Relativistic Quantum Spin

Relativistic quantum mechanics of spin-$1/2$ particles described by Dirac theory.

The Dirac matrix operators are

$$\hat{\gamma}_0 = \begin{pmatrix} \mathsf{I} & 0 \\ 0 & -\mathsf{I} \end{pmatrix}, \hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}, \hat{\gamma}_5 = \begin{pmatrix} 0 & \mathsf{I} \\ \mathsf{I} & 0 \end{pmatrix}$$

where $\hat{\gamma}_5 = -i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$ and I is the 2×2 identity matrix.

Act on Dirac spinors. 4 complex components (8 real degrees of freedom).

Follow same procedure as Pauli case. Map spinors onto elements of the 8-dimensional even subalgebra of the STA. First write

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix},$$

where $|\phi\rangle$ and $|\eta\rangle$ are 2-component spinors. Know how to represent the latter.

Full map is simply

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix} \leftrightarrow \psi = \phi + \eta\sigma_3$$

Uses both Pauli-even and Pauli-odd terms.

Explicitly have: A Dirac column spinor $|\psi\rangle$ is placed in one-to-one correspondence with an 8-component even element of the STA via

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \\ -b^3 + ib^0 \\ -b^1 - ib^2 \end{pmatrix} \quad \leftrightarrow \psi = a^0 + a^k I\sigma_k + I(b^0 + b^k I\sigma_k).$$

The action of the operators $\{\hat{\gamma}_\mu, \hat{\gamma}_5, i\}$ (where $\hat{\gamma}_5 = \hat{\gamma}^5 = -i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$)

translates as

$$\begin{aligned}\hat{\gamma}_\mu|\psi\rangle &\leftrightarrow \gamma_\mu\psi\gamma_0 \qquad (\mu = 0,\ldots,3)\\ i|\psi\rangle &\leftrightarrow \psi\, I\sigma_3\\ \hat{\gamma}_5|\psi\rangle &\leftrightarrow \psi\sigma_3,\end{aligned}$$

which are verified by simple computation; for example

$$\hat{\gamma}_5|\psi\rangle = \begin{pmatrix} -b^3 + ib^0 \\ -b^1 - ib^2 \\ a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \begin{array}{c} -b^3 + b^0\sigma_3 + b^1 i\sigma_2 - b^2 i\sigma_1 \\ +a^0\sigma_3 + a^3 i - a^2\sigma_1 + a^1\sigma_2 \end{array} = \psi\sigma_3$$

Complex conjugation in this representation translates as

$$|\psi\rangle^* \quad \leftrightarrow -\gamma_2\psi\gamma_2.$$

What about the Dirac equation equation itself? Conventionally this is

$$\hat{\gamma}^\mu(i\partial_\mu - eA_\mu)|\psi\rangle = m|\psi\rangle,$$

where the A_μ is the vector potential for any EM fields which are around — we will come back to the origin of this term later.

Applying our translation technique we find, upon postmultiplying by γ_0,

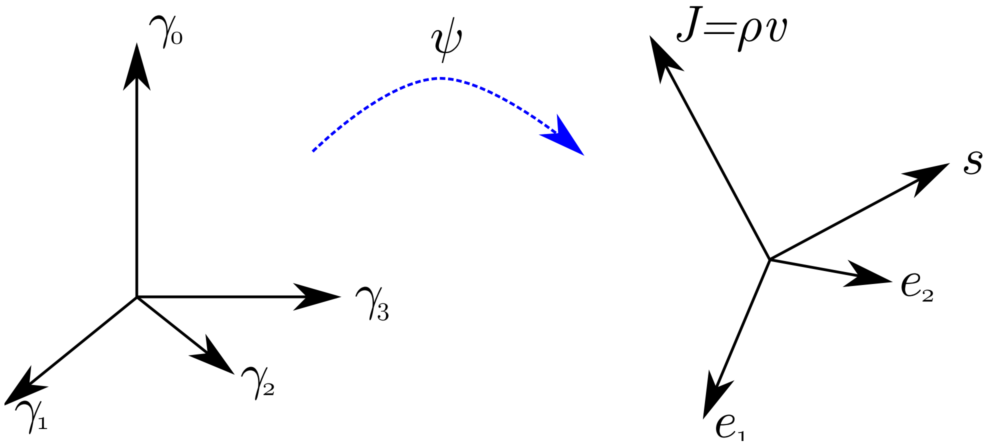
$$\nabla\psi I\sigma_3 - eA\psi = m\psi\gamma_0, \tag{12}$$

