# AGACSE 2018

## Universidade de Campinas

# Waldyr Alves Rodrigues Jr.: Fruits of a unifying philosophy

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# Introduction

Gauss (1777-1855), Ampère (1775-1836), Faraday (1791-1867), Maxwell (1831.1879), Gibbs (1839-1903), Hetz (1857-1894), Heaviside (1850-1925), Lorentz (1853-1928)

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}=\rho \qquad \text{(Gauss law for } \boldsymbol{E}\text{)} \tag{1}$$

$$\boldsymbol{\nabla}\times\boldsymbol{B}-\partial_t\boldsymbol{E}=\boldsymbol{j} \qquad \text{(Ampère-Maxwell law)} \tag{2}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}+\partial_t\boldsymbol{B}=0 \qquad \text{(Faraday's induction law)} \tag{3}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}=0 \qquad \text{(Gauss law for } \boldsymbol{B}\text{)} \tag{4}$$

$$\partial_t\rho+\boldsymbol{\nabla}\cdot\boldsymbol{j}=0 \qquad \text{(Charge conservation)} \tag{5}$$

$$\boldsymbol{F}=q(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}) \qquad \text{(Lorentz force law)} \tag{6}$$

| 4-Vector | Meaning |
|---|---|
| $\mathbf{r} = [\boldsymbol{r}, t]$ | Position |
| $\mathbf{u} = \frac{d\mathbf{r}}{d\tau} = [\gamma\boldsymbol{u}, \gamma]$ | Velocity |
| $\mathbf{p} = m_0\mathbf{u} = [m_0\gamma\boldsymbol{u}, m_0\gamma] = [m\boldsymbol{u}, m] = [\boldsymbol{p}, E]$ | Momentum-energy |
| $\mathbf{a} = \frac{d\mathbf{u}}{d\tau} = [\gamma^4\boldsymbol{u}\cdot\boldsymbol{a} + \gamma^2\boldsymbol{a}, \gamma^4\boldsymbol{u}\cdot\boldsymbol{a}]$ | Acceleration |
| $\mathbf{f} = \frac{d\mathbf{p}}{d\tau} = m_0\mathbf{a} = [\gamma\boldsymbol{f}, \gamma\dot{m}]$ | Force |
| $\mathbf{j} = [\boldsymbol{j}, \rho] = \rho_0\mathbf{u}$ | Current density |
| $\mathbf{a} = [\boldsymbol{A}, \phi]$ | Potential |
| $\Box\mathbf{a} = -\mathbf{j}$ | Wave equation |
| $\mathbf{k} = [\boldsymbol{k}, \omega]$ | Wave vector |

# GAC 2.0

- $E$ real vector space of dimension $n$.
- $\bigwedge^k E$: $k$-th *exterior* power of $E$.
- $\bigwedge E$: exterior algebra of $E$.
- $q$: metric on $E$ of signature $(r, s)$. *Notations*: $q(\boldsymbol{x}, \boldsymbol{y})$, $q(\boldsymbol{x}) = q(\boldsymbol{x}, \boldsymbol{x})$.
- Natural extension of $q$ to $\bigwedge E$ (still denoted $q$), uniquely determined by requiring that $\bigwedge^j E$ and $\bigwedge^k E$ are orthogonal for $j \neq k$ and that, for example,
$$q(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2) = \begin{vmatrix} q(\boldsymbol{x}_1) & q(\boldsymbol{x}_1, \boldsymbol{x}_2) \\ q(\boldsymbol{x}_2, \boldsymbol{x}_1) & q(\boldsymbol{x}_2) \end{vmatrix}. \tag{7}$$
- If $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is an *orthonormal* basis of $E$ (so $q(\boldsymbol{e}_i, \boldsymbol{e}_j) = 0$ when $i \neq j$ and $q(\boldsymbol{e}_i) = \pm 1$), then the $\binom{n}{k}$ *blades* $\boldsymbol{e}_{\hat{I}} = \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_k}$, where $1 \leqslant i_1 < \cdots < i_k \leqslant n$, form an orthonormal basis of $\bigwedge^k E$.

*Hodge linear isomorphism*: $*: \bigwedge^k E \to \bigwedge^{n-k} E$, determined by

$$\alpha \wedge \beta = q(*\alpha, \beta)\boldsymbol{e}_{\hat{N}}, \quad \alpha \in \bigwedge^k E,\ \beta \in \bigwedge^{n-k} E \tag{8}$$

Practical calculation:

$$*\,\boldsymbol{e}_{\hat{I}} = \sigma_I \boldsymbol{e}_{\hat{\bar{I}}}, \quad \bar{I} = N - I, \quad \sigma_I = (-1)^{t(I,\bar{I})+s} q_I, \tag{9}$$

where $t(I, \bar{I})$ is the number of inversions in the permutation $I, \bar{I}$ of $N$.

*Note:* $*$ is an isometry if $q_N = 1$ and an antiisometry if $q_N = -1$.

Original reference: Hodge-1941 [1]

Let $(t, x, y, z)$ be the lab coordinates of an inertial frame $\boldsymbol{e}_t, \boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$ (an orthonormal basis of the Minkowski space $E_{1,3}$). We define $\Lambda = \langle dt, dx, dy, dz\rangle$, which is nothing but the space of linear maps $E_{1,3} \to \mathbf{R}$, and its exterior powers

$$\Lambda^0 = \mathbf{R}$$

$$\Lambda^1 = \Lambda = \langle dt, dx, dy, dz\rangle$$

$$\Lambda^2 = \langle dx \wedge dt, dy \wedge dt, dz \wedge dt, dy \wedge dz, dz \wedge dx, dx \wedge dy\rangle$$

$$\Lambda^3 = \langle dx \wedge dy \wedge dz, dt \wedge dy \wedge dz, dt \wedge dz \wedge dx, dt \wedge dx \wedge dy\rangle$$

$$\Lambda^4 = \langle dt \wedge dx \wedge dy \wedge dz\rangle.$$

A 3-vector $\boldsymbol{k} = (k_x, k_y, k_z)$ is represented in $\Lambda^1$ and $\Lambda^2$ as follows:

$$\begin{aligned}\widehat{\boldsymbol{k}} &= k_x dx + k_y dy + k_z dz \in \Lambda^1\\ \widetilde{\boldsymbol{k}} &= k_x dy \wedge dz + k_y dz \wedge dx + k_z dx \wedge dy \in \Lambda^2.\end{aligned}$$

For example, if $f = f(x, y, z)$, then $\boldsymbol{\nabla} f$ is the 3-gradient of $f$ and $\widehat{\boldsymbol{\nabla} f} = df$.

