The shape of a galaxy is not much influenced by the potential as it might seem[1]

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ABSTRACT

The dynamics of a galaxy as a stellar system in statistical equilibrium is usually obtained from the superposition principle, based on the linearity of the Boltzmann collisionless equation (BCE) in regard to the phase space density function. The term statistical equilibrium is a notion coming from statistical dynamics, although, from an analytical dynamics viewpoint, it should be associated with an invariant density function under the BCE in the phase space. Dissipative forces like dynamical friction, which are essential to statistical dynamics, emerge in analytical dynamics via non steady-state phase density functions and/or potentials as solutions of the BCE. When some kinematic knowledge about the stellar integrals of motion or the velocity distribution function is already known, Jeans' inverse problem leads, from an statistical viewpoint, to the most probable time-dependent potential function.

For a mixture of several galactic components, the natural approach is the Jeans' inverse problem, by associating a generalised quadratic velocity distribution with each stellar population. Then, the BCE relates the dynamics of each stellar population to a potential, which is shared by all of the population components. Therefore, in solving the BCE, the coexistence of several stellar populations introduces a set of integrability conditions, which are *conditions of consistency for a population mixture*, that forces the potential function to adopt a relatively simple functional form, while the velocity or mass distributions, or the number of stellar populations, have a higher number of degrees of freedom.

Axially symmetric stellar systems have been mostly used to describe general features of galaxies, although they cannot account for spiral or bar structures. Nevertheless, due to the conditions of consistency, axially symmetric potentials are proven to be still suitable to describe non-axially symmetric stellar systems. A paradigm of this situation is the point-axial symmetry model, with rotational symmetry of order two, devoted to allow mass or velocity distributions consistent with spiral or bar structures. In such a case, the BCE yields an axially symmetric potential, although the mass and velocity distributions still maintain point-axial symmetry.

1. Introduction

In wide regions of a galaxy the phase space density function can be approximated as depending on an integral of motion quadratic in the peculiar velocities, by leaving free the functional dependency in time and space (Chandrasekhar 1960). However, the symmetry of this distribution does not allow non-null odd-order central moments, so that a mixture of populations is needed to account for other informative statistics of the velocity distribution. For the whole three dimensional space, under the axial symmetry hypothesis, Sala (1990) determined the family of potential functions that are consistent with such a quadratic integral of motion, and Cubarsi (1990) studied what restrictions would apply to the potential for a mixture of stellar populations under the same hypothesis. When the axial symmetry hypothesis relaxes toward a point-axial model, i.e. rotational symmetry of 180°, to account for mass distributions consistent with ellipsoidal, spiral or bar structures, Chandrasekhar's equations also yield an axisymmetric potential

(Sanz-Subirana 1987, Juan-Zornoza 1995). The parameters involved in the distribution function of a single population are detailed in the Appendix. By using the variables $\tau = \frac{1}{2}\varpi^2$ and $\zeta = \frac{1}{2}z^2$, the potential satisfies

$$2K_4\left[\tau\frac{\partial^2 U}{\partial\tau^2} + 2\frac{\partial U}{\partial\tau} - \zeta\frac{\partial^2 U}{\partial\zeta^2} - 2\frac{\partial U}{\partial\zeta} - (\tau - \zeta)\frac{\partial^2 U}{\partial\tau\partial\zeta}\right] + (K_1 - k_3)\frac{\partial^2 U}{\partial\tau\partial\zeta} = 0$$

$$2K_4\zeta\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial\tau} - \frac{\partial U}{\partial\zeta}\right) + \dot{K}_1\tau\frac{\partial^2 U}{\partial\tau^2} + K_1\frac{\partial^2 U}{\partial t\partial\tau} + 2\dot{K}_1\frac{\partial U}{\partial\tau} + \frac{1}{2}\dddot{K}_1 + \dot{k}_3\zeta\frac{\partial^2 U}{\partial\tau\partial\zeta} = 0 \qquad (1)$$

$$2K_4\tau\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial\zeta} - \frac{\partial U}{\partial\tau}\right) + \dot{k}_3\zeta\frac{\partial^2 U}{\partial\zeta^2} + k_3\frac{\partial^2 U}{\partial t\partial\zeta} + 2\dot{k}_3\frac{\partial U}{\partial\zeta} + \frac{1}{2}\dddot{k}_3 + \dot{K}_1\tau\frac{\partial^2 U}{\partial\zeta\partial\tau} = 0$$