which is the form first discovered by David Hestenes

- This translation is direct and unambiguous, leading to an equation which is not only coordinate-free (since the vectors $\nabla = \gamma^\mu\partial_\mu$ and $A = \gamma^\mu A_\mu$ no longer refer to any frame) but is representation-free as well!
- In manipulating (12) one needs only the algebraic rules for multiplying spacetime multivectors, and the equation can be solved completely without ever having to introduce a matrix representation.
- Equation (12) therefore expresses the intrinsic geometric content of the Dirac equation.
- Now find similar insights occur as in the non-relativistic case

- Instead of the wavefunction being a weighted spatial rotor, it's now a full Lorentz spinor: $\psi = \rho^{1/2} e^{I\beta/2} R$ with the addition of a slightly mysterious β term related to antiparticle states.

- Five observables in all, including the current, $J = \psi\gamma_0\tilde{\psi} = \rho R\gamma_0\tilde{R}$, and the spin vector $s = \psi\gamma_3\tilde{\psi} = \rho R\gamma_3\tilde{R}$, and we can picture these using the diagram we showed earlier



- The other 'observables' are related to the e_1, e_2 vectors which lie in the 'spin plane' of the electron, and rotate within this plane with a phase which is observable via e.g. interference

- It's a rotated version of the $\gamma_2\gamma_1 = I\sigma_3$ plane which is of course the plane in which rotations due to the complex phase happen conventionally

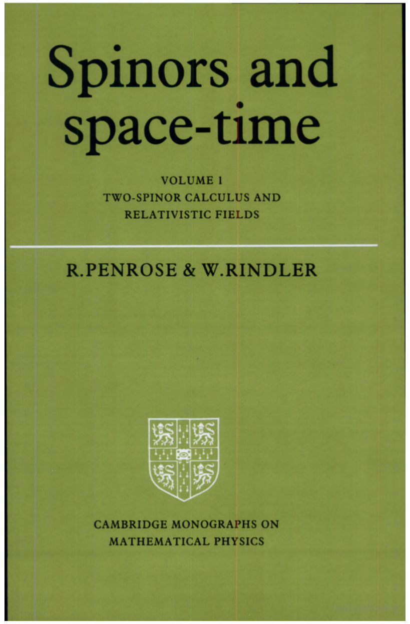
THE WEYL REPRESENTATION AND 2-SPINOR CALCULUS

- Having been explicit about our translation of quantum Dirac and Pauli spinors, we are now in a position to begin the translation of 2-spinor theory

- For the latter we adopt the notation and conventions of the standard exposition, Penrose & Rindler, Spinors and Spacetime, Vols 1 and 2

- This has been very influential in mathematics and in theories of gravitation and elementary particles

- The two volumes of Penrose & Rindler span nearly 1000 pages of dense mathematics and difficult formulae



- With the STA, we can compress things enormously, and still work with just the geometric entities of spacetime

- Probably still lots of interesting things to decode, however — this only scratches the surface (particularly as regards twistors)
- The basic translation is as follows. In 2-spinor theory, a spinor can be written either as an abstract index entity κ^A, or as a complex spin vector in spin-space (just like a quantum Pauli spinor) $\underline{\boldsymbol{\kappa}}$
- We put a 2-spinor κ^A in 1-1 correspondence with a Clifford spinor κ via

$$\kappa^A \quad \leftrightarrow \kappa(1+\sigma_3), \tag{13}$$

where κ is the Clifford Pauli spinor in one to one correspondence with the column spinor $\underline{\boldsymbol{\kappa}}$ (via 11)

- This introduces a very useful object, the projector $(1+\sigma_3)$
- Actually easier to work with 1/2 this. Call $P = \frac{1}{2}(1+\sigma_3)$. Then see

$$P^2 = \frac{1}{4}(1+\sigma_3)(1+\sigma_3) = \frac{1}{4}\left(1+\sigma_3+\sigma_3+\sigma_3^2\right) = \frac{1}{2}(1+\sigma_3) = P$$