In addition we have:

$$d\widehat{\boldsymbol{k}} = \widetilde{\boldsymbol{\nabla} \times \boldsymbol{k}} - \widehat{\partial_t \boldsymbol{k}} \wedge dt, \quad d\widetilde{\boldsymbol{k}} = (\boldsymbol{\nabla} \cdot \boldsymbol{k})dx \wedge dy \wedge dz + \widetilde{\partial_t \boldsymbol{k}} \wedge dt.$$

Given a 4-vector $\mathbf{k} = [\boldsymbol{k}, \kappa]$, let

$$\mathbf{k}^\sharp = \widehat{\boldsymbol{k}} - \kappa dt \in \Lambda^1.$$

This form is Lorentz invariant. In particular we have

$$\mathbf{j}^\sharp = \widehat{\boldsymbol{j}} - \rho dt, \quad \mathbf{a}^\sharp = \widehat{\boldsymbol{A}} - \phi dt.$$

It follows that the 2-form $F = d\mathbf{a}^\sharp$ is Lorentz invariant. A short computation using the tools developed so far shows that

$$F = \widehat{\boldsymbol{E}} \wedge dt + \widetilde{\boldsymbol{B}}. \tag{10}$$

The Lorentz invariance of $F$, which deserves being called the *electromagnetic form*, is equivalent to the textbook relations expressing $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of the $\boldsymbol{E}'$ and $\boldsymbol{B}'$ as seen in another lab.

In this approach, we have the tautological relation $dF = 0$, because $d^2 = 0$. But in terms of Eq. (10) we have

$$\begin{aligned} dF &= d(\widehat{\boldsymbol{E}} \wedge dt + \widetilde{\boldsymbol{B}}) \\ &= d(\widehat{\boldsymbol{E}}) \wedge dt + d\widetilde{\boldsymbol{B}} \\ &= \widetilde{\boldsymbol{\nabla} \times \boldsymbol{E}} \wedge dt + (\boldsymbol{\nabla} \cdot \boldsymbol{B}) dx \wedge dy \wedge dz + \widetilde{\partial_t \boldsymbol{B}} \wedge dt \end{aligned}$$

and so the vanishing of $dF$ is equivalent to Maxwell's equations (3) and (4) (the homogeneous pair).

The two non-homogeneous Maxwell's equations, (1) and (2), can also be recast as a single equation, namely

$$\delta F = -\mathbf{j}^\sharp,$$

where $\delta$, the *codiferential operator*, stands for $*d* : \Lambda^2 \to \Lambda^1$ (it is in fact the adjoint of $d$ with repect to the Minkowski metric). Indeed, Eq. (9) implies that $* : \Lambda^2 \to \Lambda^2$ is determined by the relations

$$*\widetilde{\boldsymbol{k}} = -\widehat{\boldsymbol{k}} \wedge dt, \quad *(\widehat{\boldsymbol{k}} \wedge dt) = \widetilde{\boldsymbol{k}}$$

and $* : \Lambda^3 \to \Lambda^1$ by the relations

$$*(\widetilde{\boldsymbol{k}} \wedge dt) = \widehat{\boldsymbol{k}}, \quad *(dx \wedge dy \wedge dz) = dt,$$

from which we deduce

$$\begin{aligned} *F &= \widetilde{\boldsymbol{E}} - \widehat{\boldsymbol{B}} \wedge dt \\ d*F &= (\boldsymbol{\nabla}\cdot\boldsymbol{E})dx \wedge dy \wedge dz + \widetilde{\partial_t \boldsymbol{E}} \wedge dt - \widetilde{\boldsymbol{\nabla}\times\boldsymbol{B}} \wedge dt \\ &= -\big(\widetilde{\boldsymbol{\nabla}\times\boldsymbol{B}} - \widetilde{\partial_t \boldsymbol{E}}\big) \wedge dt + (\boldsymbol{\nabla}\cdot\boldsymbol{E})dx \wedge dy \wedge dz \end{aligned}$$

Finally

$$\delta F = -\left(\widehat{\boldsymbol{\nabla} \times \boldsymbol{B}} - \widehat{\partial_t \boldsymbol{E}}\right) + (\boldsymbol{\nabla} \cdot \boldsymbol{E})dt = -(\widehat{\boldsymbol{j}} - \rho dt) = -\mathbf{j}^{\sharp}.$$

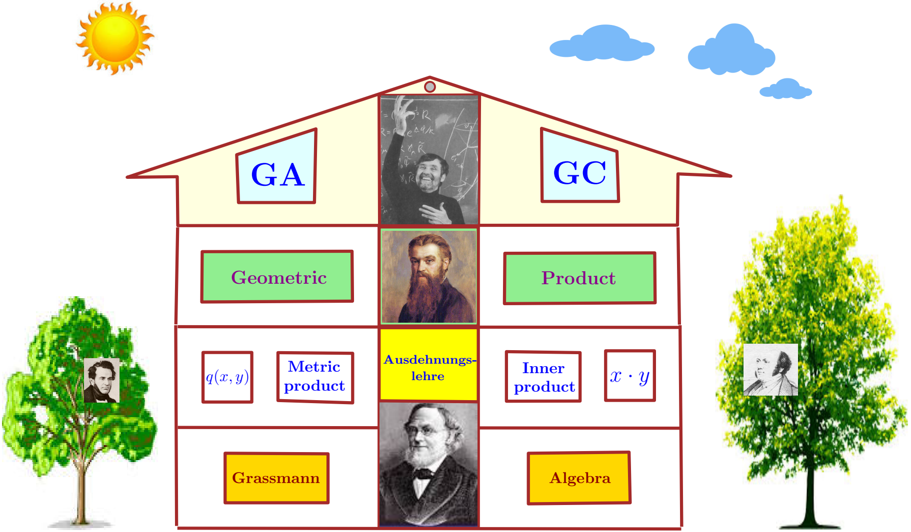
Subtracting $\delta F = -\mathbf{j}^{\sharp}$ from $df = 0$, which makes sense in $\Lambda$, we get that the Maxwell's equations are equivalent to the single equation

$$(d - \delta)F = \mathbf{j}^{\sharp}$$

which is getting closer to GAC 3.0, but not quite because the operator $d - \delta$ does not have a representation as an element of the algebra $\Lambda$.

