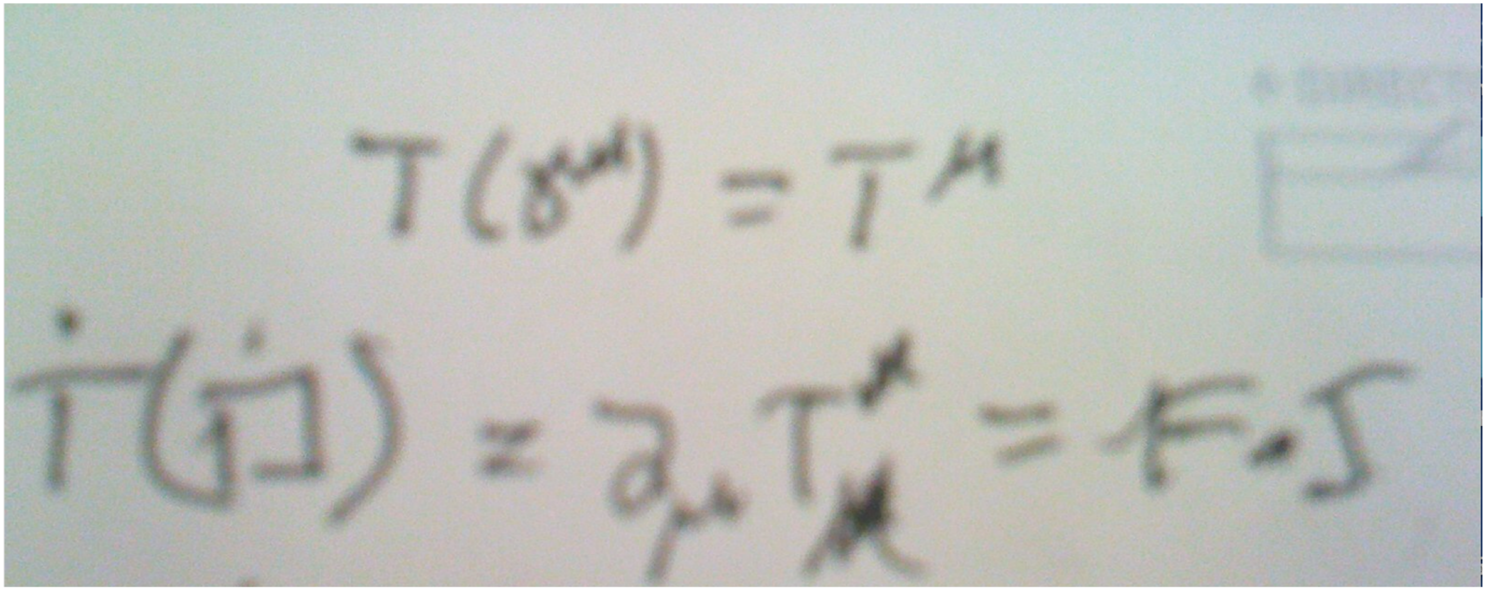
The vector algebra war
a historical perspective

Derek Abbott

I got started in ~2010



David Hestenes

IEEE Access, **12:**1997–2004, 2016

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The Vector Algebra War: A Historical Perspective

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ABSTRACT There are a wide variety of different vector formalisms currently utilized in engineering and physics. For example, Gibbs' three-vectors, Minkowski four-vectors, complex spinors in quantum mechanics, and quaternions used to describe rigid body rotations and vectors defined in Clifford geometric algebra. With such a range of vector formalisms in use, it thus appears that there is as yet no general agreement on a vector formalism suitable for science as a whole. This is surprising, in that, one of the primary goals of 19th century science was to suitably describe vectors in 3-D space. This situation also has

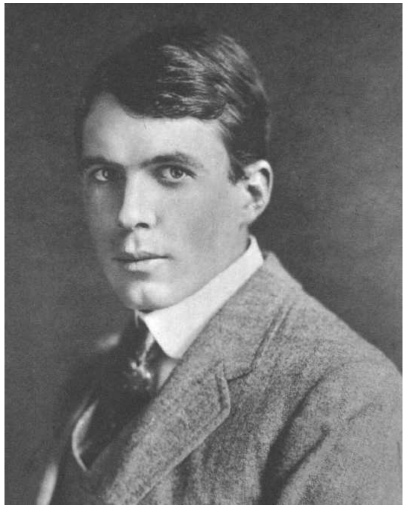
Where is Adelaide?

