Billiards with a given number of (*k*,*n*)-orbits

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We consider billiard dynamics inside a smooth strictly convex curve. For each pair of integers (*k*,*n*), we focus our attention on the billiard trajectory that traces a closed polygon with *n* sides and makes *k* turns inside the billiard table, called a (*k*,*n*)-orbit. Birkhoff proved that a strictly convex billiard always has at least two (*k*,*n*)-orbits for any relatively prime integers *k* and *n* such that $1 \leq k < n$. In this paper, we show that Birkhoff's lower bound is optimal by presenting examples of strictly convex billiards with exactly two (*k*,*n*)-orbits. We generalize the result to billiards with given even numbers of orbits for a finite number of periods. © *2012 American Institute of Physics.*
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We look at the following problem: A particle moves with constant speed in a region enclosed by a curve Γ, reflecting elastically at the impacts with the boundary, called Billiard Problem. The trajectories described by the particle are polygonal lines and a (*k*,*n*)-trajectory is a particle path that closes after *n* hits with the boundary, making *k* windings before closing. On the beginnings of the 20th century, Birkhoff proved that billiards on strictly convex curves have at least two (*k*,*n*)-trajectories, for any relatively prime integers *k* and *n* such that $1 \leq k < n$. The proof of this theorem suggests that for each pair (*k*,*n*), there actually exists a strictly convex billiard having exactly two of such trajectories. In this paper, we show that this is true and produce examples of curves satisfying this property. We also deal with the question of finding billiards with given even numbers of orbits for a finite number of periods.

I. INTRODUCTION

Let Γ be a planar, closed, regular, simple, oriented counterclockwise C^l curve, $l \geq 2$, with strictly positive curvature and given in polar coordinates by $r = r(\theta)$.

The billiard problem on Γ consists in the free motion of a point particle in the plane region enclosed by Γ, being reflected elastically at the impacts with the boundary. The motion is completely determined by the point of impact at Γ, given by the polar angle θ, and the direction of motion immediately after each reflection, defined by $p = \cos \alpha$, where α is the angle between the direction of motion and the oriented tangent to the boundary at the impact point. Therefore, we can define a billiard map $f : \mathbb{T} \times (-1, 1) \to \mathbb{T} \times (-1, 1)$, $f(\theta_0, p_0) = (\theta_1, p_1)$, which maps each initial condition (θ_0, p_0) to the next impact and direction (θ_1, p_1). Here, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Billiards inside strictly convex C^l-curves have several useful properties.[1–4] We just recall that the billiard map f is

a C^{l-1}-diffeomorphism that preserves the measure $d\mu = \|\Gamma'(\theta)\| dp d\theta$. Besides, f is a monotone twist map with Lagrangian function

$$h(\theta, \tilde{\theta}) = \|\Gamma(\theta) - \Gamma(\tilde{\theta})\|,$$

so the billiard dynamics satisfies the implicit equations

$$\begin{cases} p_0 = -\|\Gamma'(\theta_0)\|\partial_1 h(\theta_0, \theta_1), \\ p_1 = \|\Gamma'(\theta_1)\|\partial_2 h(\theta_0, \theta_1). \end{cases}$$

Let k and n be two relatively prime integers such that $1 \leq k < n$. We say that a billiard orbit

$$(\theta_i, p_i) = f^i(\theta_0, p_0), \quad i \in \mathbb{Z},$$

is *n-periodic* when n is the smallest positive integer such that $(\theta_n, p_n) = (\theta_0, p_0)$. In addition, we say that it is a *(k,n)-orbit* if it is *n*-periodic and the angular variable verifies the relation $\theta_n = \theta_0 + 2\pi k$ when lifted to the universal cover $\mathbb{R}$. All points of a (*k*,*n*)-orbit are called *(k,n)-periodic*.

A point particle following any (*k*,*n*)-orbit traces a polygon with *n* sides that makes *k* turns inside the billiard table. Two (*k*,*n*)-orbits are *geometrically distinct* when they give rise to different polygons. Four examples of (*k*,*n*)-orbits are shown in Fig. 1.

About hundred years ago, Birkhoff introduced convex billiards and stated the following result:[1]

Theorem 1. *If the boundary* Γ *is a strictly convex C^l-curve, $l \geq 2$, then the billiard on* Γ *has at least two geometrically distinct (k,n)-orbits for any relatively prime integers k and n such that $1 \leq k < n$.*

Birkhoff's proof is based on Poincaré's Last Geometric Theorem; see Chap. VI of Ref. 1. Kozlov and Treshchëv derived the same lower bound by means of a variational method; see Chap. II of Ref. 2. Similar variational proofs can be found in Refs. 3 and 4. Kozlov and Treshchëv noted that the lower bound, obtained from completely different methods agree, which suggests the following problem.