References: [2], [3].

# GAC 3.0



GA

GC

Geometric

Product

$q(x,y)$

Metric product

Ausdehnungslehre

Inner product

$x \cdot y$

Grassmann

Algebra

A constructive view of GAC

- $E_{r,s} = (E, q)$, real vector space with a quadratic form $q$ of signature $(r, s)$ and dimension $n = r + s$.
- $\mathcal{G} = \mathcal{G}_{r,s}$ GA of $E_{r,s}$

**1.** $\mathcal{G}$ is a unital *associative* real algebra. Its product is called *geometric product* and is denoted by juxtaposition of the factors.

**2** (Clifford's relations)**.** For any $\boldsymbol{x}, \boldsymbol{y} \in E = \mathcal{G}^1$,

$$\boldsymbol{x}\boldsymbol{y} + \boldsymbol{y}\boldsymbol{x} = 2q(\boldsymbol{x}, \boldsymbol{y}) = 2\boldsymbol{x} \cdot \boldsymbol{y}, \quad \boldsymbol{x}^2 = q(\boldsymbol{x}).$$

In particular, $\boldsymbol{x}\boldsymbol{y} = -\boldsymbol{y}\boldsymbol{x}$ if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ are $q$-orthogonal.

**3** (The map $\wedge^k E \to \mathcal{G}$)**.** There is a canonical linear map $\wedge^k E \to \mathcal{G}$

$$\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k \mapsto \mathrm{g}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k)$$

where $\mathrm{g}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k) = \frac{1}{k!}\sum_p (-1)^{t(p)} \boldsymbol{x}_{p_1} \cdots \boldsymbol{x}_{p_k}$, the sum extended to all permutations $p = [p_1, \ldots, p_k]$ of $1, \ldots, k$. Note that $\boldsymbol{x} \wedge \boldsymbol{y} \mapsto \frac{1}{2}(\boldsymbol{x}\boldsymbol{y} - \boldsymbol{y}\boldsymbol{x})$.

**4.** Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be a basis of $E$. For each subset $I = \{i_1, \ldots, i_k\}$ of $N = \{1, \ldots, n\}$, let $\boldsymbol{e}_I = \boldsymbol{e}_{i_1} \cdots \boldsymbol{e}_{i_k}$. Then $B = \{\boldsymbol{e}_I\}_{I \subseteq N}$ is a linear basis of $\mathcal{G}$.

**5.** If $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is an *orthogonal* basis of $E$, then

$$\boldsymbol{e}_{\hat{I}} = \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_k} \mapsto \boldsymbol{e}_I,$$

so $\mathrm{g} : \bigwedge^k E \simeq \mathcal{G}^k$, $\mathcal{G}^k = \langle B_k \rangle$, $B_k = \{\boldsymbol{e}_I\}_{|I|=k}$.

In particular, $\mathcal{G}^k$ does not depend on the orthogonal basis used to describe it and therefore the linear grading

$$\mathcal{G} = \mathcal{G}^0 \oplus \mathcal{G}^1 \oplus \mathcal{G}^2 \oplus \cdots \oplus \mathcal{G}^n,$$

is canonical. Any $x \in \mathcal{G}$ can be uniquely written in the form $x = \langle x \rangle_0 + \langle x \rangle_1 + \cdots + \langle x \rangle_n$, with $x_j \in \mathcal{G}^j$ $(j = 0, 1, \ldots, n)$, or just $x = x_0 + x_1 + \cdots + x_n$.

We have $\dim \mathcal{G}^k = \binom{n}{k}$ and $\dim \mathcal{G} = 2^n$.

**6** (Outer product)**.** The isomorphism **5** allow us to graft the exterior product of $\wedge E$ to a product of $\mathcal{G}$, which will be called *exterior* or *outer* product of $\mathcal{G}$ and will be denoted with the same symbol $\wedge$. By the very definition, it is a unital and associative product and the basic rule for its computation is that

$$\boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k = \mathrm{g}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k).$$

In particular, $\boldsymbol{x} \wedge \boldsymbol{y} = \frac{1}{2}(\boldsymbol{x}\boldsymbol{y} - \boldsymbol{y}\boldsymbol{x})$ for any $\boldsymbol{x}, \boldsymbol{y} \in E$ and $\boldsymbol{e}_{\hat{I}} = \boldsymbol{e}_I$ whenever $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is orthogonal.

Note that

$$\boldsymbol{e}_I \wedge \boldsymbol{e}_J = \begin{cases} 0 & \text{if } I \cap J \neq \emptyset \\ (-1)^{t(I,J)}\boldsymbol{e}_{I+J} & \text{otherwise} \end{cases}$$

where $t(I, J)$ is the number of order inversions in the sequence $I, J$ and $I + J$ is the result of reordering $I, J$ in increasing order.