The University of Adelaide

University opened in 1874

Engineering building

The University of Adelaide

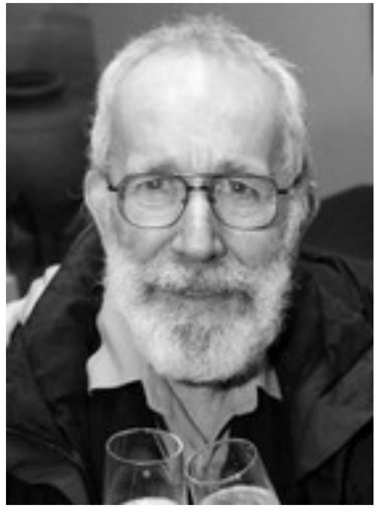


W. L. Bragg
1915 Nobel prize
X-ray crystalogr.

M. L. E. Oliphant
Deuterium
Tritium
Helium 3

H. W. Florey
1945 Nobel prize
Penicillin

R. A. Fisher
Fisher info.
Max. likelihood
F-K Equation



J. R. Warren
2005 Nobel prize
Ulcers

Proc. IEEE, **101**(1):2147–2153, 2013

The Reasonable Ineffectiveness of Mathematics

By DEREK ABBOTT
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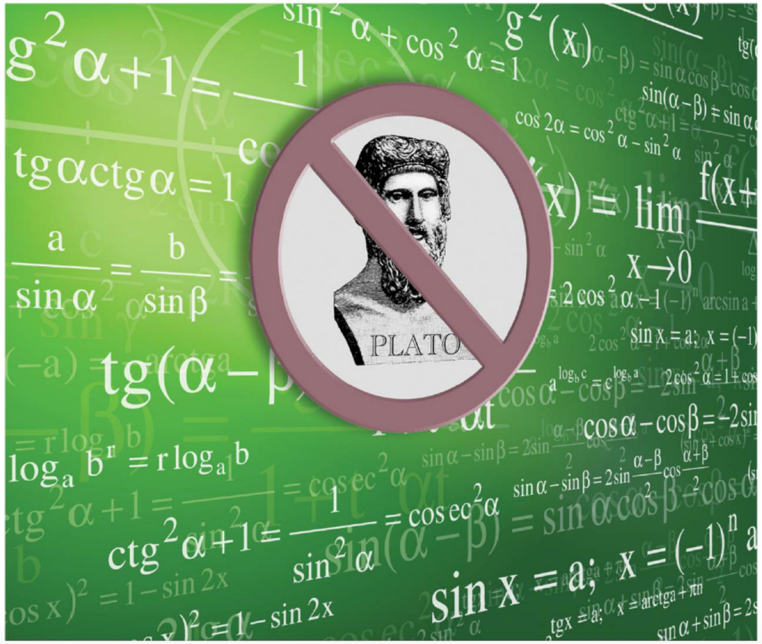
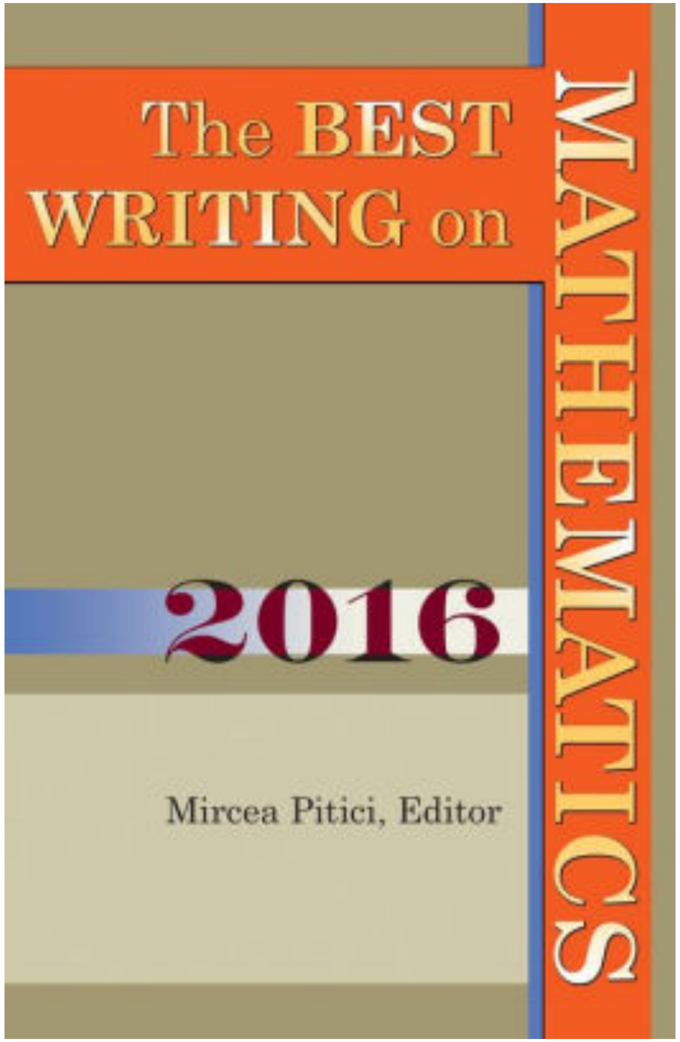


Hamming raised four interesting propositions that he believed fell short of providing a conclusive explanation [3]. Thus, like Wigner before him, Hamming resigned himself to the idea that mathematics is unreasonably effective. These four points are: 1) we see what we look for; 2) we select the kind of mathematics we look for; 3) science in fact answers comparatively few problems; and 4) the evolution of man provided the model.

In this article, we will question the presupposition that mathematics is as effective as claimed and thus remove the quandary of Wigner's "miracle," leading to a non-Platonist viewpoint.[1] We will also revisit Hamming's four propositions and show how they may

Chapter 4



Princeton University Press

Euler's formula

$$e^{-j\pi} = -1$$

Proc. IEEE, **102**(9):1340–1363, 2014

INVITED PAPER

Geometric Algebra for Electrical and Electronic Engineers

This tutorial paper provides a short introduction to geometric algebra, starting with its history and then presenting its benefits and exploring its applications.

By James M. Chappell, Samuel P. Drake, Cameron L. Seidel, Lachlan J. Gunn, *Student Member IEEE*, Azhar Iqbal, Andrew Allison, and Derek Abbott, *Fellow IEEE*

PLOSONE, **10**(3):e0116943, 2015

Functions of Multivector Variables

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School of Electrical and Electronic Engineering, University of Adelaide, Adelaide, South Australia, Australia

* james.chappell@adelaide.edu.au

Abstract

As is well known, the common elementary functions defined over the real numbers can be generalized to act not only over the complex number field but also over the skew (non-commuting) field of the quaternions. In this paper, we detail a number of elementary functions extended to act over the skew field of Clifford multivectors, in both two and three dimensions. Complex numbers, quaternions and Cartesian vectors can be described by the various components within a Clifford multivector and from our results we are able to demonstrate new inter-relationships between these algebraic systems. One key relationship that we discover is that a complex number raised to a vector power produces a quaternion thus combining these systems within a single equation. We also find a single formula that produces the square root, amplitude and inverse of a multivector over one, two and three dimensions. Finally, comparing the functions over different dimension we observe that $C\ell(\Re^3)$ provides a particularly versatile algebraic framework.