- We call such a P an idempotent
- The function of the 'fiducial projector' $\frac{1}{2}(1+\sigma_3)$ relates to what happens under a 'spin transformation' represented by an arbitrary complex spin matrix $\underline{\boldsymbol{R}}$
- The new spin vector is $\underline{\boldsymbol{R\kappa}}$ and has only 4 real degrees of freedom, whereas an arbitrary Lorentz rotation specified by a Clifford R applied to a Clifford κ gives the quantity $R\kappa$, which contains 8 degrees of freedom
- However, applying R to $\kappa(1+\sigma_3)$ limits the degrees of freedom back to 4 again, in conformity with what happens in the 2-spinor formulation
- Can see this explicitly as follows. We can write a general STA spinor as

$$\psi = a^0 + a^1 I\sigma_1 + a^2 I\sigma_2 + a^3 I\sigma_3 + \left(b^0 + b^1 I\sigma_1 + b^2 I\sigma_2 + b^3 I\sigma_3\right)\sigma_3$$

hence applying $(1+\sigma_3)$ at the right results in a quantity we can write as

$\psi(1+\sigma_3) = \kappa(1+\sigma_3)$ where κ is the Pauli spinor

$$\kappa = (a^0 + b^0) + (a^1 + b^1)I\sigma_1 + (a^2 + b^2)I\sigma_2 + (a^3 + b^3)I\sigma_3$$

hence having just 4 degrees of freedom

- The complex conjugate spinor $\overline{\kappa}^{A'}$ introduced by P&R we find belongs to the opposite ideal under the action of the projector $(1+\sigma_3)$,

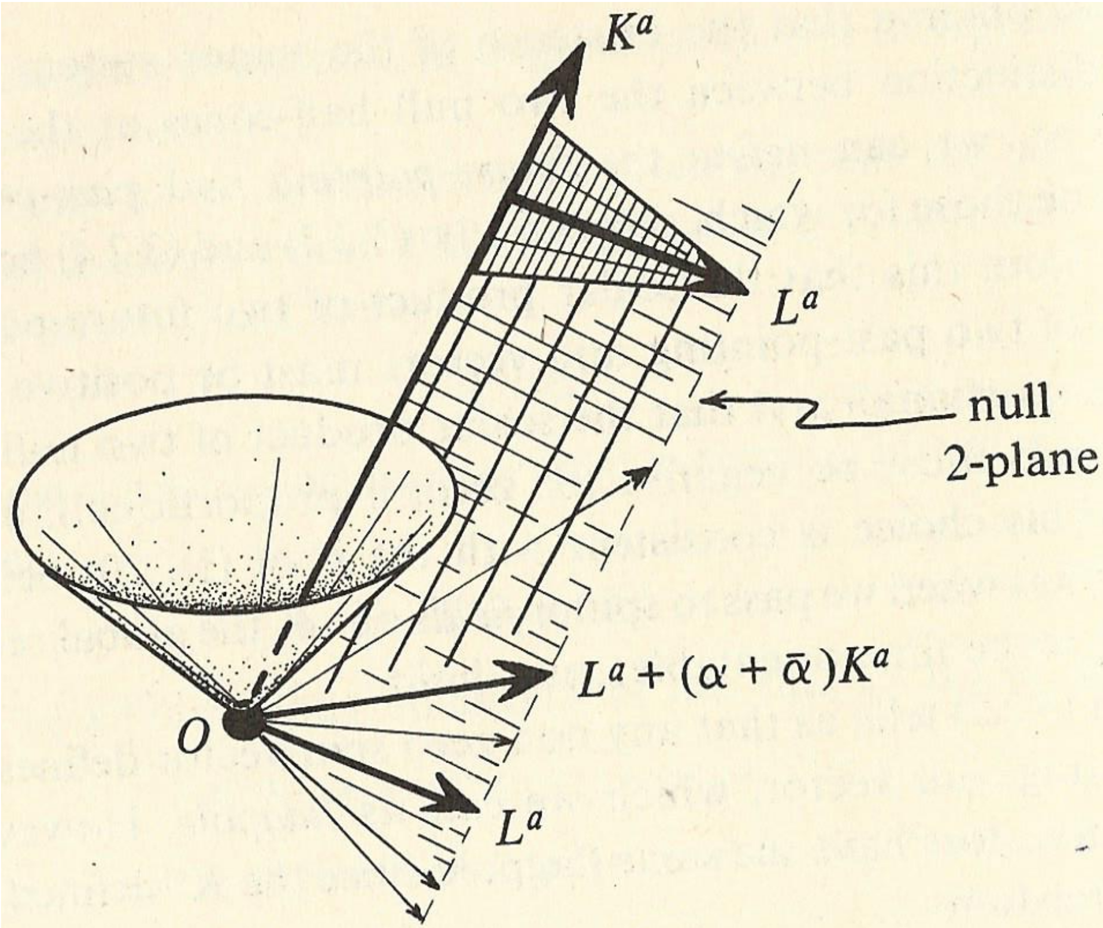
$$\overline{\kappa}^{A'} \quad \leftrightarrow -\kappa I\sigma_2(1-\sigma_3)$$

