MULTIVECTOR DERIVATIVES WITH RESPECT TO FUNCTIONS

Note to avoid any confusion: The first paper dealing with multivector differentiation for physics, and in particular the concept of multivector derivatives with respect to functions, was

A. N. Lasenby, C. J. L. Doran and S. F. Gull, A Multivector Derivative Approach to Lagrangian Field Theory, Found. Phys. 23(10), 1295-1327 (1993)

Our approach and notation evolved a bit after this point, and the first definitive treatment was in Appendix B of the GTG paper

'Gravity, Gauge Theories and Geometric Algebra' by Lasenby, Doran & Gull (Phil.Trans.Roy.Soc.Lond. A356 (1998) 487-582)

referred to above

The site `http://geometry.mrao.cam.ac.uk/` contains these papers and many more

Later will need a formalism for the derivative with respect to a linear function

Given the linear function $\underline{h}(a)$ and the fixed frame $\{e_i\}$, we define the scalar coefficients

$$h_{ij} \equiv e_i \cdot \underline{h}(e_j)$$

The individual partial derivatives $\partial_{h_{ij}}$ are assembled into a frame-free derivative by defining

$$\partial_{\underline{h}(a)} \equiv a \cdot e_j e_i \partial_{h_{ij}}$$

The fundamental property of $\partial_{\underline{h}(a)}$ is that

$$\begin{aligned}
\partial_{\underline{h}(a)} \underline{h}(b) \cdot c = a \cdot e_j e_i \partial_{h_{ij}} \big(h(b^k e_k) \cdot (c^l e_l)\big) &= a \cdot e_j e_i \partial_{h_{ij}} (h_{lk} b^k c^l) \\
&= a \cdot e_j \, e_i c^i b^j \\
&= a \cdot b \, c
\end{aligned}$$

which, together with Leibniz' rule, is sufficient to derive all the required properties of the $\partial_{\underline{h}(a)}$ operator.

For example, if B is a fixed bivector,

$$\begin{aligned}
\partial_{\underline{h}(a)} \langle \underline{h}(b \wedge c) B \rangle &= \dot{\partial}_{\underline{h}(a)} \langle \underline{\dot{h}}(b) \underline{h}(c) B \rangle - \dot{\partial}_{\underline{h}(a)} \langle \underline{\dot{h}}(c) \underline{h}(b) B \rangle \\
&= a \cdot b \, \underline{h}(c) \cdot B - a \cdot c \, \underline{h}(b) \cdot B \\
&= \underline{h} \, (a \cdot b \, c - a \cdot c \, b) \cdot B \\
&= \underline{h}[a \cdot (b \wedge c)] \cdot B
\end{aligned}$$

This result extends immediately to give

$$\partial_{\underline{h}(a)} \langle \underline{h}(A_r) B_r \rangle = \langle \underline{h}(a \cdot A_r) B_r \rangle_1$$

In particular,

$$\begin{aligned}
\partial_{\underline{h}(a)} \det(\underline{h}) &= \partial_{\underline{h}(a)} \langle \underline{h}(I) I^{-1} \rangle \\
&= \underline{h}(a \cdot I) I^{-1} = \left[\underline{h}(a \cdot I) I^{-1} \det(\underline{h})^{-1} \right] \det(\underline{h}) \\
&= \overline{h}^{-1}(a) \, \det(\underline{h}),
\end{aligned}$$

where the definition of the inverse you have already seen from Joan has been employed. This derivation affords a remarkably direct proof of the formula for the derivative of the determinant w.r.t. the linear function (and makes sense of formulae for differentiating determinants w.r.t. matrices — e.g. in GR need to differentiate the metric determinant w.r.t. the metric — very hard to make sense of conventionally)

The above results hold equally if $\underline{h}$ is replaced throughout by its adjoint $\overline{h}$. Note, however, that

$$\begin{aligned}\partial_{\underline{h}(a)}\overline{h}(b) &= \partial_{\underline{h}(a)}\langle\underline{h}(c)b\rangle\partial_c \\ &= a\cdot c\, b\,\partial_c = ba\end{aligned}$$

while

$$\begin{aligned}\partial_{\underline{h}(a)}\underline{h}(b) &= \partial_{\underline{h}(a)}\langle\underline{h}(b)c\rangle\partial_c \\ &= a\cdot b\, c\,\partial_c = 4a\cdot b\end{aligned}$$

Thus the derivatives of $\underline{h}(b)$ and $\overline{h}(b)$ give different results, regardless of any symmetry properties of $\underline{h}$. This has immediate implications for the symmetry (or lack of symmetry) of the functional stress-energy tensors for certain fields.

We finally need some results for derivatives with respect to the bivector-valued linear function $\Omega(a)$. The extensions are straightforward, and we just give the required results:

$$\partial_{\Omega(a)} \langle \Omega(b) M \rangle = a\cdot b \langle M \rangle_2$$
$$\partial_{\Omega(b),a} \langle c\cdot\nabla\Omega(d) M \rangle = a\cdot c\, b\cdot d\, \langle M \rangle_2$$

EULER-LAGRANGE EQUATIONS AND NOETHER'S THEOREM

Suppose that the system of interest depends on a field $\psi(x)$, where x is a spacetime position vector.

The action is now defined as an integral over a region of spacetime by

$$S = \int d^4x\, \mathcal{L}(\psi, \partial_\mu \psi, x)$$

where $\mathcal{L}$ is the *Lagrangian density* and x^μ are a set of fixed orthonormal coordinates for spacetime. More general coordinate systems are easily accomodated with the inclusion of suitable factors of the Jacobian.