IEEE Trans. TST, **6**(6):793-802

Terahertz Signal Classification Based on Geometric Algebra

Shengling Zhou, Dimitar G. Valchev, Alex Dinovitser, *Member, IEEE*, James M. Chappell, Azhar Iqbal, Brian Wai-Him Ng, *Member, IEEE*, Tak W. Kee, and Derek Abbott, *Fellow, IEEE*

***Abstract*—This paper presents an approach to classification of substances based on their terahertz spectra. We use geometric algebra to provide a concise mathematical means for attacking the classification problem in a coordinate-free form. For the first time, this allows us to perform classification independently of dispersion and, hence, independently of the transmission path length through the sample. Finally, we validate the approach with experimental data. In principle, the coordinate-free transformation can be extended to all types of pulsed signals, such as pulsed microwaves or even acoustic signals in the field of seismology. Our source code for classification based on geometric algebra is publicly available at: https://github.com/swuzhousl/Shengling-zhou/blob/geometric-algebra-classifier/GAclassifier/.**

capture THz spectra both in reflection and in transmission simultaneously. Dual THz systems that perform simultaneous transmission and reflection measurements have been developed [4]. Reflective measurements avoid the disadvantages of transmission; however, this is at the expense of a signal scattered over a wide capture angle, thereby reducing the signal-to-noise ratio (SNR). Transmission measurements offer improved collimation of energy; however, SNR will rapidly fall off as a function of the thickness of the sample. Thus, in practice, a dual reflection and transmission mode system assists in maximizing the total information that can be obtained from an arbitrary

Quantum Inf. Proc., **12**(4):1719–1735, 2013

Quantum Inf Process (2013) 12:1719–1735
DOI 10.1007/s11128-012-0483-7

An improved formalism for quantum computation based on geometric algebra—case study: Grover's search algorithm

James M. Chappell · Azhar Iqbal · M. A. Lohe · Lorenz von Smekal · Derek Abbott

Received: 23 May 2012 / Accepted: 28 August 2012 / Published online: 21 September 2012

Abstract The Grover search algorithm is one of the two key algorithms in the field of quantum computing, and hence it is desirable to represent it in the simplest and

Int. J. Modern Physics, **33**:1460355, 2014

Hot Topics in Physical Information (HoTPI-2013)
International Journal of Modern Physics: Conference Series
Vol. 33 (2014) 1460355 (8 pages)

DOI: 10.1142/S201019451460355X

The double-padlock problem: Is secure classical information transmission possible without key exchange?

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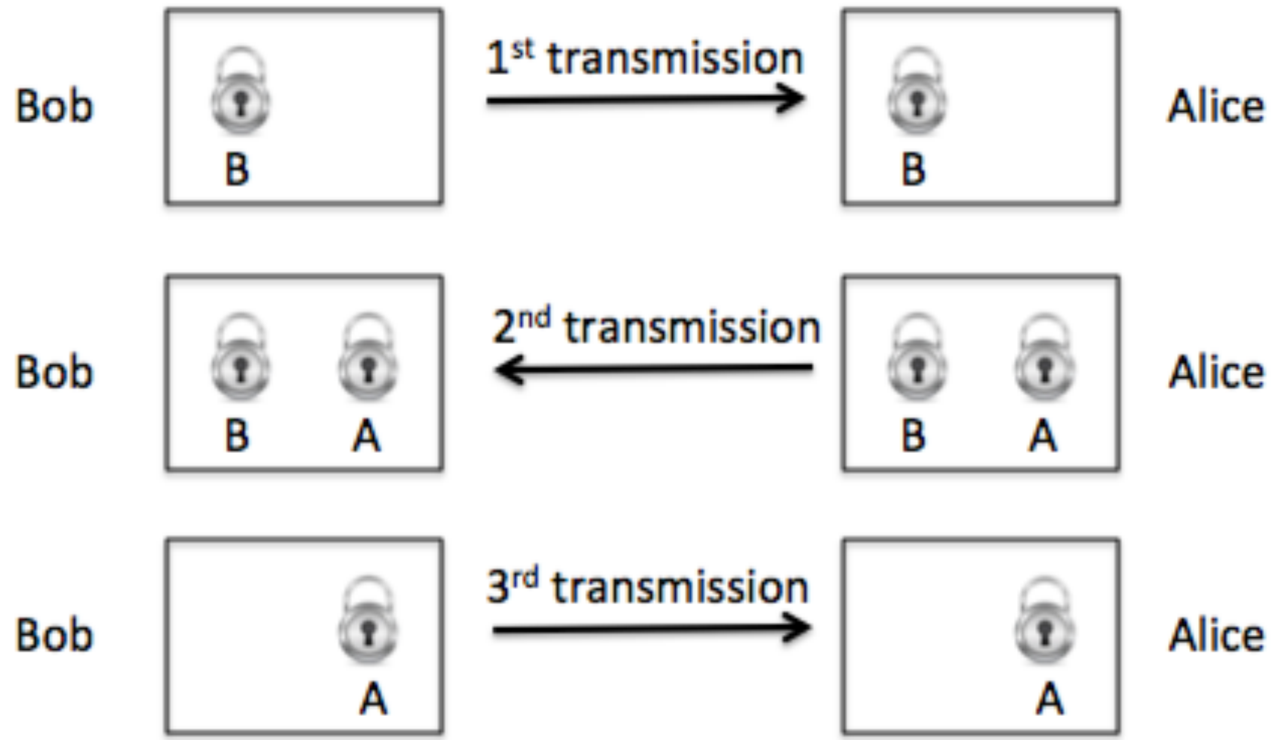
‡derek.abbott@adelaide.edu.au