- This explains why κ^A and its complex conjugate have to be treated as belonging to different 'modules' in the Penrose and Rindler theory

- Note that in more conventional quantum notation our projectors $(1 \pm \sigma_3)$ would correspond to the chirality operators $(1 \pm \hat{\gamma}_5)$, or in the notation of the appendix of Penrose & Rindler, Vol II, to (multiples of) $\underline{\mathbf{\Pi}}$ and $\underline{\tilde{\mathbf{\Pi}}}$

- We do not use these alternative notations since it is a vital part of what we are doing that the projection operators should be constructed from ordinary spacetime entities.

- The most important quantities associated with a single 2-spinor κ^A are its *flagpole* $K^a = \kappa^A \overline{\kappa}^{A'}$, and the *flagplane* determined by the bivector $P^{ab} = \kappa^A \kappa^B \epsilon^{A'B'} + \epsilon^{AB} \overline{\kappa}^{A'} \overline{\kappa}^{B'}$



From Penrose & Rindler, Vol. I, p128

- Here we use the Penrose notation in which a is a 'lumped index' representing the spinor indices AA' *etc.*
- What are the STA equivalents?
- Firstly, if we write $\psi = \kappa(1+\sigma_3)$, the flagpole of the 2-spinor κ^A is just (up to a factor 2) the Dirac current associated with the wavefunction ψ,

$$K = \frac{1}{2}\psi\gamma_0\tilde{\psi} = \kappa(\gamma_0+\gamma_3)\tilde{\kappa}. \tag{14}$$

- We see that the projector $(1+\sigma_3)$ has produced a massless (null) current.
- Secondly, the flagplane bivector is a rotated version of the fiducial bivector σ_1:

$$P = \frac{1}{2}\psi\sigma_1\tilde{\psi} = \kappa(\gamma_1\wedge(\gamma_0+\gamma_3))\tilde{\kappa}. \tag{15}$$

- Since σ_1 anticommutes with $I\sigma_3$, while γ_0 commutes, P responds at double rate to phase rotations $\kappa \mapsto \kappa e^{I\sigma_3\theta}$, whilst the flagpole is unaffected. A convenient spacelike vector L, perpendicular to the flagpole and satisfying

$P = L \wedge K$, is $L = (\kappa\tilde{\kappa})^{-1/2}\kappa\gamma_1\tilde{\kappa}$, that is, just the 'body' 1-direction.

- Another very important concept in 2-spinor theory, is that of a 'spin-frame', usually written o^A, ι^A, but for notational reasons, and to draw out the parallel with twistors, we prefer to write these as ω^A, π^A.

- In our translation, a spin-frame ω^A, π^A is packaged together to form a Clifford Dirac spinor ϕ via

$$\phi = \omega\tfrac{1}{2}(1+\sigma_3) - \pi I\sigma_2\tfrac{1}{2}(1-\sigma_3). \tag{16}$$

- Now

$$\phi\tilde{\phi} = \tfrac{1}{2}\kappa(1+\sigma_3)I\sigma_2\tilde{\omega} + \text{ reverse} = \lambda + I\mu \qquad \text{say.} \tag{17}$$