These equations depend on the population specific functions $K_1(\theta, t)$, $k_3(t)$ and $K_4(\theta)$. In addition, there are three integrability conditions depending on the θ-derivatives $K_1'(\theta, t)$ and $K_4'(\theta)$.

2. Conditions of consistency

2.1. Axisymmetric potential

The constant k_2 does not appear in Eq. 1, so that the axisymmetric potential does not constraint the velocity distribution in the rotation direction. The common solution of Eq. 1, planned for each population, is of course valid for populations having the parameters K_1, k_3 and K_4 proportional. However, we must reject such a case because it leads to extremely constrained populations, with the same differential movement and proportional velocity ellipsoids, but for the rotation direction.

2.2. Point-axial symmetric distribution

One or several populations may have a point-axial distribution even with an axisymmetric potential. Therefore, the potential does not depend on the specific population parameter $K_4(\theta)$. Then, we are led to the following conditions,

$$\tau\frac{\partial^2 U}{\partial\tau^2} + 2\frac{\partial U}{\partial\tau} - \zeta\frac{\partial^2 U}{\partial\zeta^2} - 2\frac{\partial U}{\partial\zeta} - (\tau - \zeta)\frac{\partial^2 U}{\partial\tau\partial\zeta} = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial\tau} - \frac{\partial U}{\partial\zeta}\right) = 0 \qquad (2)$$

which yield a potential $U = U_1(\tau + \zeta, t) + \frac{1}{\tau+\zeta} U_2(\zeta/\tau)$

It is worth noticing that the same conditions are obtained by imposing the consistency with one or several populations having a flat velocity distribution, $K_4 \to 0$, which is also equivalent to a velocity distribution isothermic in the z direction.

2.3. Separability of the potential

According to the first expression in Eq. 1 and Eq. 2, we have

$$(K_1 - k_3)\frac{\partial^2 U}{\partial\tau\partial\zeta} = 0 \qquad (3)$$
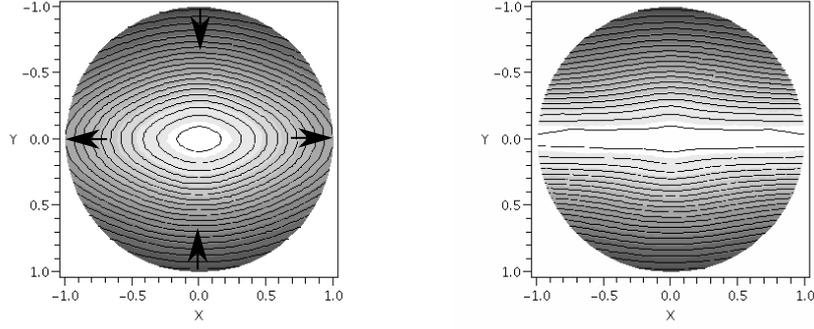
Fig. 1. Rupture of axisymmetry in the Galactic plane produced by tidal forces. The bar structure acts as a quasi-static density wave for the other components of the rotating disc.

which implies either $K_1(\theta, t) = k_3(t)$ or $\dfrac{\partial^2 U}{\partial \tau \partial \zeta} = 0$. In the first case, the resulting potential is

$$U = A(t)(\tau + \zeta) + \frac{1}{k_3} U_1\left(\frac{\tau + \zeta}{k_3}\right) + \frac{U_2(\zeta/\tau)}{\tau + \zeta} \tag{4}$$

and, in the separable case, it is

$$U = A(t)(\tau + \zeta) + \frac{B}{\tau}, \text{if } K_1' = 0; \quad U = A(t)(\tau + \zeta), \text{if } K_1' \neq 0 \tag{5}$$

2.4. Unconstrained centroid motions

The separable potentials, Eq. 5, allow unconstrained centroid motions in all directions. However, the non-separable case, Eq. 4, requires the potential to be independent from the quantity $\dot{k}_3/k_3$. This condition implies a potential with the following form

$$U = A(t)(\tau + \zeta) + \frac{U_2(\zeta/\tau)}{\tau + \zeta} \tag{6}$$

In all this cases the velocity and mass distributions depend on θ through K_4 (and only for the unrealistic harmonic potential of Eq. 5 through K_1). In any case, the potential depends on time through $A(t)$ and $k_3(t)$.