Published 17 September 2014

Double padlock protocol



Significance

Klappenecker has proved that if there is a solution then:

$$P \neq NP$$

A. Klappenecker, *Fluctuation and Noise Letters*, **4**(4):C25–C26, 2015

Quantum Inf. Proc., **12**(4):1719–1735, 2013

Quantum Inf Process (2013) 12:1719–1735
DOI 10.1007/s11128-012-0483-7

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PLOSONE, **(7)**5:e36404, 2012

OPEN ACCESS Freely available online

PLoS one

N-Player Quantum Games in an EPR Setting

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School of Electrical and Electronic Engineering, University of Adelaide, Adelaide, South Australia, Australia

Abstract

The N-player quantum games are analyzed that use an Einstein-Podolsky-Rosen (EPR) experiment, as the underlying physical setup. In this setup, a player's strategies are not unitary transformations as in alternate quantum game-theoretic frameworks, but a classical choice between two directions along which spin or polarization measurements are made. The players' strategies thus remain identical to their strategies in the mixed-strategy version of the classical game. In the EPR setting the quantum game reduces itself to the corresponding classical game when the shared quantum state reaches zero entanglement. We find the relations for the probability distribution for N-qubit GHZ and W-type states, subject to general measurement directions, from which the expressions for the players' payoffs and mixed Nash equilibrium are determined. Players' $N \times N$ payoff matrices are then defined using linear functions so that common two-player games can be easily extended to the N-player case and permit analytic expressions for the Nash equilibrium. As a specific example, we solve the Prisoners' Dilemma game for general $N \geq 2$. We find a new property for the game that for an even number of players the payoffs at the Nash equilibrium are equal, whereas for an odd number of players the cooperating players receive higher payoffs. By dispensing with the standard unitary transformations on state vectors in Hilbert space and using instead rotors and multivectors, based on Clifford's geometric algebra (GA), it is shown how the N-player case becomes tractable. The new mathematical approach presented here has wide implications in the areas of quantum information and quantum complexity, as it opens up a powerful way to tractably analyze N-partite qubit interactions.

RSOS **5**:180526, 2018

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Research

Check for updates

Cite this article: Christian J. 2018 Quantum correlations are weaved by the spinors of the Euclidean primitives. *R. Soc. open sci.* **5**: 180526. http://dx.doi.org/10.1098/rsos.180526