- If one now calculates the 2-spinor inner product for the same spin-frame one finds

$$\{\underline{\boldsymbol{\omega}}, \underline{\boldsymbol{\pi}}\} = \omega_A\pi^A = -(\lambda + i\mu). \tag{18}$$

- Thus the complex 2-spinor inner product is in fact a disguised version of the quantity $\phi\tilde{\phi}$
- The 'disguise' consists of representing something that is in fact a pseudoscalar (the I in $\lambda + I\mu$) as an uninterpreted scalar i
- The condition for a spin frame to be normalized, $\omega_A \pi^A = 1$, is in our approach the condition for ϕ to be a Lorentz transformation, that is $\phi\tilde{\phi} = 1$ (except for a change of sign which in twistor terms corresponds to negative helicity)
- We can thus say "*a normalized spin frame is equivalent to a Lorentz transformation*"
- The orthonormal real tetrad, t^a, x^a, y^a, z^a, determined by such a spin-frame (P&R, Vol 1, p120), is in fact the same (up to signs) as the frame of 'body axes' $e_\mu = \phi\gamma_\mu\tilde{\phi}$ which we drew attention to in standard Dirac theory, whilst the null tetrad is just a rotated version of a certain 'fiducial' null tetrad as follows:

$$
\begin{aligned}
l^a &= \frac{1}{\sqrt{2}}(t^a + z^a) = \omega^A \overline{\omega}^{A'} & \leftrightarrow \phi(\gamma_0 + \gamma_3)\tilde{\phi}, \\
n^a &= \frac{1}{\sqrt{2}}(t^a - z^a) = \pi^A \overline{\pi}^{A'} & \leftrightarrow \phi(\gamma_0 - \gamma_3)\tilde{\phi}, \\
m^a &= \frac{1}{\sqrt{2}}(x^a - iy^a) = \omega^A \overline{\pi}^{A'} & \leftrightarrow -\phi(\gamma_1 + I\gamma_2)\tilde{\phi}, \\
\overline{m}^a &= \frac{1}{\sqrt{2}}(x^a + iy^a) = \pi^A \overline{\omega}^{A'} & \leftrightarrow -\phi(\gamma_1 - I\gamma_2)\tilde{\phi}.
\end{aligned}
\tag{19}
$$

- Note that the x or y axis is inverted with respect to the world vector equivalents, which is a feature that occurs throughout our translation of 2-spinor theory.

- (This also applies to quaternions whose algebra is isomorphic to the even sub-algebra of the 3d GA, the specific correspondence being

$$1, i, j, k \leftrightarrow 1, I\sigma_1, -I\sigma_2, I\sigma_3.)$$

- Note also that $\gamma_1 - I\gamma_2$ and $\gamma_1 + I\gamma_2$ involve *trivector* components. This is how complex world vectors in the Penrose & Rindler formalism appear when translated down to equivalent objects in a single-particle STA space. These are very interesting, since they can function as supersymmetry generators!

FIELD SUPERSYMMETRY GENERATORS

A common version of the field supersymmetry generators required for the Poincaré super-Lie algebra uses 2-spinors Q_α with Grassmann entries:

$$Q_\alpha = -i\left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\alpha'}\overline{\theta}^{\alpha'}\partial_\mu\right),$$

where the θ^α and $\overline{\theta}^\alpha$ are Grassmann variables, and μ is a spatial index

A translation of Q_α into STA basically amounts to finding real spacetime representations for the θ^α variables

We use the 4 quantities $\gamma_0 \pm \gamma_3$ and $\gamma_1 \pm I\gamma_2$ as effective Grassmann variables, with the anticommutator $\{A, B\}$ replaced by the symmetric product $\langle\tilde{A}B\rangle$. With

$$\begin{array}{ll} \theta_1 = \gamma_0 + \gamma_3 & \overline{\theta}_1 = \gamma_0 - \gamma_3 \\ \theta_2 = \gamma_1 + I\gamma_2 & \overline{\theta}_2 = -\gamma_1 + I\gamma_2 \end{array}$$

it is a simple exercise to verify that the θ_α satisfy the required supersymmetry algebra (with $\{A, B\} \equiv \langle\tilde{A}B\rangle$)

$$\{\theta_\alpha, \theta_\beta\} = \{\overline{\theta}_\alpha, \overline{\theta}_\beta\} = 0, \qquad \{\theta_\alpha, \overline{\theta}_\beta\} = 2\delta_{\alpha\beta}. \tag{20}$$

This raises interesting new possibilites of being able to reduce the arena of 'superspace' to ordinary spacetime — however, this is still under-developed as yet — an opportunity for the future!