**7** (Artin's formula)**.** If $I, J$ are multiindices, then

$$\boldsymbol{e}_I \boldsymbol{e}_J = (-1)^{t(I,J)} q_{I\cap J}\, \boldsymbol{e}_{I \,\triangle\, J},$$

where $I \triangle J$ is the *symmetric difference* of $I$ and $J$, and

$$q_K = q(\boldsymbol{e}_{k_1}) \cdots q(\boldsymbol{e}_{k_l})$$

for any index sequence $K = k_1, \ldots, k_l$. In particular,

$$\boldsymbol{e}_J^2 = (-1)^{|J|/\!\!/2} q_J.$$

Note that the minimum grade $m$ of $\boldsymbol{e}_I \boldsymbol{e}_J$, given $k = |I|$ and $l = |J|$, is obtained precisely when either $I \subseteq J$ or $J \subseteq I$, and that in these cases $m = |k - l|$.

*Commutation formula*: $\boldsymbol{e}_J \boldsymbol{e}_I = (-1)^{|I|\cdot|J|+c}$, $c = |I \cap J|$.

*Clifford's group of an orthonormal basis*. Artin's formula shows that the set $B^{\pm} = \{\pm \boldsymbol{e}_I\}_{I \subseteq N}$ is a group if $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is orthonormal. Its order is $2^{n+1}$.

**8** (Grades of a geometric product)**.** Let $x \in \mathcal{G}^k$ and $y \in \mathcal{G}^l$. If $p \in \{0, 1, \ldots, n\}$ and $(xy)_p \neq 0$, then $p = |k - l| + 2i$ with $i \geqslant 0$ and $p \leqslant k + l$. Moreover, $(xy)_{k+l} = x \wedge y$.

Next we introduce the inner product, but it has to be stressed that it is not the metric product on $\mathcal{G}$ induced from the metric product $q$ of $E$ (see **12** below).

**9** (Inner product)**.** If $k = 0$ or $l = 0$, the only grade appearing in $xy$ is $k + l = |k - l|$, and $xy = x \wedge y$. On the other hand, if $k, l > 0$, then $|k - l| \leqslant k + l - 2$ and so we can define the bilinear product $x \cdot y$ by the relation $x \cdot y = (xy)_{|k-l|}$. In this case we have $xy = x \cdot y + \cdots + x \wedge y$, where $\cdots$ stands for terms of grade $p$ such that $|k - l| + 2 \leqslant p \leqslant k + l - 2$, if any. In order to insure that this equality also holds for $k = 0$ or $l = 0$, we are bound to set $x \cdot y = 0$ in any of these cases.

For example, if $\boldsymbol{e} \in E$ and $x \in \mathcal{G}^k$, then $\boldsymbol{e}x = \boldsymbol{e} \cdot x + \boldsymbol{e} \wedge x$, even for $k = 0$. Similarly, $x\boldsymbol{e} = x \cdot \boldsymbol{e} + x \wedge \boldsymbol{e}$.

Since the inner product is bilinear, its computation is straightfoward on noticing that if $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is an orthogonal basis of $E$ and $I, J$ are non-empty multiindices, then we get, using **7**,

$$\boldsymbol{e}_I \cdot \boldsymbol{e}_J = \begin{cases} \boldsymbol{e}_I \boldsymbol{e}_J & \text{if } I \subseteq J \text{ or } J \subseteq I \\ 0 & \text{otherwise} \end{cases}$$

This, together with the commutation formula in **7**, gives the commutation property of the inner product of $x \in \mathcal{G}^k$ and $y \in \mathcal{G}^l$:

$$y \cdot x = (-1)^{kl+m} x \cdot y, \quad m = \min(k, l). \tag{11}$$

In particular $x \cdot y = y \cdot x$ if $k = l$. In general, $x \cdot y = y \cdot x$ precisely when $k$ and $l$ have the same parity or else $m$ is even. Otherwise, which means that $k$ and $l$ have different parity and $m$ is odd, we have $x \cdot y = -y \cdot x$.

**10** (Parity involution). The linear map $\mathcal{G} \to \mathcal{G}$, $x \mapsto \hat{x}$, such that $\hat{x} = (-1)^k x$ for $x \in \mathcal{G}^k$, is an *involution* (i.e, $\hat{\hat{x}} = x$ for all $x \in \mathcal{G}$) and

$$\widehat{xy} = \hat{x}\hat{y}, \quad \widehat{x \wedge y} = \hat{x} \wedge \hat{y}, \quad \widehat{x \cdot y} = \hat{x} \cdot \hat{y}$$

for all $x, y \in \mathcal{G}$. We say that it is an *automorphism* of $\mathcal{G}$.

Among the properties of the inner product related to the parity involution, let us mention that for any vector $\boldsymbol{e}$ and any $x, y \in \mathcal{G}$,

$$\boldsymbol{e} \cdot (xy) = (\boldsymbol{e} \cdot x)y + \hat{x}(\boldsymbol{e} \cdot y) \text{ and } \boldsymbol{e} \cdot (x \wedge y) = (\boldsymbol{e} \cdot x) \wedge y + \hat{x} \wedge (\boldsymbol{e} \cdot y).$$

In other words, the map $\mathcal{G} \to \mathcal{G}$, $x \mapsto \boldsymbol{e} \cdot x$, is a (left) skew-derivation of both the geometric and the outer products. And $x \mapsto x \cdot \boldsymbol{e}$ is a right skew-derivation of both products:

$$(xy) \cdot \boldsymbol{e} = x(y \cdot \boldsymbol{e}) + (x \cdot \boldsymbol{e})\hat{y}.$$

and similarly for the outer product. This can be established by using the left skew-derivation property and the reverse involution introduced next.

**11** (Reverse involution)**.** The linear map $\mathcal{G} \to \mathcal{G}$, $x \mapsto \tilde{x}$, such that $\tilde{x} = (-1)^{k/\!\!/2}x$ for $x \in \mathcal{G}^k$, is an *involution* (i.e, $\tilde{\tilde{x}} = x$ for all $x \in \mathcal{G}$) and satisfies the relations

$$\widetilde{xy} = \tilde{y}\tilde{x}, \quad \widetilde{x \wedge y} = \tilde{y} \wedge \tilde{x}, \quad \widetilde{x \cdot y} = \tilde{y} \cdot \tilde{x}$$

for all $x, y \in \mathcal{G}$. We say that it is an *anti-automorphism* of $\mathcal{G}$.

