelar y permite visualizar la contribución de varios parámetros a la forma de las órbitas, siendo un excelente ejemplo de cómo la enseñanza y el aprendizaje de las ciencias pueden beneficiarse de la modelización matemática.

The solutions of the equations of motion for a point mass particle under a conservative force field are generally constrained by a basic set of integrals of motion, which depend on the knowledge about the potential function and on time and space properties and symmetries satisfied by the dynamical system. The epicycle approximation is a particular case of integration of the equations of motion under a minimum set of hypotheses, such as symmetry plane and axial symmetry, allowing to obtain solutions for nearly circular orbits in the three dimensional space. Under this approach, any orbit projected onto the symmetry plane describes an ellipse with origin at a guiding centre or epicentre, which moves uniformly in a circular orbit around the centre of the system. The approach is easy to model and allows to visualise the contribution of several parameters to the shape of the orbits, being an excellent example of how education and learning of science can take profit of mathematical modelling.

Keywords: Mecánica Celeste, sistemas dinámicos.
Celestial mechanics, dynamical systems

1 Introduction

Gravitational forces play an important role in lectures and laboratory practices of any first course of Physics in scientific or technical studies and in more advanced courses devoted to Astronomy and Astrophysics. In this context, when studying either the motion of a star in a galaxy or the elliptical Kepler orbit of a planet around the Sun, the epicycle approach helps to understand what an integral of motion is and how it is related to the orbits. This approach is easy to model and allows to visualise the solutions of the equations of motion as well as to simulate the contribution of several parameters to the shape of star or planet orbits. Therefore, it is an excellent example of introducing mathematical modelling in science education and learning for a better comprehension of the subjects.

The solution of the equations of motion of a point mass particle under a conservative force field is determined by the knowledge we have about the potential function. In a galaxy, for example, the mutual gravitational interactions of the stars determine their orbits. Leaving aside stellar encounters, theses interactions arise from the smoothed-out stellar distribution of matter, which are given through a gravitational potential. In addition, some symmetry properties may be generally assumed providing us with a basic set of integrals of motion.

The epicycle approximation is a particular case of integration of the equations of motion under a minimum set of hypotheses, allowing to obtain solutions for nearly circular orbits in the three dimensional space. Under this approach, the orbit of any star projected on the galactic plane describes an ellipse with origin at a guiding centre or epicentre, which moves uniformly in a circular orbit around the galactic centre.

The model leads to interesting properties about the star motion, like the epicycle frequency and the axial ratio of the epicycle, that only depend on local properties of the potential, as well as properties about the local stellar velocity distribution.

2 Equations of motion

In absence of collisions, the motion of a star in a galaxy is derived from a gravitational potential $\mathcal{U}$, generally assumed to be stationary. In a cylindrical coordinates system, if we note the star position as (r, θ, z) and the velocity of the star as $(\Pi, \Theta, \mathrm{Z}) = (\dot{r}, r\dot{\theta}, \dot{z})$, the equations of motion may be written as

$$\begin{aligned} \frac{d^2 r}{dt^2} &= r\dot{\theta}^2 - \frac{\partial \mathcal{U}}{\partial r} \\ \frac{d}{dt}(r^2\dot{\theta}) &= -\frac{\partial \mathcal{U}}{\partial \theta} \\ \frac{d^2 z}{dt^2} &= -\frac{\partial \mathcal{U}}{\partial z} \end{aligned} \tag{1}$$

Two isolating integrals of motion exist for all orbits under steady-state and axisymmetric potentials. That is,

$$\frac{\partial \mathcal{U}}{\partial t} = \frac{\partial \mathcal{U}}{\partial \theta} = 0 \tag{2}$$

One is the energy integral

$$I = \Pi^2 + \Theta^2 + \mathrm{Z}^2 + 2\,\mathcal{U}(r,z) \tag{3}$$

and the other one is the axial component of the angular momentum,

$$J = r\Theta = r^2\dot{\theta} \tag{4}$$

For a fixed integral of motion J, the energy integral may be written as

$$I = \Pi^2 + \mathrm{Z}^2 + 2\,\mathcal{V}(r,z); \quad \mathcal{V}(r,z) = \frac{J^2}{2r^2} + \mathcal{U}(r,z) \tag{5}$$

where $\mathcal{V}(r,z)$ is the effective potential energy.