Same applies to

Twistors

These were first developed during the 70s and 80s, with initial hopes of providing a completely new route through to the physics of elementary particles

Basic idea is to replace ordinary spacetime structure with incidence relations for null vectors

Has recently been enjoying a renaissance within string theory and holography theory, e.g. the famous AdS_5/CFT correspondence

Again we think that a Geometric Algebra version could be very valuable, and make the maths much simpler (note e.g. the Twistor Equation — quite complicated in the two spinor formulation, ends up in translation as $\boldsymbol{\nabla} \boldsymbol{r} = 4$!)

Some results in Arcaute, Lasenby & Doran, Advances in Applied Clifford Algebras, Vol. 18, p373 (2008) and more details in math-ph/0604048 and math-ph/0603037 (both 2006 and unpublished)

Here's a fundamental and interesting point:

On page 47 of Penrose & Rindler, Vol 2, the authors state '*Any temptation to identify a twistor with a Dirac spinor should be resisted. Though there is a certain formal resemblance* at one point*, the vital twistor dependence on position has no place in the Dirac formalism*.'

We argue on the contrary that a twistor *is* a Dirac spinor, with a particular dependence on position imposed.

Our fundamental translation is

$$Z = \phi - r\phi\,\gamma_0\, I\sigma_3\, \tfrac{1}{2}(1+\sigma_3), \tag{21}$$

where ϕ is an arbitrary constant relativistic STA spinor, and $r = x^\mu \gamma_\mu$ is the position vector in 4-dimensions.

To start making contact with the Penrose notation, we decompose the Dirac spinor Z, quite generally, as

$$Z = \omega\, \tfrac{1}{2}(1+\sigma_3) - \pi\, I\sigma_2\, \tfrac{1}{2}(1-\sigma_3)$$

Then the pair of Pauli spinors ω and π are the translations of the 2-spinors ω^A and $\pi_{A'}$ appearing in the usual Penrose representation

$$Z^\alpha = (\omega^A, \pi_{A'}). \tag{22}$$

In (22) $\pi_{A'}$ is constant and ω^A is meant to have the fundamental twistor dependence on position

$$\omega^A = \omega_0^A - i x^{AA'} \pi_{A'},$$

where ω_0^A is constant. We thus see that the arbitary constant spinor ϕ in (21) is

$$\phi = \omega_0 \, \tfrac{1}{2}(1 + \sigma_3) - \pi \, I\sigma_2 \, \tfrac{1}{2}(1 - \sigma_3)$$

We note this is identical to the STA representation of a spin-frame.

This ability, in the STA, to package the two parts of a twistor together, and to represent the position dependence in a straightforward fashion, leads to some remarkable simplifications in twistor analysis.

This applies both with regard to connecting the twistor formalism with physical properties of particles (spin, momentum, helicity, etc.), and to the sort of computations required for establishing the geometry associated with a given twistor.

Here's an interesting example

CONFORMAL GEOMETRY AND TWISTORS

- In the Arcaute *et al.* papers we showed how to embed twistors in the conformal geometric algebra of 4 dimensional spacetime
- This is just like the CGA for 3d Euclidean space you have been hearing about, but we start with the 4d STA, signature $(1,3)$ and add two further vectors, e and $\bar{e}$ with squares $e^2 = +1$ and $\bar{e}^2 = -1$
- So signature of the CGA is $(2,4)$ — a 6d space
- In this space, we can instantiate the complete 15-dimensional group of conformal transformations of spacetime

- Counting is 4 translations, 6 Lorentz boosts and rotations, 4 special conformal transformations and 1 dilation
- Nature appears to know about this — Maxwell equations are invariant under full conformal group!
- In CGA approach, everything achieved with combinations of rotations and reflections, in a fully covariant fashion
- We embed twistors of the form we have just been discussing into this via

$$T = ZW_1W_2$$

where W_1 and W_2 are the following projection operators

$$W_1 = \frac{1}{2}\left(1 - I\gamma_3 e\right), \quad W_2 = \frac{1}{2}\left(1 - I\gamma_3 \bar{e}\right)$$

[Exercise: show that these are indeed projection operations.]