**12** (Metric formulas 3.0)**.** The metric on $\mathcal{G}$ obtained by grafting the metric $q$ on $\wedge E$ given by the Gram rule (cf. Eq. (7)) is determined by the geometric product and the grading as follows: For all $x, y \in \mathcal{G}$,

$$q(x, y) = (\tilde{x}y)_0 = (x\tilde{y})_0. \tag{12}$$

In particular we have $q(x) = (\tilde{x}x)_0 = (x\tilde{x})_0$ for all $x \in \mathcal{G}$.

In the case that $x, y \in \mathcal{G}^k$, $(\tilde{x}y)_0 = \tilde{x} \cdot y = (-1)^{k/\!\!/2}x \cdot y$. Thus we conclude that $x \cdot y = (-1)^{k/\!\!/2}q(x, y)$. Note finally that if $x \in \mathcal{G}^k$, $y \in \mathcal{G}^l$ and $k \neq l$, then $q(x, y) = 0$ but $x \cdot y$ need not be zero.

**13** (Invertible blades)**.** If $X = \boldsymbol{x}_1 \wedge \cdots \wedge \boldsymbol{x}_k \neq 0$ (we say that $X$ is a *k-blade*), then $\tilde{X}X$ and $X\tilde{X}$ are scalars, because we can express $X$ in the form $\boldsymbol{y}_1 \cdots \boldsymbol{y}_k$ with $\boldsymbol{y}_1, \dots, \boldsymbol{y}_k$ pairwise orthogonal, and hence $\tilde{X}X = \boldsymbol{y}_1^2 \cdots \boldsymbol{y}_k^2 = X\tilde{X}$. Therefore

$$q(X) = \tilde{X}X = X\tilde{X} = (-1)^{k/\!\!/2}X^2.$$

In particular we see that $X$ is invertible if and only if $X^2 \neq 0$, or if and only if $q(X) \neq 0$, and if this is the case,

$$X^{-1} = X/X^2 = \tilde{X}/q(X).$$

*Example.* Let $\mathbf{e} = \boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be an orthonormal basis of $E = E_{r,s}$ and define

$$\omega_{\mathbf{e}} = \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n \in \mathcal{G}^n.$$

We will say that $\omega_{\mathbf{e}}$ is the *pseudoescalar* associated to $\mathbf{e}$. Note that the metric formula gives us that

$$q(\omega_{\mathbf{e}}) = q(\boldsymbol{e}_1) \cdots q(\boldsymbol{e}_n) = (-1)^s.$$

If $\mathbf{e}' = e'_1, \ldots, e'_n$ is another orthonormal basis of $E$, then

$$\omega_{\mathbf{e}'} = \delta\omega_{\mathbf{e}},$$

where $\delta = \det_{\mathbf{e}}(\mathbf{e}')$ is the determinant of the matrix of the vectors $\mathbf{e}'$ with respect to the basis $\mathbf{e}$. Now the equalities

$$q(\omega_{\mathbf{e}}) = q(\omega_{\mathbf{e}'}) = q(\delta\omega_{\mathbf{e}}) = \delta^2 q(\omega_{\mathbf{e}})$$

allow us to conclude that $\delta = \pm 1$. This means that, up to sign, there is a unique pseudoscalar. The distinction of one of them amounts to choose an *orientation* for $E$.

**14** (Properties of the pseudoscalar). Let $\omega \in \mathcal{G}^n$ be a pseudoescalar and $\mathcal{G}^\times$ the group of invertible multivectors with respect to the geometric product. Then

1. $\omega \in \mathcal{G}^\times$, $\quad \omega^{-1} = (-1)^s\,\tilde{\omega} = (-1)^s(-1)^{n//2}\omega$,
   $\omega^2 = (-1)^{n//2}(-1)^s$.
2. *Hodge duality 3.0*. For any $x \in \mathcal{G}^k$, $\omega x, x\omega \in \mathcal{G}^{n-k}$ and the maps $x \mapsto \omega x$ and $x \mapsto x\omega$ are linear isomorphisms $\mathcal{G}^k \to \mathcal{G}^{n-k}$. The inverse maps are given by $x \mapsto \omega^{-1}x$ and $x \mapsto x\omega^{-1}$, respectively.
3. If $n$ is odd, $\omega$ commutes with all the elements of $\mathcal{G}$ *(this is expressed by saying that $\omega$ is a central element of $\mathcal{G}$)*. If $n$ is even, $\omega$ commmutes (anticommutes) with even (odd) multivectors.
4. If $q(\omega) = 1$ *($q(\omega) = -1$)*, the Hodge duality maps are *isometries (antiisometries)*.

What relation it there between Hodge duality 3.0 and Hodge duality 2.0? The quickest answer is to use Eq. (9), which now can be written

$$*\boldsymbol{e}_I = (-1)^{t(I,\bar{I})+s} q_I \boldsymbol{e}_{\bar{I}}, \quad \bar{I} = N - I.$$

Comparing with $\boldsymbol{e}_I\omega = \boldsymbol{e}_I\boldsymbol{e}_N = (-1)^{t(I,N)} q_I \boldsymbol{e}_{\bar{I}}$, we immediately get

$$x\omega = (-1)^{k/\!\!/2+s}(*x)$$

for all $x \in \mathcal{G}^k$. We note that the condition that $* : \mathcal{G}^k \to \mathcal{G}^{n-k}$ is an isometry (antiisometry) agrees with the condition that $x \mapsto x\omega$ ($x \in \mathcal{G}^k$) is an isometry or an antiisometry (the sign $(-1)^{k/\!\!/2+s}$ is irrelevant for this question), but that the latter was considerable easier to establish (with 3.0 tools!).

**15** ($\mathrm{Spin}_{r,s}$)**.** This is the group of *spinors*, i.e. the group of elements $u$ of $\mathcal{G}$ that can be expressed as a product $u = \boldsymbol{u}_1 \cdots \boldsymbol{u}_{2k}$ of an even number of unit vectors of $E_{r,s}$ (thus, by definition, $q(\boldsymbol{u}_j) = \pm 1$ for $j = 1, 2, \ldots, 2k$).