Problem 1: Let k and n be two relatively prime integers such that $1 \leq k < n$. Is there some strictly convex billiard

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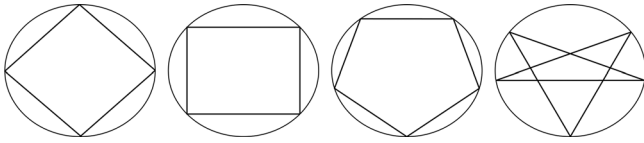b)Electronic mail: rafael.ramirez@upc.edu.

FIG. 1. Two geometrically distinct (1,4)-orbits, a (1,5)-orbit, and a (2, 5)-orbit.

with exactly two geometrically distinct (k,n)-orbits? If so, how to find such a billiard?

As a first observation, we recall that billiards inside ellipses have exactly two (1, 2)-orbits, which correspond to the minor and major axes of the ellipse.

In this paper, we give an affirmative answer to this problem. Our key tool is a function $L_\epsilon : \mathbb{T} \to \mathbb{R}$, called *radial potential*,[5] whose critical points are in 1-to-1 correspondence with the (k,n)-periodic points of the billiard map.

As a particular case of our result, one can show that the small perturbation of the unitary circle given in polar coordinates by $r = 1 + \epsilon \cos n\theta$ has exactly two geometrically distinct (k,n)-orbits when the perturbative parameter ϵ is small enough. This curve was already studied by Duzhin on his MS Thesis,[6] where he proved the existence of exactly two (1,3)-orbits and claims that the proof will work for the other periods.

We also tackle out the following generalized problem.

Problem 2: Let $q_1, \ldots, q_m$ be some positive integers. Let $(k_1, n_1), \ldots, (k_m, n_m)$ be some couples of relatively prime integers such that $1 \leq k_i < n_i$. Is there some strictly convex billiard with exactly $2q_i$ geometrically distinct (k_i, n_i)-orbits for each $i = 1, \ldots, m$? If so, how to find such a billiard?

That is, we look for billiards with given even numbers of orbits for a finite number of periods. We note that billiards with an odd number of (k,n)-orbits are very degenerate and restrict our attention to the generic case.

II. PERTURBATIONS OF THE CIRCULAR BILLIARD

When Γ_0 is the unit circle $r \equiv 1$, its associated billiard map

$$f_0(\theta, p) = (\theta + \omega(p), p), \quad \omega(p) = 2 \arccos(p)$$

is an integrable twist map with Lagrangian function

$$h_0(\theta, \tilde{\theta}) = \|\Gamma_0(\theta) - \Gamma_0(\tilde{\theta})\| = 2\sin(|\theta - \tilde{\theta}|/2).$$

It leaves invariant all horizontal circles $T = \mathbb{T} \times \{\cos(\alpha_0)\}$, $0 < \alpha_0 < \pi$, of the cylinder $\mathbb{T} \times (-1, 1)$ and $f_0|_T$ is just a rigid rotation of angle $2\alpha_0$. Given two relatively prime integers k and n such that $1 \leq k < n$, we consider the *(k,n)-resonant horizontal circle* $T_0 = \mathbb{T} \times \{\cos(k\pi/n)\}$ where every orbit is a (k,n)-orbit of f_0.

We wonder what happens to this resonant horizontal circle under a small smooth perturbation $\Gamma_\epsilon = \Gamma_0 + O(\epsilon)$ of the unit circle, written in polar coordinates as

$$r = r_\epsilon(\theta) = 1 + \epsilon r_1(\theta) + O(\epsilon^2) \tag{1}$$

for some smooth function $r_1 : \mathbb{T} \to \mathbb{R}$.

To begin with, we note that horizontal invariant circles are unusual structures for smooth convex billiard tables. For instance, Gutkin[7] proved that a noncircular billiard has an horizontal invariant circle of the form $\mathbb{T} \times \{\cos(\alpha_0)\}$, $\alpha_0 \neq \pi/2$, if and only if $\tan(j\alpha_0) = j\tan(\alpha_0)$ for some integer $j > 1$. Besides, only the curves of constant width give rise to billiard tables with $\mathbb{T} \times \{0\}$ as an horizontal invariant circle.[7,8] Finally, it is known that resonant invariant curves generically break up, and, if the perturbation is small enough, some periodic orbits always persist in a small neighborhood of each resonant curve.[9] Therefore, the expected behavior is that the (k,n)-resonant horizontal circle T_0 breaks up in a finite number of periodic orbits under the above perturbation.