A third integral of motion, the so-called Oort's integral, exists if a separable potential $\mathcal{U} = \mathcal{U}_1(r) + \mathcal{U}_2(z)$ is assumed. It is valid near the galactic plane. However, to our purposes, we shall only assume a less restrictive hypothesis: a galactic plane of symmetry, $z = 0$. Thus, the potential satisfies

$$\left.\frac{\partial \mathcal{U}}{\partial z}\right|_{z=0} = 0 \tag{6}$$

which is equivalent to saying that $\mathcal{U}$ is a function even in z.

3 Bounded orbits in the galactic plane

The epicycle approximation is explained in many standard books on astronomy (e.g., Binney & Tremaine 1987, [2] Gilmore, King & van der Kruit 1989 [4]), however, here we shall follow a slightly different viewpoint. Let us assume a star moving in a stable orbit on the galactic plane $z = 0$, with a vertical velocity component $\dot{z} = 0$. The third equation of Eq. 1, combined with Eq. 6, tells us that there is no acceleration in the vertical direction. Therefore, the motion of this star is restricted to the galactic plane. We now fix integral value J. By taking into account the energy integral of Eq. 5 we have

$$I = \dot{r}^2 + 2\,\mathcal{V}(r,0) \tag{7}$$

Therefore, for each value of the energy integral I, the orbits, and in particular the values of r, are constrained by the condition (see Fig. 1)

$$\mathcal{V}(r,0) \leq \frac{1}{2}\,I$$

In general, the equation $\mathcal{V}(r,0) = \frac{1}{2}\,I$ provides the extreme values of r for which $\dot{r} = 0$, by delimiting a annular region $r_{min} \leq r \leq r_{max}$ where the motion takes place (e.g. Arnold 1989, p35 [1]). In particular, by diminishing the value of I we reach a minimum value $I = I_c$, for which the pericentre and the apocentre coincide, say $r_c \equiv r_{min} = r_{max}$. In this case the orbit becomes circular, and satisfies $r = r_c$, $\dot{r} = 0$, where r_c is a local minimum satisfying

$$\left.\frac{\partial \mathcal{V}(r,0)}{\partial r}\right|_{r_c} = 0 \tag{8}$$

under the condition

$$\left.\frac{\partial^2 \mathcal{V}(r,0)}{\partial r^2}\right|_{r_c} > 0 \tag{9}$$

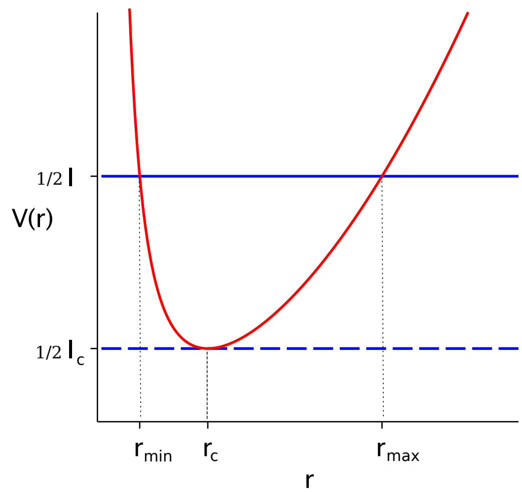

Figure 1: Effective potential energy producing bounded orbits and a circular orbit at the local minimum.

4 Circular orbits

For a circular orbit there is obviously no radial acceleration. To integrate the equations of motion, we make use of the angular momentum integral Eq. 4 in the first expression of Eq. 1, and take into account the identity

$$\frac{\partial \mathcal{V}}{\partial r} = -\frac{J^2}{r^3} + \frac{\partial \mathcal{U}}{\partial r} \tag{10}$$

Then, by introducing the condition given by Eq. 8 we get

$$\ddot{r}_c = -\left.\frac{\partial \mathcal{V}(r,0)}{\partial r}\right|_{r_c} = 0 \tag{11}$$