The map $\mathcal{G} \to \mathcal{G}$, $x \mapsto uxu^{-1}$ is an automorphism of $\mathcal{G}$ and it is easy to see that it is a *grade-preserving isometry* of $\mathcal{G}$. In particular it induces an isometry $\underline{u}$ of $E_{r,s}$: $\underline{u}(\boldsymbol{x}) = u\boldsymbol{x}u^{-1}$. So we have a map $\mathrm{Spin}_{r,s} \to \mathrm{SO}_{r,s}$, $u \mapsto \underline{u}$, which turns out to be surjective and with kernel $\pm 1$, i.e, $\underline{u'} = \underline{u}$ if and only if $u' = \pm u$.

For any spinor $u$, $\tilde{u}u = u\tilde{u} = \epsilon_u$, with $\epsilon_u = \pm 1$. If $\epsilon_u = 1$, $u$ is called a *rotor*, and rotors form a subgroup $\mathrm{Spin}^{+}_{r,s}$ of $\mathrm{Spin}_{r,s}$. Since $\epsilon_u = -1$ does not occur for Euclidean or anti-Euclidean signatures, we have

$$\mathrm{Spin}_n = \mathrm{Spin}^{+}_n = \mathrm{Spin}_{\bar{n}}.$$

For $n \geqslant 3$, this group is simply connected and $\mathrm{SO}_n$ is connected.

If $r, s \geqslant 1$, and $r+s \geqslant 3$, $\mathrm{Spin}^+_{r,s}$ is simply connected and its image $\mathrm{SO}^+_{r,s}$ in $\mathrm{SO}_{r,s}$ is the connected component of the identity.

An important especial case is that $\mathrm{Spin}_3 = \mathrm{SU}_2$, the group of unit quaternions, and the map $\mathrm{Spin}_3 \to \mathrm{SO}_3$ is the familiar construction of rotations by means of unit quaternions.

Another important case is that $\mathcal{L} = \mathrm{SO}^+_{1,3}$ is the restricted Lorentz group (orthochronous orientation-preserving isometries) and $\mathrm{Spin}^+_{1,3} = \mathrm{SL}_2(\mathbf{C})$. The 2:1 map $\mathrm{Spin}^+_{1,3} \to \mathrm{SO}^+_{1,3}$ is represented as the map $\mathrm{SL}_2(\mathbf{C}) \to \mathcal{L}$, $U \mapsto \underline{U}$, such that

$$h(\underline{U}\mathbf{a}) = Uh(\mathbf{a})U^{-1}$$

where $h$ is the familiar representation of a vector

$$\mathbf{a} = a^0\boldsymbol{e}_0 + a^1\boldsymbol{e}_1 + a^2\boldsymbol{e}_2 + a^3\boldsymbol{e}_3 \in E_{1,3}$$

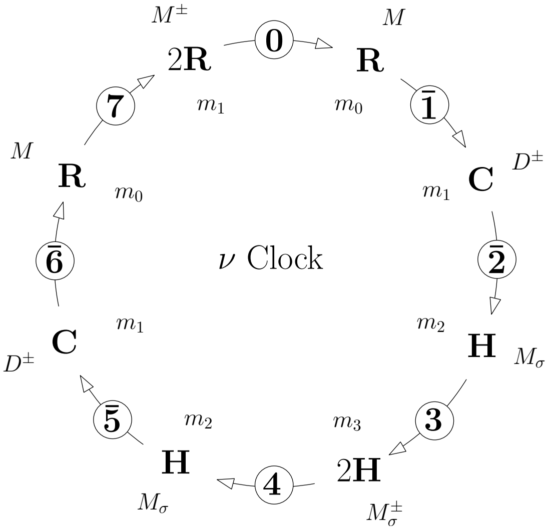
by the Hermitian matrix

$$h(\mathbf{a}) = \begin{pmatrix} a^0 + a^1 & a^2 + ia^3 \\ a^2 - ia^3 & a^0 - a^1 \end{pmatrix}.$$

**16** (Classification of $\mathcal{G}_{r,s}$). We summarize the results on the classification of $\mathcal{G}_{r,s}$ in terms of $\nu = s - r \mod 8$.

| $\nu$ | 0, 6 | 1, 5 | 2, 4 | 7 | 3 |
|---|---|---|---|---|---|
| $F_\nu$ | $\mathbf{R}$ | $\mathbf{C}$ | $\mathbf{H}$ | $2\mathbf{R}$ | $2\mathbf{H}$ |
| $m$ | $m_0$ | $m_1$ | $m_2$ | $m_1$ | $m_3$ |
| Label | $M$ | $D^\pm$ | $M_\sigma$ | $M^\pm$ | $M_\sigma^\pm$ |

Synopsis of the $\mathcal{G}_{r,s}$ forms in terms of $\nu = s - r \mod 8$. By definition, $m_k = 2^{(n-k)/2}$ ($n = r + s$), the order of the matrices in the algebra class or the dimension of the ground space on which these matrices act. In the lables, $M$ and $D$ stand for Majorana and Dirac, respectively.


$\nu$ Clock
$M^{\pm}$
$2\mathbf{R}$
$m_1$
$\mathbf{0}$
$M$
$\mathbf{R}$
$m_0$
$\bar{\mathbf{1}}$
$D^{\pm}$
$\mathbf{C}$
$m_1$
$\bar{\mathbf{2}}$
$m_2$
$\mathbf{H}$
$M_\sigma$
$\mathbf{3}$
$m_3$
$2\mathbf{H}$
$M_\sigma^{\pm}$
$\mathbf{4}$
$\mathbf{H}$
$M_\sigma$
$m_2$
$\bar{\mathbf{5}}$
$\mathbf{C}$
$D^{\pm}$
$m_1$
$\bar{\mathbf{6}}$
$M$
$\mathbf{R}$
$m_0$
$\mathbf{7}$

N

# On WARJr work

In general terms, the scientific endeavors of WR Jr were physically motivated and aimed at raising the established practices in differential geometry (DG), quite successful in mathematics, and to a good extend also in physics, to the 3.0 level.