We want to know how many periodic orbits persist. We will count them by using the *radial Melnikov potential*. For the sake of completeness, we will sketch the main steps leading to its construction in our particular setup of billiards inside perturbed circles. The construction for general twist maps can be found in Refs. 5 and 10.

Let $f_\epsilon = f_0 + O(\epsilon)$ be the billiard map associated to the perturbed circle $\Gamma_\epsilon = \Gamma_0 + O(\epsilon)$ given by Eq. (1).

Lemma 1: If $\epsilon > 0$ is small enough, then there exist two unique smooth functions $g_\epsilon, \tilde{g}_\epsilon : \mathbb{T} \mapsto (-1, 1)$ such that $g_\epsilon(\theta) = \cos(k\pi/n) + O(\epsilon)$ and $\tilde{g}_\epsilon(\theta) = \cos(k\pi/n) + O(\epsilon)$ uniformly in $\theta \in \mathbb{T}$, and $f_\epsilon^n(\theta, g_\epsilon(\theta)) = (\theta, \tilde{g}_\epsilon(\theta))$.

Proof: Let $p_0 = \cos(k\pi/n)$. We consider the function

$$\Xi(p, \epsilon; \theta) := \Pi_1(F_\epsilon^n(\theta, p)) - \theta - 2\pi k,$$

where F_ϵ is a lift of f_ϵ and $\Pi_1(\theta, p) = \theta$ is the projection onto the angular coordinate. The function $\Xi(p, \epsilon; \theta)$ is 2π-periodic in θ and verifies the hypotheses

$$\Xi(p_0, 0; \theta) = 0, \partial_1 \Xi(p_0, 0; \theta) = n\omega'(p_0) \neq 0$$

of the Implicit Function Theorem at $(p, \epsilon) = (p_0, 0)$. Here, p and ϵ are the variables, whereas θ is considered a parameter. Besides, we have used the twist condition

$$\omega'(p_0) = -2(1 - (p_0)^2)^{-1/2} = \frac{-2}{\sin(k\pi/n)} \neq 0.$$

Therefore, there exists a unique 2π-periodic function $g_\epsilon(\theta) = p_0 + O(\epsilon)$ such that $\Xi(g_\epsilon(\theta), \epsilon; \theta) \equiv 0$. Then, we determine $\tilde{g}_\epsilon(\theta)$ from relation $f_\epsilon^n(\theta, g_\epsilon(\theta)) = (\theta, \tilde{g}_\epsilon(\theta))$. Uniformity in θ follows from the compactness of $\mathbb{T}$.   □

Using the preservation of the measure $d\mu$ and the uniqueness of g_ϵ, we have the following result.

Lemma 2: Let $T_\epsilon = \{(\theta, g_\epsilon(\theta)) : \theta \in \mathbb{T}\} = T_0 + O(\epsilon)$ and $\tilde{T}_\epsilon = \{(\theta, \tilde{g}_\epsilon(\theta)) : \theta \in \mathbb{T}\} = T_0 + O(\epsilon)$. Then,

$$T_\epsilon \cap \tilde{T}_\epsilon = \{(k, n)-\text{periodic points of } f_\epsilon\} \neq \emptyset,$$

for any $\epsilon > 0$ small enough.

Let $h_\epsilon(\theta, \tilde{\theta}) = \|\Gamma_\epsilon(\theta) - \Gamma_\epsilon(\tilde{\theta})\|$ be the Lagrangian function of the perturbed billiard map f_ϵ. Rewriting Lemma 6 of Ref. 5 in polar coordinates, we get the lemma bellow.

Lemma 3: The radial distance between T_ϵ and $\tilde{T}_\epsilon$ is

$$\tilde{g}_\epsilon(\theta) - g_\epsilon(\theta) = \|\Gamma'_\epsilon(\theta)\| L'_\epsilon(\theta),$$

where $L_\epsilon : \mathbb{T} \to \mathbb{R}$, $L_\epsilon(\theta) = \sum_{j=1}^n h_\epsilon(\bar{\theta}_{j-1}(\theta, \epsilon), \bar{\theta}_j(\theta, \epsilon))$, *and* $\bar{\theta}_j(\theta, \epsilon) = \Pi_1(f_\epsilon^j(\theta, g_\epsilon(\theta)))$ *for* $j = 1, \ldots, n$.

We say that $L_\epsilon : \mathbb{T} \to \mathbb{R}$ is the *radial potential* of the resonant horizontal circle $T_0 = \mathbb{T} \times \{\cos(k\pi/n)\}$ under the perturbation (1).