Quantum correlations are weaved by the spinors of the Euclidean primitives

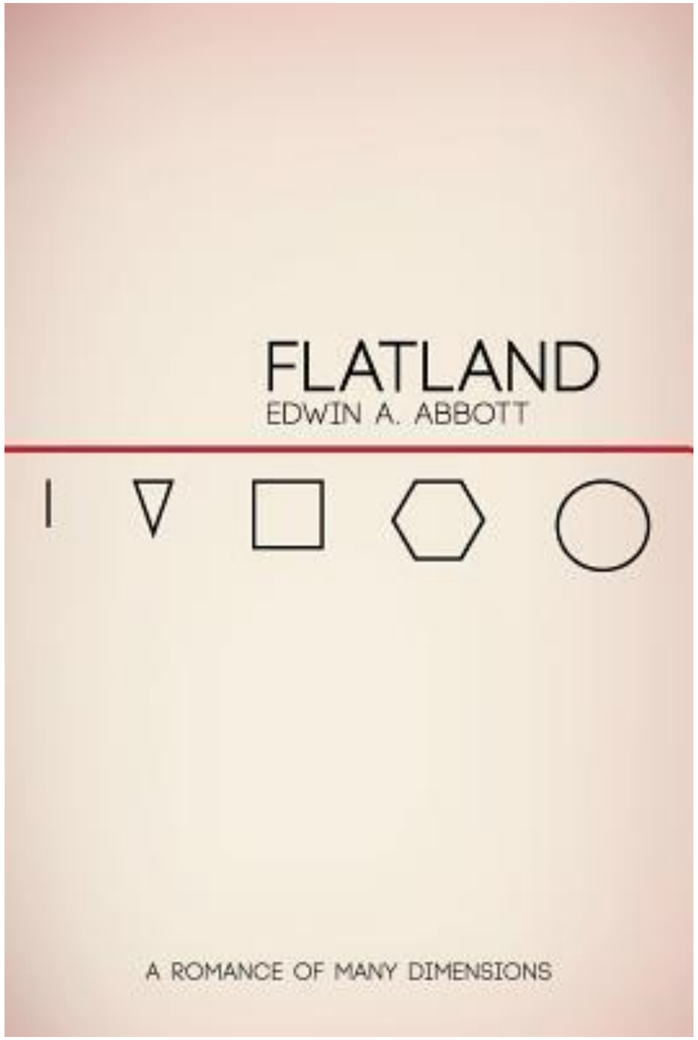
Joy Christian

Einstein Centre for Local-Realistic Physics, 15 Thackley End, Oxford OX2 6LB, UK

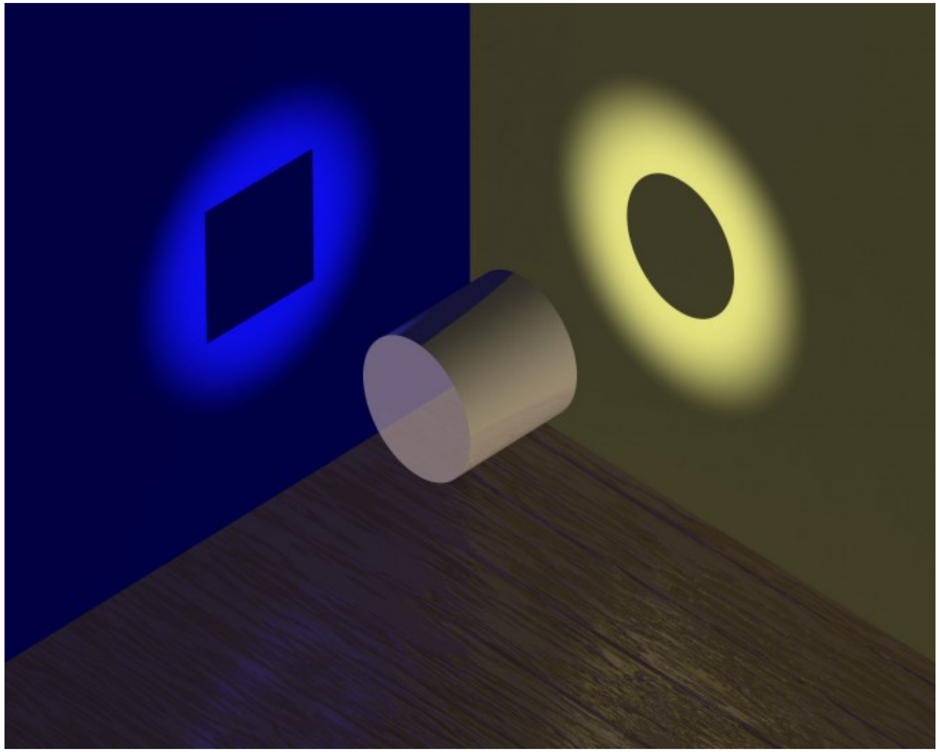
JC, 0000-0002-8741-6943

The exceptional Lie group E_8 plays a prominent role in both mathematics and theoretical physics. It is the largest symmetry group associated with the most general possible normed division algebra, namely, that of the non-associative real

A book written in 1884

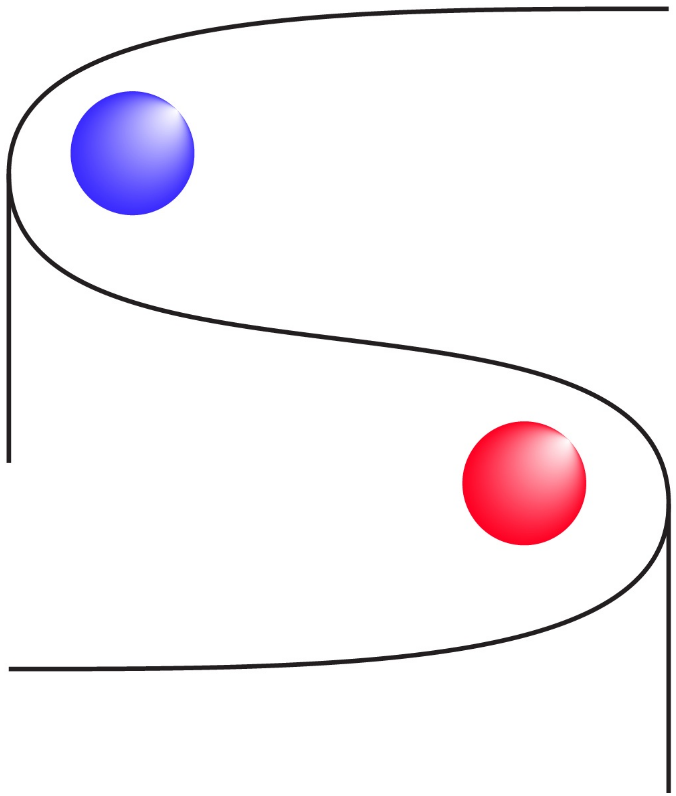


Spookiness is simple in higher dimensions



An internet meme

Spookiness can arise from twists



Inspired by Escher

Joy Christian's paper

- Uses GA to represent S^7
- Recovers known quantum correlations
- Twists in Hopf bundles arise
- Locality is restored (!)
- Non-locality is recovered when $S^7 \longrightarrow \mathbb{E}^3$

End of Introduction

Let the war begin……

Varieties of vector notation

Complex numbers

$$x + iy, \quad i = \sqrt{-1}$$

Gibbs vectors

$$[x, y, z]$$

Quaternions

$$a + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

Minkowski 4-vectors

$$[t, x, y, z]$$

Row and column vectors

$$\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

Pauli spinors

$$\psi = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Dirac spinors

$$\psi_{\mathrm{Dirac}} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

Beginnings of vectors

Plato/Aristotle
4 Century B.C.

Descartes,1637

"Just as arithmetic consists of only four or five operations, namely, addition, subtraction, multiplication, division and the extraction of roots....so in geometry, to find required lines it is merely necessary to add or subtract lines."