Therefore, according to Eq. 10, a star in circular motion in the plane $z = 0$, with angular momentum integral $J = J_c$, has a constant radius $r = r_c$ that can be obtained from the relationship

$$\left.\frac{\partial \mathcal{U}(r,0)}{\partial r}\right|_{r_c} = \frac{J_c^2}{r_c^3} \tag{12}$$

The condition of minimum, Eq. 9, is now given, by taking into account Eq. 12, through

$$\left.\frac{\partial^2 \mathcal{V}(r,0)}{\partial r^2}\right|_{r_c} = \left(\frac{3}{r}\frac{\partial \mathcal{U}(r,0)}{\partial r} + \frac{\partial^2 \mathcal{U}(r,0)}{\partial r^2}\right)_{r_c} \equiv n^2 > 0 \tag{13}$$

The angular and circular velocities are constant, such that

$$\dot{\theta}_c \equiv \Omega_c = \frac{J_c}{r_c^2}; \quad \Theta_c = \frac{J_c}{r_c} \tag{14}$$

According to Eq. 12 and Eq. 14, the value of the angular velocity is related to local properties of the potential as follows

$$\Omega_c^2 = \frac{1}{r_c}\left.\frac{\partial \mathcal{U}(r,0)}{\partial r}\right|_{r_c} \tag{15}$$

Thus, an orbit in the galactic plane with circular motion and constant angular velocity corresponds to a minimum value of the energy integral,

$$I_c = \frac{J_c^2}{r_c^2} + 2\,\mathcal{U}(r_c, 0) \tag{16}$$

and other orbits with the same angular momentum integral J_c are non-circular and have $I > I_c$ energy integral.

5 Nearly circular orbits

Linblad's approach consists in to refer the orbit of a star with position (r, θ, z) near the galactic plane to a reference frame with centre in the position $(r_c, \theta_c, 0)$ of a star in the galactic plane in circular motion with the same angular momentum integral J_c. Let us call it point C.

Thus, for the first two coordinates, we may write

$$r = r_c + \varepsilon; \quad \theta = \theta_c + \delta \tag{17}$$

The first requirement for the validity of the model is

$$\varepsilon << r_c \tag{18}$$

The second, since θ_c may have an arbitrary origin, by differentiating Eq. 17 we get

$$\dot{r} = \dot{\varepsilon}; \quad \dot{\theta} = \Omega_c + \dot{\delta} \tag{19}$$

and we assume

$$\dot{\delta} << \Omega_c \tag{20}$$

The values ε and $\dot{\delta}$ are not free since the angular momentum integral is fixed. Thus, by taking into account Eq. 4, Eq. 17, and Eq. 19 we have

$$J_c = (\Omega_c + \dot{\delta})(r_c + \varepsilon)^2 = \Omega_c r_c^2 + 2\Omega_c r_c \varepsilon + r_c^2 \dot{\delta} + \Omega_c \varepsilon^2 + 2 r_c \varepsilon \dot{\delta} + \varepsilon^2 \dot{\delta}$$

Being $J_c = \Omega_c r_c^2$, by considering only the terms up to first order, we get

$$2\Omega_c r_c \varepsilon + r_c^2 \dot{\delta} = 0$$

Therefore, we get the constraint

$$\dot{\delta} = -\frac{2\Omega_c}{r_c}\,\varepsilon \tag{21}$$

Given the position and velocity of a star, according to previous section, we may obtain r_c and Ω_c for the centre C of the new reference frame, which are specific of each star.

6 Position referred to the circular orbit

To obtain the orbit of the star in the circular motion reference frame we also write the first expression of Eq. 1 in terms of the effective potential energy gradient, Eq. 10, so that

$$\ddot{r} = -\frac{\partial \mathcal{V}}{\partial r} \tag{22}$$

and we expand it up to first order around the point C,

$$\frac{\partial \mathcal{V}}{\partial r} \approx \left.\frac{\partial \mathcal{V}}{\partial r}\right|_c + \left.\frac{\partial^2 \mathcal{V}}{\partial r^2}\right|_c \varepsilon$$

From Eq. 17, Eq. 11, and Eq. 13, the radial component of the equations of motion becomes

$$\ddot{\varepsilon} = -\left.\frac{\partial^2 \mathcal{V}}{\partial r^2}\right|_c \varepsilon = -n^2\,\varepsilon \tag{23}$$

Therefore, the condition of a local minimum of the effective potential energy at r_c, Eq. 9, ensures the orbit to be stable.