Corollary 1: The critical points of the radial potential L_ϵ are in 1-to-1 correspondence with the (k,n)-orbits of the perturbed billiard map f_ϵ.

Once the radial potential

$$L_\epsilon(\theta) = L_0(\theta) + \epsilon L_1(\theta) + \mathrm{O}(\epsilon^2)$$

has been introduced, we extract information from its low-order terms. A straightforward computation[5] shows that $L_0(\theta) \equiv 2n \sin(k\pi/n)$ and

$$L_1(\theta) = 2\sin(k\pi/n) \sum_{j=1}^n r_1(\theta + j2\pi k/n). \qquad (2)$$

The function $L_1(\theta)$ is the *radial Melnikov potential* of the (k,n)-resonant horizontal circle $T_0 = \mathbb{T} \times \{\cos(k\pi/n)\}$ under the perturbation (1). It is 2π-periodic; even more, it is $2\pi/n$-periodic.

Next, we relate its nondegenerate critical points on the interval $[0, 2\pi/n)$ with the nondegenerate (k,n)-orbits of the perturbed map f_ϵ. We also determine the linear stability of these (k,n)-orbits.

If (θ_0, p_0) is a n-periodic point of a planar map f that preserves area and orientation, then the determinant of the tangent map $Df^n(\theta_0, p_0)$ is equal to one. Thus, the eigenvalues of this tangent map are either both equal to 1 or –1 (*degenerate* periodic point), or they are real but different (*hyperbolic* periodic point), or they are complex conjugates of modulus one (*elliptic* periodic point).

Proposition 1: If θ_0 is a nondegenerate critical point of $L_1(\theta)$, the perturbed billiard map f_ϵ has a nondegenerate (k,n)-periodic point $\mathrm{O}(\epsilon)$-close to $(\theta_0, \cos(k\pi/n))$ for any $\epsilon > 0$ small enough. Moreover, if θ_0 is a nondegenerate maximum (resp., minimum) of $L_1(\theta)$, then the previous (k,n)-periodic point is hyperbolic (resp., elliptic).

Proof: If θ_0 is a nondegenerate critical point of $L_1(\theta)$, then the radial potential $L_\epsilon(\theta) = L_0 + \epsilon L_1(\theta) + \mathrm{O}(\epsilon^2)$ has a nondegenerate critical point $\theta_\epsilon = \theta_0 + \mathrm{O}(\epsilon)$ for any $\epsilon > 0$ small enough. Then, $(\theta_\epsilon, p_\epsilon)$, $p_\epsilon = g_\epsilon(\theta_\epsilon) = p_0 + \mathrm{O}(\epsilon)$, is a (k,n)-periodic point of f_ϵ. Note that $p_0 = \cos(k\pi/n)$.

Next, we study the linear stability of $(\theta_\epsilon, p_\epsilon)$, which is determined by the value of the trace of the tangent map $A_\epsilon = Df_\epsilon^n(\theta_\epsilon, p_\epsilon)$, since $\det[A_\epsilon] \equiv 1$. Let $\tau_\epsilon = \mathrm{tr}[A_\epsilon]$. The (k,n)-periodic point $(\theta_\epsilon, p_\epsilon)$ is hyperbolic if and only if $|\tau_\epsilon| > 2$, whereas it is elliptic if and only if $|\tau_\epsilon| < 2$. The degenerate case corresponds to $\tau_\epsilon = \pm 2$.

Set $A_\epsilon = A_0 + \epsilon A_1 + \mathrm{O}(\epsilon^2)$ and $\tau_\epsilon = \tau_0 + \epsilon \tau_1 + \mathrm{O}(\epsilon^2)$. Since $f_\epsilon = f_0 + \mathrm{O}(\epsilon)$ and $f_0^n(\theta, p) = (\theta + n\omega(p), p)$, we get that $\tau_0 = 2$, and

$$A_0 = \begin{pmatrix} 1 & n\omega_0 \\ 0 & 1 \end{pmatrix}, \quad \omega_0 = \omega'(p_0) = \frac{-2}{\sin(k\pi/n)} < 0.$$

By equating the $\mathrm{O}(\epsilon)$-terms of the identity $\det[A_\epsilon] \equiv 1$, we deduce that $\tau_1 = \mathrm{tr}[A_1] = a_{11} + a_{22} = n\omega_0 a_{21}$, where

$$A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

On the other hand, we recall from Lemmas 1 and 3 that

$$f_\epsilon^n(\theta, g_\epsilon(\theta)) = (\theta, \tilde{g}_\epsilon(\theta)) = (\theta, g_\epsilon(\theta) + \|\Gamma'_\epsilon(\theta)\| L'_\epsilon(\theta)).$$