We will try to illustrate this assessment by explaining the main ideas of his approach to the study of gravity and it particuly its analogies with Mawwell's equation

$$\partial F = J.$$

N

Manifolds are ubiquitous. Even in restricted contexts, they may appear as submanifolds, as symmetry groups, as auxiliary structures to understand simple objects or transformations, as parameter spaces of the configurations of some system, or to provide theoretical possibilities to carry on research.

Around each point, a manifold $M$ is similar to a small open set of a real vector space, and with a little care a number of related concepts and operations can be conceived. Among them, the algebra of smooth functions defined on an open set $U$ of $M$, the vector space $T_xM$ of *tangent vectors* to $M$ at $x \in M$, or its dual, $T_x^*M$, the space of *tangent covectors*. All structures that can be defined for a real vector space (the exterior algebras, for example) can be defined at each point of a manifold.

A *field* $\psi$ on a manifold $M$ is a map that assigns to each point $x \in M$ and object $\psi(x)$ of some kind that varies smoothly with $x$. If $\psi_x \in T_xM$, we have a *vector field*.

If $\psi_x \in \wedge^k T_x^*M$, we have a field of $k$-forms, or simply a *$k$-form*. A metric $q$ is a field such that $q_x$ is a metric of $T_xM$. In this case, we can consider fields such that $\psi(x) \in \mathcal{G}(T_xM)$ (*multivector fields*), or $\psi(x) \in \mathcal{G}^+(T_xM)$ (*even multivector fields*), or $\psi(x) \in \mathcal{G}^2(T_xM)$ (*bivector fields*).

Since a metric on $T_xM$ defines a metric on $T_x^*M$, we may also consider *multicovector fields*. In any case, the linear graded isomorphism $\wedge(T_xM) \simeq \mathcal{G}(T_xM)$ allows us to use all the machinery of GA as explained before, including the geometric, outer and inner products and the involutions.

We can 'bundle' the tangent spaces $T_x M$ into $TM = \sqcup_x T_x M$, so that we may imagine a point of $TM$ as a pair $(x, v)$ with $x \in M$ and $v \in T_x M$. Then we have a map $\pi : TM \to M$, $(x, v) \mapsto x$, and a vector field $\psi$ is a *section* of $\pi$ (or of $TM$), as $\psi(x) \in T_x M = \pi^{-1}(x)$.

The same can be applied to the other linear constructions and we get *vector bundles* like $T^*M$, $\wedge^k(TM)$, $\wedge(TM)$, $\mathcal{G}^+(TM)$, and so on.

In general, a *vector bundle* is a 'bundle' of vector spaces $E = \sqcup E_x$, where $E_x$ is a vector space, maybe with some extra structure (a metric, for example).

The simplest case is a *trivial bundle*, which is a product $E = M \times F$, $F$ a vector space, possibly with some extra structure. This is sufficient in many interesting situations, like when $M$ is an Euclidean or a Minkowskian affine space. In general, however, vector bundles are required to be *locally trivial*.

When the 'fiber' $F$ stands for the internal states of some system, possibly quantum states (for example spin states), then the sections of $E$ are fields with (locally) values in that space. Examples of that are the Pauli, Dirac, or Hestenes-Dirac fields.

We will write $\Gamma E$ to denote the vector space of the vector bundle $E$. In the case of the sections of $\mathcal{G}_{r,s}(T^*M)$, we will simply write $\Gamma_{r,s}(T^*M)$, with obvious adaptations in other similar cases, like for example $\Gamma^k_{r,s}(T^*M)$ for the sections of $\mathcal{G}^k_{r,s}(T^*M)$

- *Background:* $\mathcal{M}$ a parallelizable 4-dimensional manifold.
- *Potentials:* $g^0, g^1, g^2, g^3 \in \Gamma^1(T^*M)$ such that $\boldsymbol{\omega} = g^0 \wedge g^1 \wedge g^2 \wedge g^3$ is non-zero everywhere.
- *Metric $\boldsymbol{\eta}$:* The unique metric for which $\mathbf{g}$ is an orthonormal frame of signature $(1,3)$ at any point. Note that $\omega$ is the pseudoscalar of this metric.
- *Field strength:* If we $\mathbf{g} = [g^0, g^1, g^2, g^3] \in \Gamma^1(T^*M)^4$, the field strength is $\mathbf{f} = d\mathbf{g} \in \Gamma^2(T^*M)^4$.
- *Tautological equation:* $d\mathbf{f} = 0$.
- *Lagrangian density:* $\mathcal{L} = \mathcal{L}_{\mathbf{g}} + \mathcal{L}_m$, $\mathcal{L} \in \Gamma^4(T^*M)$. $\mathcal{L}_{\mathbf{g}}$ is a Lorentz invariant expression of the potentials only (and using only 2.0 tools). $\mathcal{L}_m = \rho\boldsymbol{\omega}$, where function $\rho$ encodes the energy density.

For simplicity, today we will consider only pure gravity ($\mathcal{L}_m = 0$).

- *Euler-Lagrange equations of* $\mathcal{L}$*:* $\delta \mathbf{f} = -J_{\mathbf{g}} \in \Gamma^1(T^*M)$, where $J_{\mathbf{g}}$ is an explicit expression of the potentials (using 2.0 tools).
- The equations $d\mathbf{f} = 0$ and $\delta \mathbf{f} = J_{\mathbf{g}}$ (*WR equations*)can be combined as single equation in the $\mathcal{G}$ bundle.

$$(d - \delta)\mathbf{f} = J_{\mathbf{g}} \tag{13}$$

- $\mathcal{L}_g$ *is equivalent* to the Hilbert-Einstein Lagrangian density, which implies that equation (13) is equivalent to the Einstein's equations:

$$\mathrm{Ricci} - \tfrac{1}{2}Rg = 0 \ (\mathrm{or}\, T)$$

- $d - \delta = g^{\mu}\boldsymbol{\nabla}_{g^{\mu}} = \boldsymbol{\partial}$, so Einsteins equations can can be written in the Maxwell-like form
$$\boldsymbol{\partial}\mathbf{f} = J.$$

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[6] W. A. Rodrigues Jr. and E. Capelas de Oliveira, *The Many Faces of Maxwell, Dirac and Einstein Equations (Second edition)*.