The approximation given by Eq. 23 describes the radial motion of the star as an harmonic oscillator around the point C with frequency n,

$$\varepsilon = a\,\sin n(t-p) \tag{24}$$

The value n is called epicycle frequency, and a and p are integration constants.

The axial component is easily obtained by integrating Eq. 21,

$$\dot{\delta} = -\frac{2\Omega_c}{r_c}\,a\,\sin n(t-p) \tag{25}$$

$$\delta = \frac{2\Omega_c}{r_c\,n}\,a\,\cos n(t-p) \tag{26}$$

where the additive integration constant can be assumed as null since $\varepsilon \to 0$ and $\delta \to 0$ when $a \to 0$.

In a similar way, the vertical component z of the star referred to the point C in the galactic plane is obtained, in a first order approximation, by expanding the vertical gradient of the potential as

$$\frac{\partial \mathcal{U}}{\partial z} \approx \left.\frac{\partial^2 \mathcal{U}}{\partial z^2}\right|_c z$$

so that we obtain

$$z = b\,\sin m(t-q) \tag{27}$$

being

$$m^2 = \left.\frac{\partial^2 \mathcal{U}}{\partial z^2}\right|_c > 0 \tag{28}$$

where m is the vertical epicycle frequency, and b and q integration constants.

7 Epicycle motion

A star at a position S, with coordinates (x, y, z) referred to the galactic centre (GC), will be referred to the circular velocity point C in the galactic plane, with coordinates $(r_c, \theta_c, 0)$, which is moving with a circular velocity $\Theta_c = \Omega_c r_c$ (see Fig. 2). Then, the new coordinates (ξ, η, ζ)

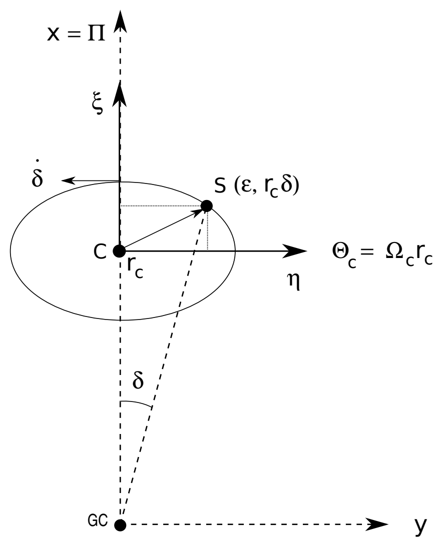

Figure 2: In the galactic plane, the star S is referred to the galactic centre (GC) with coordinates (x, y) and to the circular velocity point C, with circular velocity $\Theta_c = \Omega_c r_c$, with coordinates (ξ, η).

of the star satisfy

$$\begin{aligned} \xi &= \varepsilon = a\,\sin n(t-p) \\ \eta &= r_c\,\delta = \frac{2\Omega_c}{n}\,a\,\cos n(t-p) \\ \zeta &= z = b\,\sin m(t-q) \end{aligned} \tag{29}$$

We focus on the star's motion projected onto the galactic plane since the motion in the direction ζ is independent of the other coordinates. Vertically, the star simply oscillates about the galactic plane. By defining $\kappa = \dfrac{n}{2\Omega_c}$, which is a constant depending on local properties of the stellar system, the coordinates (ξ, η) describe the following ellipse centred at C,

$$\xi^2 + \kappa^2\eta^2 = a^2 \tag{30}$$

where κ is the ratio between axes related to eccentricity. Notice that the local velocities referred to C describe an ellipse with the same eccentricity than the previous one,

$$\dot{\xi}^2 + \kappa^2\dot{\eta}^2 = n^2a^2 \tag{31}$$

It is worth noticing that above ellipses are relative to the point C, which is specific of each star.