If we differentiate the previous identity with respect to θ and evaluate at the critical point θ_ϵ, it turns out that

$$A_\epsilon \begin{pmatrix} 1 \\ g'_\epsilon(\theta_\epsilon) \end{pmatrix} = \begin{pmatrix} 1 \\ g'_\epsilon(\theta_\epsilon) + \|\Gamma'_\epsilon(\theta_\epsilon)\| L''_\epsilon(\theta_\epsilon) \end{pmatrix}, \qquad (3)$$

since $L'_\epsilon(\theta_\epsilon) = 0$. We know from $g_\epsilon(\theta) = p_0 + \mathrm{O}(\epsilon)$ that $g'_\epsilon(\theta_\epsilon) = \epsilon g_1 + \mathrm{O}(\epsilon^2)$ for some coefficient g_1. Then, by equating the $\mathrm{O}(\epsilon)$-terms of relation (3), we obtain that

$$A_0 \begin{pmatrix} 0 \\ g_1 \end{pmatrix} + A_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ g_1 + L''_1(\theta_0) \end{pmatrix},$$

since $\|\Gamma'_0(\theta)\| \equiv 1$ and $L_0(\theta) \equiv 2n\sin(k\pi/n)$. The second component of the previous vectorial identity implies that $a_{21} = L''_1(\theta_0)$. Therefore,

$$\tau_1 = n\omega_0 a_{21} = \frac{-2nL''_1(\theta_0)}{\sin(k\pi/n)}.$$

If θ_0 is a nondegenerate maximum of $L_1(\theta)$, then $L''_1(\theta_0) < 0$, so $\tau_\epsilon = 2 + \epsilon\tau_1 + \mathrm{O}(\epsilon^2) > 2$, and the point is hyperbolic for small enough ϵ. On the contrary, if θ_0 is a nondegenerate minimum of $L_1(\theta)$, then $L''_\epsilon(\theta_0) > 0$, so $|\tau_\epsilon| = |2 + \epsilon\tau_1 + \mathrm{O}(\epsilon^2)| < 2$, and the point is elliptic for small enough ϵ. $\square$

We summarize the results of this section as follows.

Theorem 2. *Let Γ_ϵ be the perturbation of the unit circle given by Eq. (1) in polar coordinates. Let k and n be two relatively prime integers such that $1 \leq k < n$. Let $L_1(\theta)$ be the radial Melnikov potential defined in Eq. (2). If $L_1(\theta)$ is a Morse function and ϵ is small enough, then the (even) number of critical points of $L_1(\theta)$ contained in the interval $[0, 2\pi/n)$ is equal to the number of (k,n)-orbits of the billiard map inside Γ_ϵ. Moreover, all those (k,n)-orbits are $\mathrm{O}(\epsilon)$-close to the (k,n)-horizontal circle $T_0 = \mathbb{T} \times \{\cos(k\pi/n)\}$.*

III. BILLIARDS WITH A GIVEN NUMBER OF (k,n)-ORBITS

Along this section, we will denote the radial Melnikov potential (2) as $L_1^{(k,n)}(\theta)$ for the sake of clarity, because we will deal with different couples (k,n).

A. One fixed period

If $r_1(\theta) = \sum_{j\in\mathbb{Z}} c_j \exp(\mathrm{i}j\theta)$ is the Fourier expansion of the smooth function $r_1 : \mathbb{T} \to \mathbb{R}$, and k and n are two relatively prime integers such that $1 \leq k < n$, then the radial Melnikov potential (2) becomes

$$L_1^{(k,n)}(\theta) = 2n \sin(k\pi/n) \sum_{j \in n\mathbb{Z}} c_j \exp(ij\theta).$$

This, together with Theorem 2, shows how to build billiard tables with a given even number of (k,n)-orbits for a fixed period n.

Proposition 2: Given any two integers $n \geq 2$ and $l \geq 1$, the billiard inside the perturbed unit circle given in polar coordinates by

$$r = 1 + \epsilon \cos ln\theta$$

has, provided that ϵ is small enough, exactly $2l$ geometrically distinct (k,n)-orbits for every integer k relatively prime with n such that $1 \leq k < n$.

Proof: Let k be any integer relatively prime with n such that $1 \leq k < n$. Then, the radial Melnikov potential is

$$L_1^{(k,n)}(\theta) = 2n \sin(k\pi/n) \cos ln\theta.$$

This function is Morse and has $2l$ critical points on $[0, 2\pi/n)$, leading to the desired number of (k,n)-orbits, half of them hyperbolic, half elliptic.