Pioneers of vectors

Descartes
1637

$$\Re$$

Gauss
(with Argand and Wessel)
1825

$$\mathbb{C}$$

$$x + iy, \quad i = \sqrt{-1}$$

2D vectors so(2)

Hamilton
1843

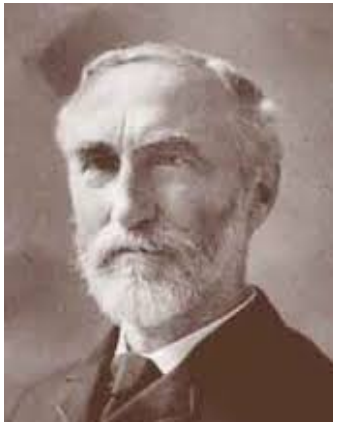
$$\mathbb{H}$$

$$a + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

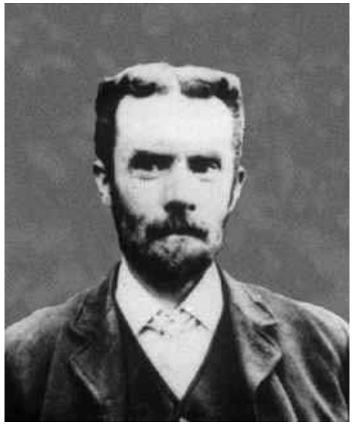
3D vectors su(2)

Two parallel vector universes

Hamilton's
vectors
1843

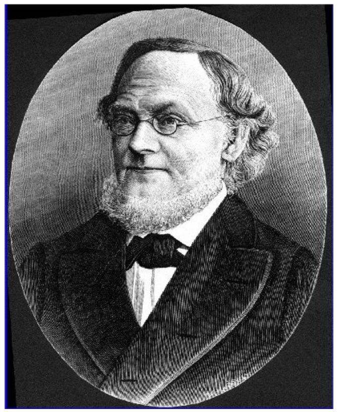


Gibbs



Heaviside

$$[x, y, z]$$



Grassman

Clifford 1879

$$C\ell\left(\Re^3\right)$$

Heaviside on quaternions

"….a positive evil of no inconsiderable magnitude; and that by its avoidance the establishment of vector analysis was made quite simple and its working also simplified, and that it could be conveniently harmonized with ordinary Cartesian work."

Gibbs/Heaviside	vs	Hamilton
$[x, y, z]$		$a + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

So what is wrong with quaternions?

Gibbs/Heaviside vs Hamilton

$$[x, y, z] \qquad\qquad a + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

1.The basis vectors square to -1 and not +1, ie. $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$

Eg., kinetic energy would be:

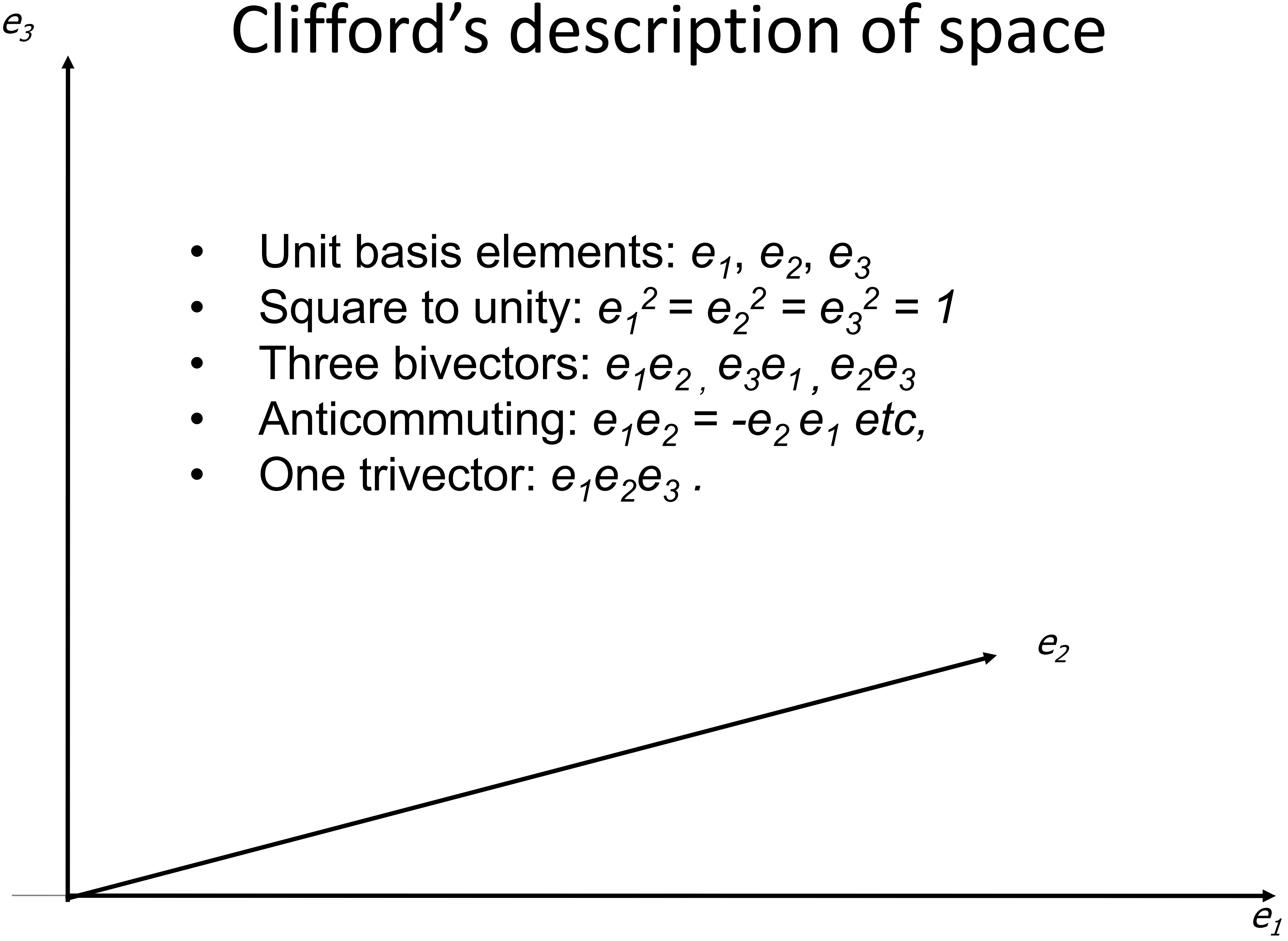
$$\frac{1}{2}m\boldsymbol{v}^2 = -\frac{1}{2}m|\boldsymbol{v}|^2$$

2. All operations are non-commutative.

3. A fourth scalar component needs to be added to the vector components.

Clifford's description of space

- Unit basis elements: e_1, e_2, e_3
- Square to unity: $e_1{}^2 = e_2{}^2 = e_3{}^2 = 1$
- Three bivectors: e_1e_2 , e_3e_1 , e_2e_3
- Anticommuting: $e_1e_2 = -e_2\, e_1$ *etc,*
- One trivector: $e_1e_2e_3$.