- We can then transform the T in our 6d space, to T' say, and ask what Z' satisfies $T' = Z'W_1W_2$

- This gives an induced transformation on twistor space itself
- Won't go through details, but just want to emphasise one result: suppose we start with our constant spinor

$$\phi = \omega_0 \, \tfrac{1}{2}(1+\sigma_3) - \pi \, I\sigma_2 \, \tfrac{1}{2}(1-\sigma_3)$$

and then translate it using the CGA by r. We get

$$Z = \phi - r\phi\,\gamma_0\, I\sigma_3 \, \tfrac{1}{2}(1+\sigma_3),$$

i.e. precisely the 'translation' (!) we had already made of what a twistor is!

- (Note special conformal transformations give the same form but with a $(1-\sigma_3)$ at the right.)
- Using this knowledge, we can unwrap some twistor geometry which is otherwise quite mysterious, by translation to the origin
- E.g. consider the following, which shows the Robinson congruence of a twistor (the flagpole directions of the ω part of Z)

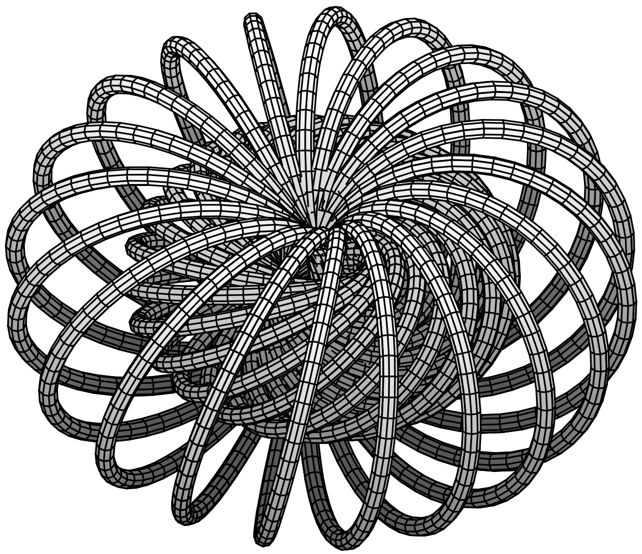


FIGURE 5. Congruences of circles.

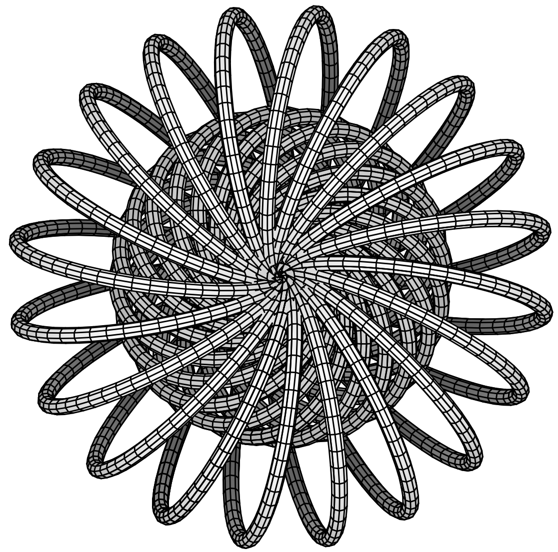


FIGURE 6. Upper view of congruences.

- This is for a non-null twistor, $\langle \tilde{Z} Z \rangle \neq 0$, where it turns out that we have to take a spherical or hyperbolic CGA as the 6d space the spinors live in
- We can do a translation back to the origin, and find

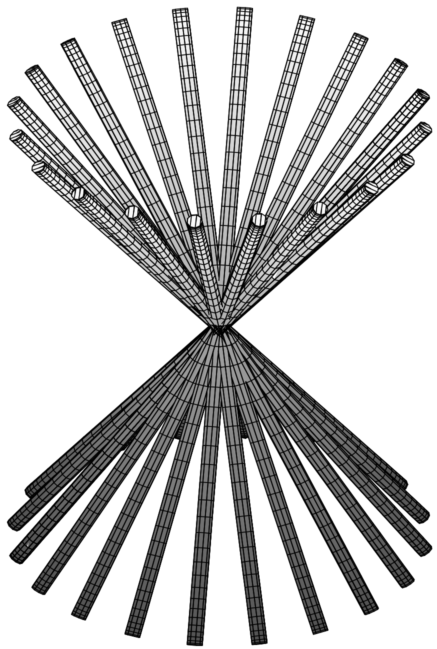


FIGURE 7. Congruence of d-lines at the origin.