The number of critical points does not depend on k, which explains why the proposition holds for any k.   □

We note that if $m \geq 2$ is an integer such that $ln \notin m\mathbb{Z}$, then $L_1^{(k,m)}(\theta) \equiv 0$ for all k, and our first-order Melnikov method does not provide information about the (k,m)-orbits of this billiard. One must look for higher order Melnikov potentials to study those orbits.

Proposition 2 answers the question of finding a billiard with exactly two orbits for given k and n. For example, given $n = 5$ and $k \in \{1, 2, 3, 4\}$, we take the billiard map associated to $r = 1 + \epsilon \cos 5\theta$. Its phase-space, for $\epsilon = 0.02$, is displayed in Fig. 2, where we clearly see the five islands corresponding to each one of the (k,5)-elliptic orbits. One can see also, on this figure, islands around other (k,n)-orbits, not predicted by our first-order method.

B. Finite number of periods

Once we know how to deal with a single period, the next natural step is to ask if the same method works for a finite number of periods. As an illustrative example, let us seek a billiard with exactly two (1,3)-orbits and two (1,5)-orbits. Once fixed some coefficients $a_3, a_5 \neq 0$, we consider the perturbed unit circle

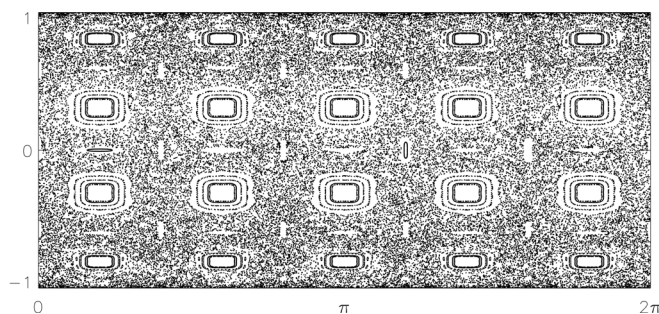$$r = 1 + \epsilon(a_3 \cos 3\theta + a_5 \cos 5\theta).$$

For each $n \in \{3, 5\}$ the radial Melnikov potential of the (k,n)-resonant horizontal circle under that perturbation is

$$L_1^{(k,n)}(\theta) = 2n \sin(k\pi/n)a_n \cos n\theta,$$

which is a Morse function with exactly one maximum and one minimum in the interval $[0, 2\pi/n)$, for every $1 \leq k < n$. Therefore, we deduce that there exists ϵ_n such that the billiard inside that perturbed circle has exactly two (k,n)-orbits, one hyperbolic, one elliptic, for each $n \in \{3, 5\}$, $1 \leq k < n$, $\gcd(k, n) = 1$ and $0 < \epsilon < \epsilon_n$. Taking $\bar{\epsilon} \leq \min\{\epsilon_3, \epsilon_5\}$, we conclude that the perturbed circle has exactly two (1,3), (2,3), (1,5), (2,5), (3,5), and (4,5)-orbits if $\epsilon < \bar{\epsilon}$, one elliptic and one hyperbolic. We display the phase-space of the associated billiard map for $a_3 = 0.2$, $a_5 = 0.3$, and $\epsilon = 0.2$ in Fig. 3. As on the first example, one can see islands around those elliptic orbits and also around other elliptic (k,n)-orbits, $n \notin \{3, 5\}$, not predicted by our first-order method.

This example is generalized in the first part of the proposition below, which also analyses other possibility.

Proposition 3: Let $n_1, ..., n_m \geq 2$ be pairwise distinct integers.

(1) *If $n_i \notin n_j\mathbb{Z}$ for all $i \neq j$, the billiard inside the perturbed unit circle given in polar coordinates by*

$$r = 1 + \epsilon \sum_{i=1}^{m} a_i \cos n_i\theta, \quad a_i \neq 0,$$

 has exactly two geometrically distinct (k, n_i)-orbits for each relatively prime integers k and n_i such that $1 \leq k < n_i$, if ϵ is small enough.

(2) *If $q_1, ..., q_m$ are some arbitrary positive integers and $w = \text{lcm}(q_1 n_1, ..., q_m n_m)$, the billiard inside the perturbed unit circle given in polar coordinates by*

$$r = 1 + \epsilon \cos w\theta$$

 has at least $2q_i$ geometrically distinct (k, n_i)-orbits for each relatively prime integers k and n_i such that $1 \leq k < n_i$, if ϵ is small enough.