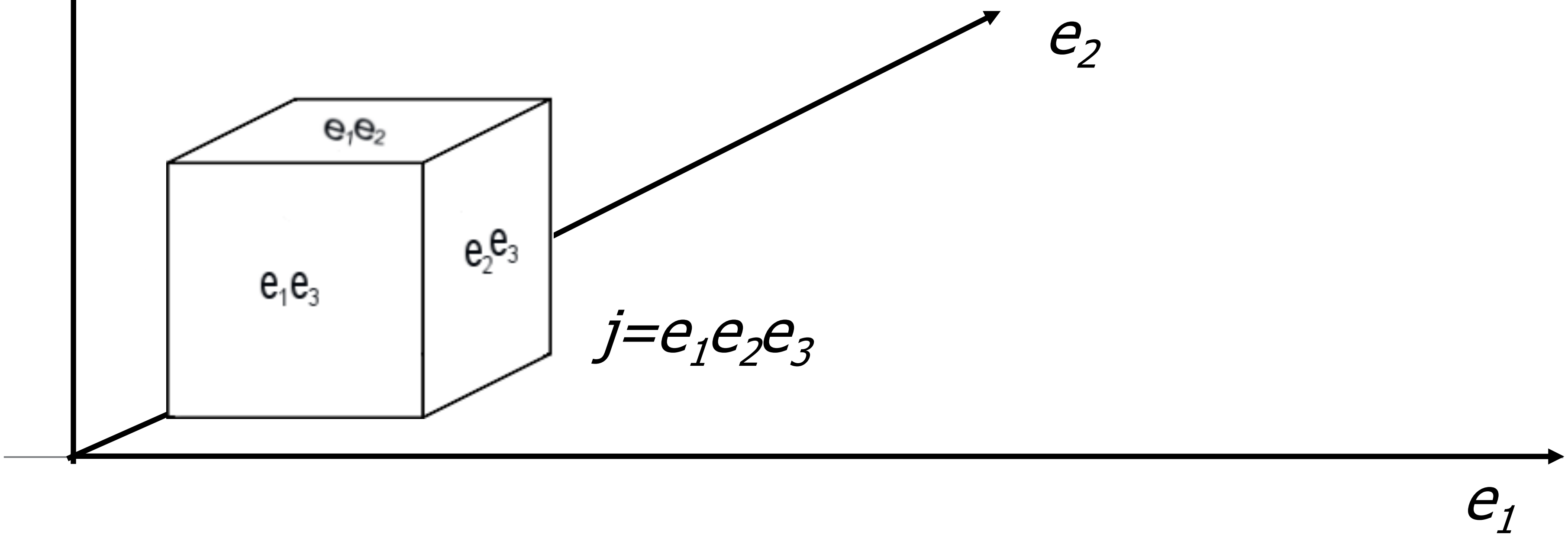


Clifford's unified vector system

Clifford multivector

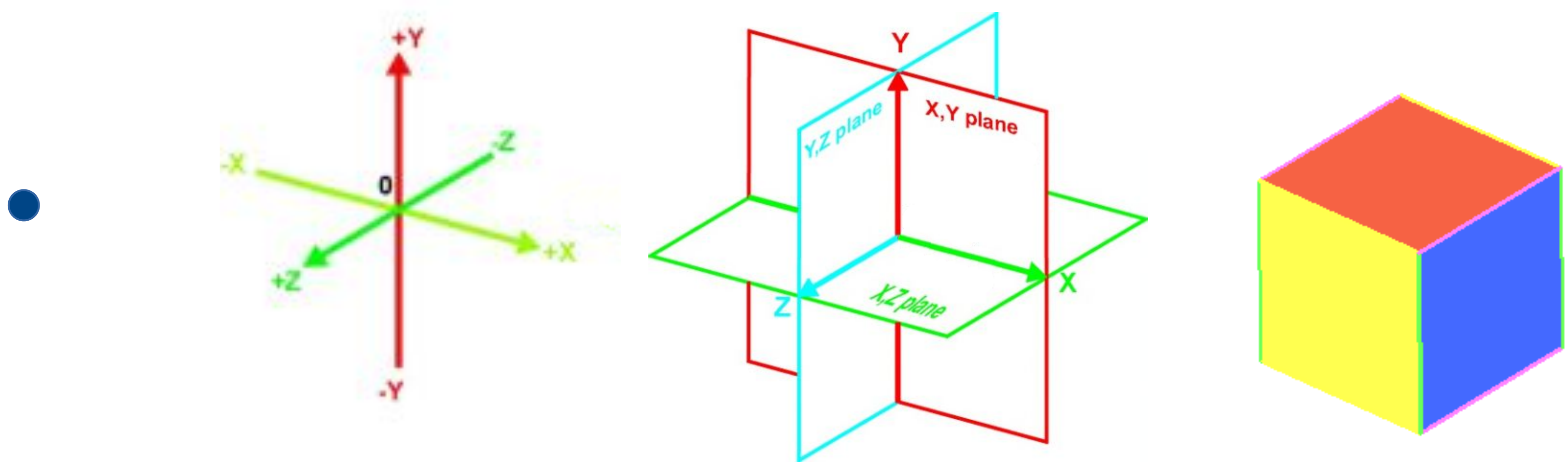
$$a + v_1 e_1 + v_2 e_2 + v_3 e_3 + w_1 e_2 e_2 + w_2 e_3 e_1 + w_3 e_1 e_2 + b e_1 e_2 e_3$$

$$M = a + \boldsymbol{v} + j\boldsymbol{w} + jb$$



Clifford's system describes physical space

$$a + v_1e_1 + v_2e_2 + v_3e_3 + w_1e_2e_2 + w_2e_3e_1 + w_3e_1e_2 + be_1e_2e_3$$