3. Bar structure

The rupture of axisymmetry takes place probably by the interaction of close galaxies, where tidal forces play a crucial role Fig. 1. They produce a transient variation in their mass distributions, by changing them toward a nearly ellipsoidal distribution. Each interacting galaxy is stretched by the gravitational field of the other, and the most vulnerable components, such as the gas and stars right at the outer edges of the disc, are sheared off from their respective galaxy. The tidal shear acts to twist and compress the gas clouds to trigger intense star formation around the major axis of the elliptical disc. Part of the gas component, in their rotation around the galactic centre, is withheld and feeds an emerging bar structure with a higher formation rate by causing more luminosity.

The tidal force has its maximum efficiency over the component of one galaxy having a synchronous rotation with the other galaxy. Although for rigid bodies only the satellite should be
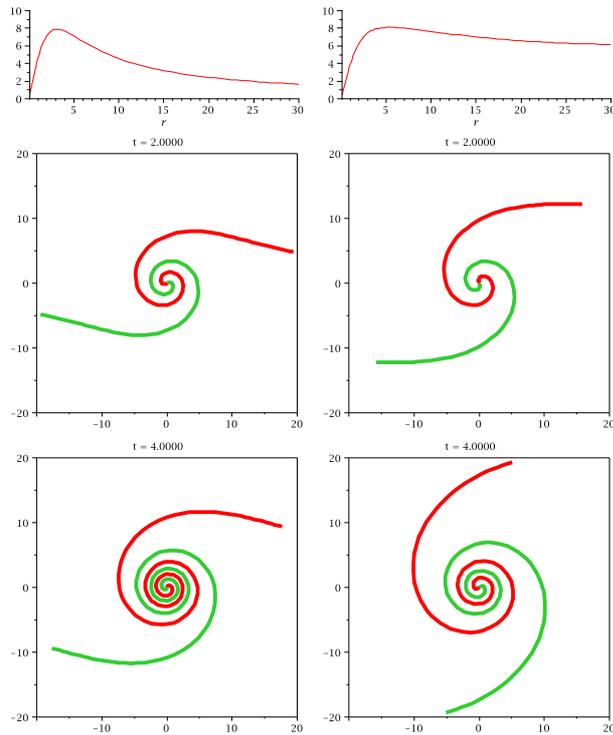
Fig. 2.
(Left) Rotation velocity curve for a single population with $\alpha = 0.1$ and $\beta = 5$ in Eq. 8 (arbitrary units). The high slope at the origin transforms the initial bar into two spiral arms. As time increases, due to the vanishing trend of the rotation velocity, the shape of the arms is apparently maintained, so that the arms act as a static density wave for the stellar disc components.
(Right) Curve for a mixture of four populations with $\alpha = 0.1, 0.01, 0.001, 0$, respectively. If the population factor is p and the values for $p\beta$ are in proportions 5:1:0.1:0.1, the bar also produces two spiral arms, but, as time goes by, the flat rotation curve provides two spiral arms. In this case, the arms act as a rotating density wave for the other disc components.

tidally locked around the larger one, for the fluid components of a galaxy it is likely that one of the stellar or gas components of the host galaxy to be also tidally locked with the satellite, especially if the discs are in a common plane. Such a situation is basically consistent with the bar structure acting as a quasi-static density wave (Lin & Shu 1964) for the other components of the rotating disc.