FIG. 2. Phase-space of the billiard map associated to $r = 1 + 0.02 \cos 5\theta$.
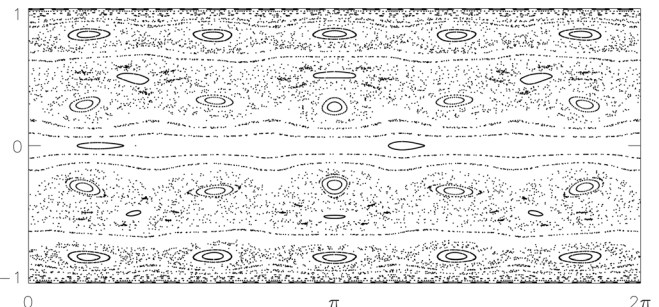

FIG. 3. Phase-space of the billiard map associated to $r = 1 + 0.04 \cos 3\theta + 0.06 \cos 5\theta$.

Proof: (1) Let k be an integer relatively prime with n_i such that $1 \leq k < n_i$. Then, the radial Melnikov potential of the (k, n_i)-resonant horizontal circle is

$$L_1^{(k,n_i)}(\theta) = 2n_i \sin (k\pi/n_i)a_i \cos n_i\theta,$$

which has exactly one maximum and one minimum on the interval $[0, 2\pi/n_i)$. Hence, there exists ϵ_i such that the perturbed billiard has two geometrically distinct (k, n_i)-orbits, one hyperbolic and elliptic, for $0 < \epsilon < \epsilon_i$. Taking $\epsilon < \min\{\epsilon_1, \ldots, \epsilon_m\}$, the result follows.

(2) Suffice it to note that the (k, n_i)-radial Melnikov potential under these hypotheses is

$$L_1^{(k,n_i)}(\theta) = 2n_i \sin (k\pi/n_i) \cos w\theta.$$

It is a Morse function with $2w/n_i \geq 2q_i$ critical points in the interval $[0, 2\pi/n_i)$. $\qquad\square$

Part (2) of this proposition only evinces the difficulty to count critical points of trigonometric polynomials with not relatively prime frequencies. As an example, let us ask if it is possible to have a certain number of (k,n)-orbits with $n \in \{4, 6\}$. Since lcm$(4, 6) = 12$, we consider the curve $r = 1 + \epsilon \cos 12\theta$. If $n \in \{4, 6\}$ and k are relatively prime integers such that $1 \leq k < n$, then the radial Melnikov potential of the (k,n)-resonant horizontal circle is

$$L_1^{(k,n)}(\theta) = 2n \sin (k\pi/n) \cos 12\theta,$$

which is a Morse function with exactly six (resp., four) critical points in the interval $[0, \pi/2)$ (resp., $[0, \pi/3)$) for $n = 4$ (resp., $n = 6$). Therefore, we get six $(k,4)$-orbits and four $(k,6)$-orbits for $\epsilon > 0$ small enough.

Next, we explain how to get six of each. To begin with, we consider $r = 1 + \epsilon(\cos 12\theta + a_{18} \cos 18\theta)$ for some $a_{18} \in \mathbb{R}$. The function $L_1^{(k,4)}(\theta)$ does not change, so there are still six $(k,4)$-orbits for any choice of a_{18}, provided that ϵ is small enough. On the other hand,

$$L_1^{(k,6)}(\theta) = 12 \sin(k\pi/6)(\cos 12\theta + a_{18} \cos 18\theta).$$

This function is Morse when $|a_{18}| \neq 4/9$. Indeed, it has four critical points in the interval $[0, \pi/3)$ when $|a_{18}| < 4/9$, but six when $|a_{18}| > 4/9$. Thus, there exist four or six $(k,6)$-orbits depending on whether $|a_{18}| < 4/9$ or $|a_{18}| > 4/9$, see Fig. 4. It turns out that $L_1^{(k,6)}(\theta)$ has still four critical points in

the interval $[0, \pi/3)$ when $|a_{18}| = 4/9$, but then one of them is degenerate, so we cannot use the Implicit Function Theorem to guarantee the existence of a periodic orbit near to this value.

As a last example, let us look for six $(k,4)$-orbits and eight $(k,6)$-orbits. First, we consider $r = 1 + \epsilon(a_{12} \cos 12\theta + a_{24} \cos 24\theta)$, for some $a_{12}, a_{24} \in \mathbb{R}$. Then, the radial Melnikov potentials are

$$L_1^{(k,n)}(\theta) = 2n \sin (k\pi/n)(a_{12} \cos 12\theta + a_{24} \cos 24\theta).$$

Here, $n \in \{4, 6\}$ and k are relatively prime integers such that $1 \leq k < n$. These functions $L_1^{(k,n)}(\theta)$ are Morse when $4|a_{24}| \neq |a_{12}|$. To be more precise, they have six (resp., twelve) critical points in $[0, \pi/2)$ and four (resp., eight) critical points in $[0, \pi/3)$, when $4|a_{24}| < |a_{12}|$ (resp., $4|a_{24}| > |a_{12}|$) for $n = 4$ (resp., $n = 6$). Therefore, no infinitesimal perturbation of the previous form gives rise to the desired result.