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One of Professor Waldyr's notable contributions, was how he used Clifford's bundle to explore striking similarities between the Maxwell, Dirac, Einstein, and Navier-Stokes equations.

In this talk we will try to explain some basic concepts concurring in this unification and state a small sample of his main results

> *If one wants to make a unified theory, the first thing one should try is to represent these fields as objects of the same mathematical nature.*
> W.A. Rodrigues Jr, 1.1.2017 (private communication)

P

Originated by Gibbs and Heaviside by recyling a few remnants of Hamilton's quaternions. The basic mathematical tools are Euclidean vector space $E_3$ and the differential and integral calculus of vector fields. Maxwell's equations are still written in that formalism in many textbooks on classical electromagnetism, e.g [4].

Equation (5) is an immediate consequence of (1) and (2). Note also that equation (3) implies that $0 = \boldsymbol{\nabla}\cdot(\partial_t \boldsymbol{B}) = \partial_t(\boldsymbol{\nabla}\cdot\boldsymbol{B})$, and this implies that $\boldsymbol{\nabla}\cdot\boldsymbol{B}$ is constant at any point in space. If there was reason to believe that this divergence vanishes for some remote past or future time at any given point, it would be 0 and hence Eq. (4) would be a consequence of Faraday's law (3).

P

The formalism was adapted to the 4-vector treatment of special relativity and relativistic eletromagnetism. Proper time is denoted by $\tau$. Time and position in the lab frame are denoted by $t$ and $\boldsymbol{r}$. The Lorentz factor for velocity $\boldsymbol{u}$ is $\gamma = (1 - \boldsymbol{u}^2)^{-1/2} = dt/d\tau$. The rest mass is denoted $m_0$, the relativistic mass by $m = \gamma m_0$, and $E = m$ is the energy by Einstein's relation. References: [4], [5].

For example, the *vector potential* $\boldsymbol{A}$ and the *scalar potential* $\phi$ satisfy $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \partial_t \boldsymbol{A}$. These potentials can be chosen (gauged) to satisfy the *Lorentz condition* $\boldsymbol{\nabla} \cdot \boldsymbol{A} + \partial_t \phi = 0$, and then the 4-vectors $\mathbf{a} = [\boldsymbol{A}, \phi]$ and $\mathbf{j} = [\boldsymbol{j}, \rho]$ satisfy the wave equation $\Box \mathbf{a} = -\mathbf{j}$, out of which the relativistic transformations of the $\boldsymbol{E}$ and $\boldsymbol{B}$ can be obtained.

P

This is the level whose design has been lead by David Hestenes and in which WR Jr, like many others, decided to live and work long ago. One figure should suffice: In the book [6], the name Hestenes appears in about fifteen entries in the table of contents and over one hundred fifty times in the text (without counting headers, nor titles of sections and subsections, nor appearences in the alphabetical index), mostly in the forms of Dirac-Hestenes (DH) equation, DH spinors, DH spinor fields, DH Lagrangian, and of course in several references.

P

Ahead of the arrow of any hour $\nu$, we have the form $F_\nu$ of $\mathcal{G}_{r,s}$, where $\nu = s - r \mod 8$. Therefore the form $F_{\nu-1}$ of $\mathcal{G}^+_{r,s}$ can be read at the tail of the $\nu$-arrow.

To specify the order $m$ of the matrices it is convenient to use the notation $m_k = 2^{(n-k)/2}$. Here $k = 0, \ldots, 3$, but later we will also need $m_4$. For example, $\mathcal{G}_{3,1} \simeq F_6(m) = \mathbf{R}(m_0)$, where $m_0 = 2^{4/2} = 4$, which tells us that $\mathcal{G}_{3,1}$ is isomorphic, as an algebra, to the matrix algebra $\mathbf{R}(4)$. On the other hand, $\mathcal{G}^+_{3,1}$ is isomorphic to $\mathbf{C}(m_1) = \mathbf{C}(2)$, because $\nu = 6$, $F_5 = \mathbf{C}$, and $m_1 = 2^{(3-1)/2} = 2$. The values $\nu = 1, 2, 5, 6$ (or $\nu = 1, 2 \mod 4$) have been marked with an overbar to indicate the $\omega^2 = -1$. The labels $M$ and $M_\sigma$ stand for *real* and *symplectic* (or quaternionic) *Majorana*, respectively, and $D$ for *Dirac*, and their significance is summarized in next slide. P

*Chapter 4 discusses aspects of differential geometry that are essential for a reasonable understanding of spacetime theories [...]. [MF, p. 4].*

*[...] the main objective of Chap. 4 is to introduce a Clifford bundle formalism, which can efficiently be used in the study of the differential geometry of manifolds an also to give an unified mathematical description of the Maxwell, Dirac and gravitational fields. [...] we also recall Cartan's formulation of differential geometry, extending it to a general Riemann-Cartan-Weyl space or spacetime (hereafter denoted RCWS) [MF, p. 6].*

*Chapter 5 gives a Clifford bundle approach to the Riemannian or semi-Riemannian differential geometry of branes understood as submanifolds of a Euclidean or pseudo-Euclidean space of large dimension. We introduce the important concept of the projection operator and define some other operators associated to it, as the shape operator and the shape biform. The shape operator is essential to define the concept of bending of a submanifold (as introduced above) and to leave it clear that a surface can be bended and yet the Riemann curvature of a connection defined in it may be null (as already mentioned for the case of the Nunes connection). [MF, p. 8]*

*Remark 4.131 It is important to observe that the operators sdel $\cdot$sdel and sdel $\wedge$sdel do not have anything analogous in the formulation of the differential geometry in the Cartan and Hodge bundles.*

P