- These are just ‘d-lines’ (geodesics) through the origin, and it is only the translation in the spherical or hyperbolic space that makes things look complicated

See Arcaute, Lasenby & Doran, Advances in Applied Clifford Algebras, Vol. 18, p373 (2008) for some details

Multivector Calculus and Lagrangian Symmetries

- You have already had an introduction to Multivector differentiation and linear algebra
- Want to show here how these topics are useful in Physical applications and lead forward from this to a Multivector Calculus approach to Lagrangian symmetries
- The following summarises what we need for physical applications, and is taken from Appendix B of ‘Gravity, Gauge Theories and Geometric Algebra’ by Lasenby, Doran & Gull (Phil.Trans.Roy.Soc.Lond. A356 (1998) 487-582), which will be one of our main references later (will call LDG)

We begin with a set of results for the derivative with respect to the vector a in an n-dimensional space

$$\partial_a a\cdot b = b \qquad \partial_a a^2 = 2a$$

$$\partial_a \cdot a = n \qquad \partial_a a\cdot A_r = rA_r$$

$$\partial_a \wedge a = 0 \qquad \partial_a a\wedge A_r = (n-r)A_r$$

$$\partial_a a = n \qquad \dot{\partial}_a A_r \dot{a} = (-1)^r(n-2r)A_r.$$

[Exercise: prove these]

The results needed for the multivector derivative here (particularly for application to the Dirac equation) are

$$\partial_X \langle XA\rangle = P_X(A)$$

$$\partial_X \langle \tilde{X}A\rangle = P_X(\tilde{A}),$$

where $P_X(A)$ is the projection of A onto the grades contained in X. These results are combined using Leibniz' rule; for example,

$$\partial_\psi \langle \psi \tilde{\psi} \rangle = \dot{\partial}_\psi \langle \dot{\psi} \tilde{\psi} \rangle + \dot{\partial}_\psi \langle \psi \dot{\tilde{\psi}} \rangle = 2\tilde{\psi}$$

Note particularly that $\tilde{\psi}$ is not taken as an independent object from ψ, which it effectively is in conventional approaches!

Our approach makes more sense!

Note also the two rules we use all the time in taking scaling parts:

Cyclic reordering:

$$\langle AB \ldots C \rangle = \langle CAB \ldots \rangle$$

Reversion:

$$\langle AB \rangle = \langle \tilde{B} \tilde{A} \rangle$$

For the action principle we also require results for the multivector derivative with respect to the directional derivatives of a field ψ

The aim is again to refine the calculus so that it becomes possible to work in a frame-free manner

We first introduce the fixed frame $\{e^j\}$, with reciprocal $\{e_k\}$, so that $e^j \cdot e_k = \delta^j_k$

The partial derivative of ψ with respect to the coordinate $x^j = e^j \cdot x$ is abbreviated to $\psi_{,j}$ so that

$$\psi_{,j} \equiv e_j \cdot \nabla \psi$$

We can now define the frame-free derivative

$$\partial_{\psi_{,a}} \equiv a \cdot e_j \, \partial_{\psi_{,j}}$$

The operator $\partial_{\psi_{,a}}$ is the multivector derivative with respect to the a-derivative of ψ. The fundamental property of $\partial_{\psi_{,a}}$ is that

$$\partial_{\psi_{,a}} \langle b \cdot \nabla \psi M \rangle = a \cdot b P_\psi(M)$$

Again, more complicated results are built up by applying Leibniz' rule. The Euler-Lagrange equations for the Lagrangian density $\mathcal{L} = \mathcal{L}(\psi, a \cdot \nabla \psi)$ can now be given in the form

$$\partial_\psi \mathcal{L} = \partial_a \cdot \nabla(\partial_{\psi_{,a}} \mathcal{L})$$

as we shall see shortly