To achieve the goal, we may add a higher frequency to the perturbation. For instance, we take $r = 1 + \epsilon r_1$ where

$$r_1 = a_{12} \cos 12\theta + a_{18} \cos 18\theta + a_{30} \cos 30\theta$$

with $a_{12}, a_{18}, a_{30} \in \mathbb{R}$. Then, $L_1^{(k,4)}(\theta)$ does not change, but

$$L_1^{(k,6)}(\theta) = 12 \sin (k\pi/6)r_1(\theta).$$

Nevertheless, this new function $L_1^{(k,6)}(\theta)$ has eight critical points only at some bifurcation values of the parameters $a_{12}, a_{18}, a_{30} \in \mathbb{R}$, and some of the critical points are degenerate, invalidating the use of our method. Thus, adding a higher frequency has not solved the problem. What could be done here is to use

$$r = 1 + \epsilon(a_{12} \cos 12\theta + a_{24} \sin 24\theta)$$

with $2|a_{24}| > |a_{12}|$, which gives six $(1,4)$-orbits and eight $(1,6)$-orbits, all nondegenerate.

IV. REMARKS AND OPEN QUESTIONS

The method we presented answers the question of how to build examples of convex billiards attaining the minimum predicted by Birkhoff's Theorem, for any relatively prime integers k and n such that $1 \leq k < n$.

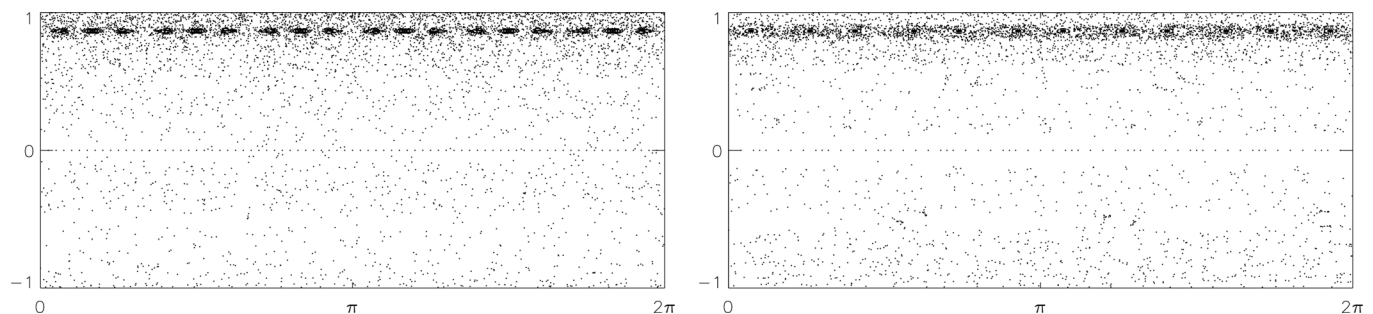
FIG. 4. Islands around elliptic $(1, 6)$-orbits of the billiard inside $r = 1 + \epsilon(\cos 12\theta + a_{18} \cos 18\theta)$ for $a_{18} > 4/9$ (left) and for $0 < a_{18} < 4/9$ (right).

We can also deal with some finite sets of periods or with specific examples, for an even number of orbits, and classify their stability.

We note that billiards with an odd number of (k,n)-orbits are very degenerate, since a periodic Morse function always has an even number of nondegerate critical points, half maxima, half minima. To search for examples with an odd number of orbits, it may be fruitful to study degenerate critical points of L_1, with odd order.

Finally, Birkhoff's Theorem says that every billiard on a strictly convex curve has (k,n)-orbits of any period n. Generically, there is only a finite number of them, for each fixed period.[11] It would be interesting to find examples with a prescribed even number of each period. However, the study of an infinite number of periods is harder than the finite case. Let us explain why. Once fixed a couple (k_i, n_i) and an integer $q_i \geq 1$, we find certain perturbations of a circular table that have exactly $2q_i$ geometrically distinct (k_i, n_i)-orbits when the parameter ϵ is small enough: $0 < \epsilon < \epsilon_i$, for some ϵ_i. Therefore, to find a perturbation with given even numbers of (k_i, n_i)-orbits for an infinite family of couples $\{(k_i, n_i) : i \in I\}$, we should check, among other things, that $\inf_{i \in I} \epsilon_i > 0$. And this is not an easy task.

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