4. Spiral arms

The bar-shaped structure is in the long term unstable for several reasons: e.g., the tidal force weakens because the satellite galaxy merges to the host, or because the interacting galaxies move away. Then the bar is left under the effects of the galactic potential by meeting its natural circular motion. In the case of galaxies moving away in parabolic orbits, the motion of the bar should depend on the relative motion and rotation directions of the interacting galaxies (Toomre & Toomre 1972), since, in moving away, a resulting torque from both extremes of the bar would also determine its rotation behaviour.

If the bar is non homogeneous, its composition may be managed through a mixture of populations. The rotation affects each population differently, since their mean motion depends on the integrals of motion which are specific of each centroid (Juan-Zornoza et al. 1990, Cubarsi et al. 1990).

5. Rotation curve

The rotation velocity curve produced by the ellipsoidal model is not consistent with the nearly flat velocity curves provided by models accounting for dark matter (Rubin et al. 1980). However,
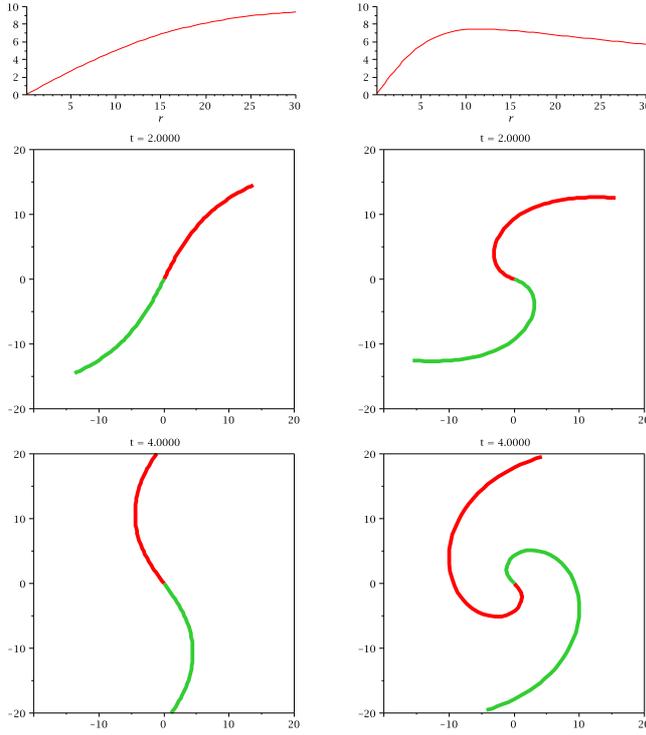
Fig. 3.
(Left) Rotation velocity curve for a two-population mixture not far from a rigid rotation, with $\alpha = 0.001, 0$, and $p\beta$ in proportions 10:1. At the origin, the linear behaviour and small slope produces a rotation of the bar with no spiral structure. The non-vanishing asymptotic trend makes the bar rotate with a slightly spiral shape far from the centre.
(Right) Curve for a two-population mixture with a higher initial slope and a slightly decreasing asymptotic trend, $\alpha = 0.01, 0.001$, and $p\beta$ in proportions 10:1. The loss of linearity at the origin produces the curvature of the bar toward a spiral. The non-vanishing rotation at large distances from the origin induces an apparent rotation of the spiral structure.

such a rotation curve is altered by the mixture nature of the stellar system, so that the total rotation curve is, at every point, the weighted mean of the population rotation curves with their respective relative stellar densities, by resulting a shape similar to the envelope of their single population rotation curves. For n populations, with respective fractions $p^{(i)}$, we have

$$\Theta_0 = \sum_{i=1}^{n} p^{(i)}(\varpi, \theta, z, t)\, \Theta_0^{(i)}(\varpi, \theta, z, t) ; \quad \sum_{i=1}^{n} p^{(i)}(\varpi, \theta, z, t) = 1 \tag{7}$$