In Fig. 3 we show the elliptic motion of the star around the point C together with the vertical motion, which produces an orbit on a cone with elliptic section. Finally, the motion of the star referred to the GC is easily obtained in cylindric coordinates according to

$$\begin{aligned} r &= r_c + \varepsilon = r_c + a\,\sin n(t-p) \\ \theta &= \theta_c + \delta = \theta_0 + \Omega_c\,(t-p) + \frac{2\Omega_c}{n}\,a\,\cos n(t-p) \\ z &= b\,\sin m(t-q) \end{aligned} \tag{32}$$

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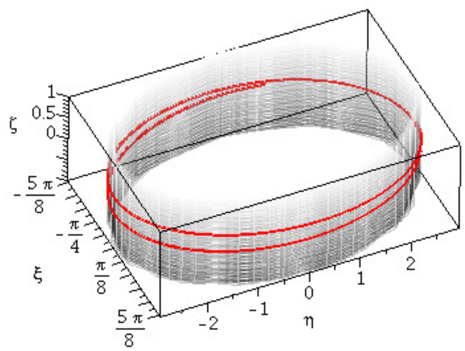

Figure 3: Local motion of the star around the point C.

We show the epicycle orbit in Fig. 4 as projected onto the galactic plane and in the three dimensional space by means of unconstrained plots, while a real orbit is shown in Fig. 5 by means of a constrained plot within its bounded region.

8 Local kinematic parameters

In the past section we have shown that the radial and transversal epicyclic frequencies are the same and, similarly to the vertical frequency, they are constant, not depending on the star but on local values of the potential. This is an important result obtained in the twenties by Oort and Lindblad. Each region in a galaxy provides the stars with a local oscillation mode. As we have adopted an axisymmetric model, the epicycle frequencies would vary depending on the distance to the galactic centre and to the plane. It is interesting to estimate how much may vary the epicycle frequencies from one point to another. If we had adopt an axisymmetric and stationary potential which is consistent with an ellipsoidal velocity distribution (Sala 1990) or a finite mixture of them (Cubarsi 1990 [3]), we should consider a potential in the general form (except an additive constant)

$$\mathcal{U} = k\,(r^2 + z^2) + \frac{1}{r^2}\,F(z/r) \tag{33}$$

with k constant and F an arbitrary function of its argument $\alpha = z/r$ (or of the polar angle $\theta = \arctan\alpha$). More general potentials exist, but they have to be explicitly time dependent.

It is easy to see that the linear operator appearing in Eq. 13 over any potential $\mathcal{U}$,

$$L_r[\mathcal{U}\,] = \left(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r}\right)\mathcal{U} \tag{34}$$

satisfies $L_r[c_1\,r^{-2} + c_2] = 0$, with c_1, c_2 constants or functions of z, and $L_r[kr^2] = 8k$. Hence, all of the potentials in the form of Eq. 33 with F constant provide constant squared frequencies $n^2 = 8k$ and $m^2 = 2k$. Therefore, the existence of bounded orbits and, in particular, of the degenerate case of circular orbits, requires the constant k to be positive.