Maxwell's equations

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon}, \quad \text{(Gauss' law);} \\
\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\partial_t \vec{E} &= \mu_0 \vec{J}, \quad \text{(Ampère's law);} \\
\vec{\nabla} \times \vec{E} + \partial_t \vec{B} &= 0, \quad \text{(Faraday's law);} \\
\vec{\nabla} \cdot \vec{B} &= 0, \quad \text{(Gauss' law of magnetism)}
\end{aligned}$$

Using the vector gradient:

$$\nabla = e_1 \partial_x + e_2 \partial_y + e_3 \partial_z$$

Maxwell's equation

$$\partial F = J$$

$$\partial = \partial_t + \nabla$$ Four-gradient operator

$$F = \boldsymbol{E} + jc\boldsymbol{B}$$ Single field variable

$$J = \mu_0 \left(c\rho - J \right)$$ Four-current

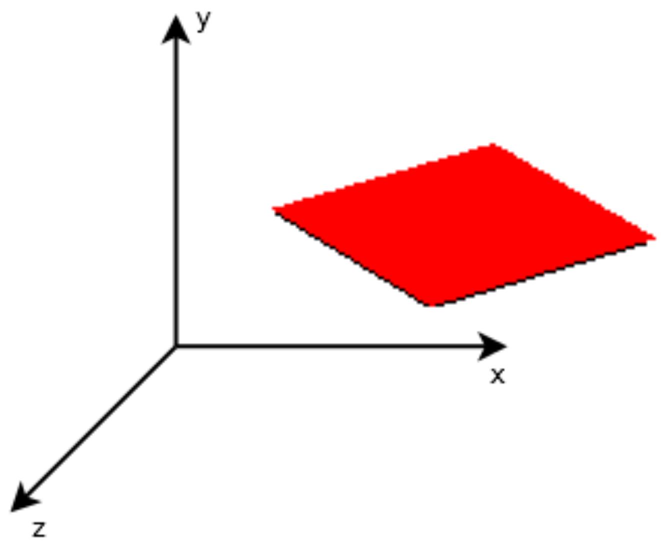
Scalar	Vector	Bivector	Trivector
Points	Lines	Areas	Volumes
Energy	Displacement r Velocity **v** Acceleration **a** Poynting vector **S**	Angular velocity **w** = **r** × **p**	
Mass m	**F** = m**a**	Torque = **r** × **F**	
Charge ρ	Curent **J**	Monopole current	Monopole charge
	Electric Dipole **p**	Magnetic dipole **m**	
	Electric Field **E**	Magnetic field **B**	
Scalar potential V	Vector potential **A**		

So what does $\sqrt{-1}$ mean?

"The true metaphysics of the square root of -1 is elusive."

Gauss, 1825

Using Clifford geometric algebra we find that a bivector e_1e_2 squares to minus one.



Clifford unifying vector systems

Clifford combines the Gibbs-like vector with the rotational algebra within a unified system.

Dimension	Clifford	Even subalgebra (Rotations)	Vector
2	$C\ell(\Re^2)$	Complex numbers $a + ib \in \mathbb{Z}$	$\boldsymbol{v} = v_1 e_1 + v_2 e_2$ $i = e_1 e_2$
3	$C\ell(\Re^3)$	Quaternions $q_0 + q_1 \boldsymbol{i} + q_2 \boldsymbol{j} + q_3 \boldsymbol{k} \in \mathbb{H}$	$\boldsymbol{v} = v_1 e_1 + v_2 e_2 + v_3 e_3$ $j = e_1 e_2 e_3$

Potential applications

- Maxwell’s equations and EM waves
- RLC circuits and transmission lines
- Anisotropic media
- Metamaterials
- Relativity equations for GPS satellites
- Image and signal processing
- Computer vision and robotics
- Quantum foundations
- Quantum computing and quantum game theory

Conclusion

Clifford's mathematical system…

"**should have gone on to dominate mathematical physics….**" **(Chris Doran)**

but….the vector algebra war continues….

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Quaternion product

$$\boldsymbol{v} \;=\; v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k} \qquad \boldsymbol{w} = w_1\boldsymbol{i}+w_2\boldsymbol{j}+w_3\boldsymbol{k}$$

$$\begin{aligned}
\boldsymbol{v}\boldsymbol{w} &= (v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k})(w_1\boldsymbol{i} + w_2\boldsymbol{j} + w_3\boldsymbol{k}) \\
&= -v_1w_1 - v_2w_2 - v_3w_3 + (v_2w_3 - v3w_2)\boldsymbol{i} \\
&\quad +(v_3w_1 - v1w_3)\boldsymbol{j} + (v_1w_2 - v2w_1)\boldsymbol{k} \\
&= -\boldsymbol{v}\cdot\boldsymbol{w} + \boldsymbol{v}\times\boldsymbol{w}
\end{aligned}$$

The dot and cross products!