In the plane $z = 0$, the rotation velocity for a single ellipsoidal population is (e.g., Sala 1990)

$$\Theta_0 = -\frac{\beta\varpi}{1 + \alpha\varpi^2} ; \quad \text{with} \quad \alpha = \frac{1}{\varpi^2}\left(\frac{\mu_{\varpi\varpi}}{\mu_{\theta\theta}} - 1\right)\Bigg|_{\varpi_0} > 0 \tag{8}$$

computed from local values of the second moments. Depending on the values of α we may have a variety of curves ranging from an asymptotically vanishing rotation velocity, as in Fig. 2 (left), to the linear rotation curve of a constant angular velocity, when $\alpha \to 0$. If $\mu_{\varpi\varpi} >> \mu_{\theta\theta}$, we obtain a rapidly increasing curve at the origin, as in Fig. 2, while if $\mu_{\theta\theta} \to \mu_{\varpi\varpi}$, the curve has a soft slope at the origin, as in Fig. 3. For a mixture of ellipsoidal populations, we may get a number of asymptotic trends, including flat rotation curves as Fig. 2 (right).

Notice that the case of nearly constant angular velocity is consistent with the subpopulation of early-type stars (named population A1 in Alcobé & Cubarsi 2005) with approximately no net radial motion (Cubarsi & Alcobé 2006). This stellar component was also found as associated with one of the prominent modes (around the Hyades stream) of the velocity distribution for disc stars with absolute velocity lower than 51 km s^{-1} (Sample IV in Cubarsi 2010) and also with one of the main subpopulations of the disc stars with eccentricity $e \leq 0.15$, associated with the Hyades and Pleiades stellar groups (Cubarsi 2010).

6. Discussion

The mixture of point-axial stellar systems with a symmetry plane maintains the axial symmetry of the velocity distribution in this plane in all the cases but for the harmonic potential, although it produces an apparent vertex deviation of the whole velocity distribution due to the unconstrained mean velocity of the populations, that may include a net radial motion. As the interaction that breaks the axial symmetry disappears, the bar evolves under the *axial* gravitational field toward an spiral arm structure, that depends on the average rotation velocity of its stellar components. The spiral arms maintain the point-axial symmetry until they are dissolved within the disc after a number of turns, and the galaxy would then recover the axial symmetry of the initial mass distribution thanks to the potential that has not lost its initial symmetry. It lasts however to evaluate whether the introduction of a symmetry plane is the cause of the nearly axial symmetry of the velocity distribution in $z = 0$, produced by $K_1' = 0$, or whether, by relaxing this hypothesis, we might get a clear point-axial velocity distribution in this plane.

Appendix

For a single stellar population, a generalised quadratic velocity distribution function in the peculiar velocities (u_1, u_2, u_3) may be written as $f(Q + \sigma(\boldsymbol{r}, t))$, $Q = \sum_{i,j} A_{ij}(\boldsymbol{r}, t) \, u_i u_j$, where A_{ij} are elements of a symmetric, positive definite matrix. Then, $Q + \sigma$ is an isolating integral of the star motion, which is a combination of some of the classical integrals. Under the point-axial symmetry hypothesis and symmetry plane $z = 0$, the elements of the second-rank tensor $\boldsymbol{A}$ are (e.g., Juan-Zornoza & Sanz-Subirana 1991):

$$
\begin{aligned}
A_{\varpi\varpi} &= K_1 + K_4 z^2, & A_{\varpi\theta} &= \tfrac{1}{2}(K_1' + K_4' z^2) \\
A_{\varpi z} &= -K_4 \varpi z, & A_{\theta\theta} &= K_1^* + k_2 \varpi^2 + K_4^* z^2 \\
A_{\theta z} &= -\tfrac{1}{2} K_4' \varpi z, & A_{zz} &= k_3 + K_4 z^2
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
K_1 &= k_1 + q \sin(2\theta + \varphi_1), & K_1^* &= k_1 - q \sin(2\theta + \varphi_1) \\
K_4 &= k_4 + n \sin(2\theta + \varphi_2), & K_4^* &= k_4 - n \sin(2\theta + \varphi_2)
\end{aligned}
\tag{10}
$$

being k_1, k_3, q, φ_1 time dependent functions, and k_2, k_4, n, φ_2 constants. The uppercase letter K represents a function depending on θ, the accents meaning derivatives with respect to it.

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