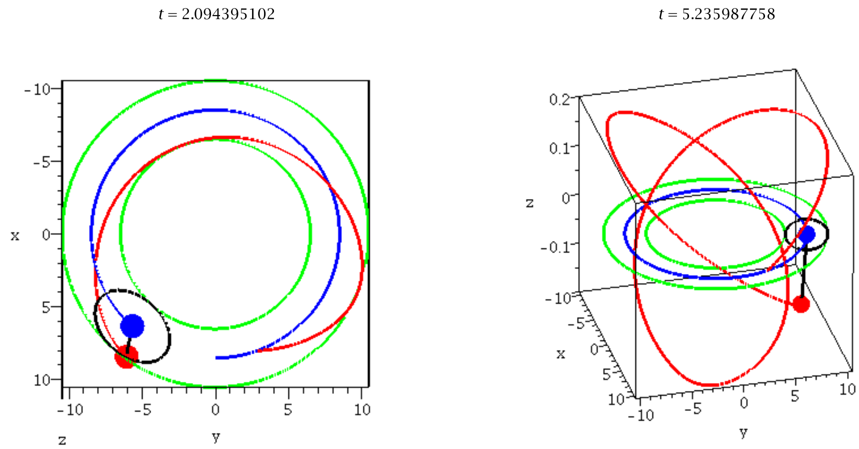

Figure 4: Epicycle orbit (red) composed of an elliptic motion (black) around a circular orbit (blue). The orbit is confined between the apocentre and the pericentre (green). The three dimensional orbit is obtained by adding oscillations around the plane $z = 0$. Plots are unconstrained. It has been used a realistic axially symmetric potential $\mathcal{U} = k\,(r^2 + z^2) + \frac{1}{r^2}\,F(z/r)$ in the galactic plane with $k > 0, F(0) < 0$. In units of kpc and 10^8 years, the simulation numerical values are $J_c = 191, r_c = 8.5, n = 3.6$, leading to the local potential constants $k = 1.62, F(0) = -9780$.

If the potential depends on the general function $F(\alpha)$, being

$$L_r[\frac{1}{r^2}\,F(\alpha)] = \frac{1}{r^4}\,\left(3\alpha F'(\alpha) + \alpha^2 F''(\alpha)\right)$$

we conclude that this term do not modify the planar epicycle frequency in the galactic plane $\alpha = 0$. Out of the galactic plane, if F is not constant, in a first order approximation the planar epicycle frequency decreases according to r^{-5}. Therefore, the axisymmetric and stationary potential of Eq. 33 provides constant epicycle frequencies in the galactic plane.

Alternatively, the planar epicycle frequency may be estimated from local kinematic parameters instead of from local potential properties. For circular orbits, Eq. 15 leads to

$$\left.\frac{\partial \mathcal{U}}{\partial r}\right|_c = \left(\frac{\Theta^2}{r}\right)_c \; ; \quad \left.\frac{\partial^2 \mathcal{U}}{\partial r^2}\right|_c = \left(\frac{2\Theta}{r}\frac{\partial \Theta}{\partial r} - \frac{\Theta^2}{r^2}\right)_c \tag{35}$$

Then we may evaluate the planar epicycle frequency in Eq. 13 by using the local circular velocity and its derivative at the point C such that

$$n^2 = \frac{2\Theta_c}{r_c}\,\left(\frac{\Theta_c}{r_c} + \left.\frac{\partial \Theta}{\partial r}\right|_c\right) \tag{36}$$

For example, in the galactic plane, for a star in circular orbit with an angular integral J, the potential Eq. 33 together with the condition Eq. 12 provides a radius satisfying

$$r_c^4 = \frac{2F(0) + J^2}{2k}$$

Being $k > 0$, the existence of bounded orbits is only possible if $J^2 > -2F(0)$. In addition, since a star moving only along the radial direction ($J = 0$) cannot have a bounded orbit, we deduce that $F(0) < 0$, by providing the minimum value of $J^2_{min} = -2F(0)$ for which bounded orbits exist.

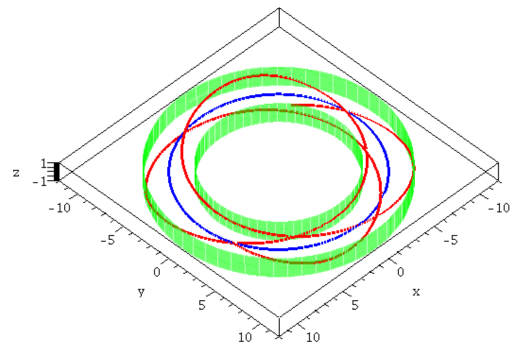

Figure 5: Real orbit (red) around the guiding circular orbit (blue) confined in a bounded region (green).

In the appendix, a MAPLE worksheet to model the epicyclic approximation is shown. It may be used to display the shape of the orbits obtained from several parameter values r_c, n, m. Indeed, from an astronomical viewpoint, those parameters depend explicitly on the potential function we had adopted, but, to understand the meaning of the model, it suffices to give some values for a, b, Ω_c, in addition to the previous ones.

9 Conclusions

Under the assumptions of stationary potential and axisymmetric stellar system, we have shown how the epicycle approximation allows to integrate of the equations of motion for any star near the galactic plane. When this motion is projected onto this plane, the star movement is well described according to a composite motion: an ellipse in the galactic plane with origin at a guiding centre or epicentre, which, at the same time, moves uniformly in a circular orbit around the galactic centre. The three dimensional motion is completed by an harmonic oscillation around the galactic plane.

The epicycle approach leads to interesting properties about the star motion, like the epicycle frequency and the axial ratio of the epicycle, that depend on local properties of the potential and are related to the integrals of motion.

The epicycle approach is easy to model and allows to understand and visualise this fundamental concepts which appear in a first course of Physics in scientific or technical studies or in studies of Astronomy and Astrophysics, being a useful tool that can contribute to an easier teaching and a better understanding of physical and mathematical concepts.

Appendix: MAPLE worksheet

```
# APROXIMACIO EPICICLICA
# Parametritzacio de superficies i corbes.
# Instruccions especifiques del MAPLE:
# spacecurve, animate, implicitplot, plottools.
#
# PARAMETRITZACIO DE CORBES EN L'APROXIMACIO EPICICLICA
# Es suposa que l'estrella es mou en el disc (amb z petita) i que el potencial
# es estacionari i te simetria cilindrica.
# Es pren com a sistema de referencia un de no inercial, pero ortogonal, lligat
# a un punt (centroide) a distancia fixa r del centre galactic, amb una posicio
# angular f, que te moviment angular constant (f'=C).
# Es suposa que la variacio radial de l'estrella respecte del centroide es molt
# petita comparada amb r.
# D'acord amb les equacions newtonianes del moviment, tot fent una aproximacio de
# primer ordre al camp de forces galactic (grad U), i sense especificar la forma
# del potencial, es pot determinar que l'estrella es mou seguint una elipse al
# voltant del centroide.
#
# En aquest sistema, en la direccio radial (positiva cap a l'anticentre) la posicio es:
# x=a sin n(t-p)
#
# En la direccio de la rotacio galactica:
# y=(2C a/n) cos n(t-p)
#
# I en la direccio vertical (positiva cap al pol galactic):
# z=b sin m(t-q)
#
# Les frequencies n i m depenen de la funcio potencial i son iguals per a totes les estrelles.
# D'aquesta manera la posicio en coordenades cilindriques  (R, F, z)  d'una estrella compleix:
# (x,y,z)=(R-r, r (F-f), z)
#
# Aixi, les velocitats (components fisiques) seran:
# PI=R'=x'=na cos n(t-p)
# rF'=rf '+y '=rC- 2C a sin n(t-p)
# THETA=RF'=Cr-Cx=Cr-Ca sin n(t-p)
# ZETA=z'=mb cos m(t-q)
#
# Valors numerics adoptats: n=3.6 e-08 /y=2Pi/1.74 e08/y=3.6/Y
# r=8.5kpc, rC=220km/s=22.5 kpc/Y,  C=2.65 rad/Y=2 Pi*0.42 voltes cada Y=1.e+08 anys,
# k=0.1 (mesurat en kpc i Y=1e+08y), invk=2C/n=1.47, b=zmax
#
> # Principi del full MAPLE V (v.15)
> restart;
> with(plots):with(plottools):with(linalg):
> # parametres inicials
> a:=2: r:=8.5:n:=3.6:m:=2.0:b:=0.2:C:=2.65:invk:=2*C/n:k:=1/invk:
> x:=(t)->a*sin(n*t);y:=(t)->(a*invk)*cos(n*t);z:=(t)->b*sin(m*t);
> # funcio auxiliar que delimita el cilindre g0 amb alada b on es mou el punt localment: alt
> alt:=h->b*h:
> g0:=plot3d([x(t),y(t),h],t=0..2*Pi,h=-1..1,shading=zgreyscale,numpoints=1500,
scaling=constrained,axes=boxed,style=wireframe,transparency=0.75):
> # Generem l'elipse (amb oscilacions en z) que descriu cada estrella al voltant del centroide
sobre el cilindre g0
> g1:=spacecurve([x(t),y(t),z(t)],t=0..Pi,scaling=constrained,axes=boxed,thickness=2,color=red,
numpoints=1500):display({g0,g1}, labelfont =["SYMBOL",10],labels = [ "x", "h" ,"z"]);
> # Cilindres amb rmin i rmax, d'alaada b, que delimiten el moviment total del centroide
> R1:=r-a;R2:=r+a;volta:=(beta)->2*Pi*beta;
> f1:=plot3d([R1,volta(beta),h],beta=0..1,h=-1...1,coords=cylindrical,shading=zgreyscale,
numpoints=3500,scaling=unconstrained,axes=boxed,style=wireframe,transparency=0.75,
view=[-12..12,-12..12,-1..1],color=green):
> f2:=plot3d([R2,volta(beta),h],beta=0..1,h=-1..1,coords=cylindrical,shading=zgreyscale,
numpoints=3500,scaling=unconstrained,axes=boxed,style=wireframe,transparency=0.75,
view=[-12..12,-12..12,-1..1],color=green):
> display({f1,f2},view=[-12..12,-12..12,-1..1]):
> # cc son circumferencies rmin i rmax sobre el pla  galactic
> cc:=spacecurve({[(r-a)*cos(t),(r-a)*sin(t),0],[(r+a)*cos(t),(r+a)*sin(t),0]},t=0..2*Pi,
numpoints=1500,color=green,scaling=unconstrained):
> #passem a posicions en cilindriques
```

```
> pi:=(t)->r+x(t);theta:=(t)->C*t+y(t)/r;theta2:=(t)->C*t;
corba:=spacecurve([pi(t),theta(t),z(t)],t=0..2*Pi,coords=cylindrical,scaling=unconstrained,
axes=boxed,thickness=3,color=red,numpoints=2500):
corba2:=spacecurve([r,theta2(t),0],t=0..2*Pi,coords=cylindrical,scaling=unconstrained,
axes=boxed,thickness=3,color=blue,numpoints=2500):
display({f1,corba,corba2,f2}, view=[-12..12,-12..12,-1..1],labels = [ "x", "y" ,"z"]);
> # orbites circulars del punt guia (blau) i de l'estrella (vemell)
> F := proc(t)
plots[display](
spacecurve([r,theta2(x),0],x=0..t,coords=cylindrical,scaling=unconstrained,axes=boxed,
thickness=3,color=blue,numpoints=2500),
spacecurve([pi(x),theta(x),z(x)],x=0..t,coords=cylindrical,scaling=unconstrained,axes=boxed,
thickness=3,color=red,numpoints=2500));
end proc:
> DF:=animate(F,[t],t=0..2*Pi):
> # moviment dels punts: punt guia (blau) i mov. estrella (vemell)
> ball:=proc(x,y,z) plots[pointplot3d]([[x,y,z]],coords=cylindrical,symbol=solidcircle,
symbolsize=40) end proc:
> DA:=animate(ball,[r,theta2(x),0], x=0..2*Pi,color=blue,scaling=unconstrained,axes=boxed,
thickness=3,numpoints=1500):
> DB:=animate(ball,[pi(x),theta(x),z(x)], x=0..2*Pi,color=red,scaling=unconstrained,axes=boxed,
thickness=3,numpoints=1500):
> # grafic de la linea que uneix el punt guia i l'estrella
> linia:=proc(t) plots[display](
plottools[line]([r*cos(theta2(t)),r*sin(theta2(t)),0], [pi(t)*cos(theta(t)),pi(t)*sin(theta(t)),
z(t)], color=black))
end proc:
> DC:=animate(linia,[t],t=0..2*Pi):
> # projeccio del mov elÂ·liptic de l'estrella al voltant del punt guia
> ombra:=proc(t)
local eliorig,g;
eliorig:=implicitplot((x)^2+(k*(y))^2=a^2, x = -5 .. 5, y = -5 .. 5, gridrefine = 2,
scaling=unconstrained,color=black,numpoints=2500);
g:=transform((x,y)->[x+r*cos(C*t),y+r*sin(C*t),0]);
g(rotate(eliorig,C*t));
end proc:
> DE:=animate(ombra,[t],t=0..2*Pi):
> display([cc,DA,DB,DC,DE,DF],orientation=[0,0,0],labels = [ "x", "y" ,"z"]);
```

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