We assume that $\psi_0(x)$ represents the extremal path, satisfying the desired boundary conditions, and look for variations of the form

$$\psi(x) = \psi_0(x) + \epsilon\phi(x)$$

Here $\phi(x)$ is a field of the same form as $\psi(x)$, which vanishes over the boundary. The first-order variation in the action will involve the term (summation convention in force)

$$\frac{d}{d\epsilon}\mathcal{L}\left(\psi_0(x) + \epsilon\phi(x), \partial_\mu\left(\psi_0(x) + \epsilon\phi(x)\right)\right) = \phi(x) * \frac{\partial\mathcal{L}}{\partial\psi} + \frac{\partial\phi}{\partial x^\mu} * \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}$$

and we have

$$\left.\frac{dS}{d\epsilon}\right|_{\epsilon=0} = \int d^4x\left(\phi(x) * \frac{\partial\mathcal{L}}{\partial\psi} + \frac{\partial\phi}{\partial x^\mu} * \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right)$$

The final term is integrated by parts to give

$$\left[\phi * \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right] - \int d^4x\,\phi(x) * \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right)$$

and the boundary term vanishes. We therefore find that

$$\left.\frac{dS}{d\epsilon}\right|_{\epsilon=0} = \int d^4x\, \phi(x) * \left(\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right)\right),$$

from which we can read off the variational equations as

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) = 0 \qquad (23)$$

- Now we would like to make the transition to frame-free notation
- This is actually very useful when deriving the equations of motion
- The answer is that (23) is equivalent to the equation we gave earlier:

$$\partial_\psi \mathcal{L} = \partial_a \cdot \nabla(\partial_{\psi_{,a}} \mathcal{L})$$

  and it is this latter equation we use. Can show they are equivalent as follows.
- Start with the r.h.s. of the equation just given, and expand everything in a

basis to get

$$\partial_a \cdot \nabla(\partial_{\psi,a}\mathcal{L}) = e^\nu \frac{\partial}{\partial a^\nu} \cdot e^\mu \frac{\partial}{\partial x^\mu} \left((a^\lambda e_\lambda) \cdot e_\alpha \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} \right)$$

$$= e^\nu \cdot e^\mu \frac{\partial}{\partial x^\mu} e_\nu \cdot e_\alpha \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} = \delta^\mu_\alpha \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \psi_{,\alpha}} = \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial \psi_{,\mu}}$$

which indeed matches with the second term in (23).

- Note the ψ here doesn't have to be a spinor — can apply to any type of object in the STA as long as we remember the fundamental rules of multivector differentiation and that

$$\partial_{\psi,a} \langle b \cdot \nabla \psi M \rangle = a \cdot b P_\psi(M) \tag{24}$$

- If more fields are present we obtain an equation of this form for each field

Deriving the Dirac equation from a Lagrangian

- So will illustrate this approach in the context of the Dirac equation
- The STA form of the Dirac Lagrangian is

$$\begin{aligned}\mathcal{L} &= \langle \nabla\psi I\gamma_3\tilde{\psi} - eA\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi}\rangle \\ &= \langle (b\cdot\nabla\psi)\, I\gamma_3\tilde{\psi}\,\partial_b\rangle - \langle eA\psi\gamma_0\tilde{\psi} + m\psi\tilde{\psi}\rangle\end{aligned}$$

where ψ is an even multivector and A is an external field (which is not varied)

- In the second line we have expanded the derivative and carried out a cyclic reordering so that we can directly use (24)
- So we first calculate $\partial_\psi\mathcal{L}$ obtaining (all steps shown)

$$\partial_\psi\mathcal{L} = (\nabla\psi I\gamma_3)\tilde{} - \gamma_0\tilde{\psi}eA - (eA\psi\gamma_0)\tilde{} - m\tilde{\psi} - (m\psi)\tilde{}$$

while

$$\partial_{\psi_{,a}} \mathcal{L} = a \cdot b \, I\gamma_3 \, \tilde{\psi} \, \partial_b = I\gamma_3 \, \tilde{\psi} \, a$$

Then we form

$$\partial_a \cdot \nabla (\partial_{\psi_{,a}} \mathcal{L}) = \partial_a \cdot \nabla (I\gamma_3 \, \tilde{\psi} \, a) = I\gamma_3 \tilde{\psi} \overleftarrow{\nabla}$$

Noting that $(I\gamma_3)\tilde{} = -I\gamma_3$, we therefore have overall

$$-I\gamma_3 \tilde{\psi} \overleftarrow{\nabla} - 2\gamma_0 \tilde{\psi} e A - 2m\tilde{\psi} = I\gamma_3 \tilde{\psi} \overleftarrow{\nabla}$$

Collecting terms, taking the reverse, and multiplying on the right by γ_0, then yields the final equation

$$\nabla \psi I \sigma_3 - eA\psi - m\psi\gamma_0 = 0 \tag{25}$$

which is the Dirac equation in STA form.

- Note particularly, there is no pretence of ψ and $\tilde{\psi}$ being independent entities, or of just knocking off the $\tilde{\psi}$ at the right in the Lagrangian to get the e.o.m. (which is sometimes suggested in the literature!)
- (Note however that (25) means the Lagrangian is in fact zero evaluated on the e.o.m — will understand generally why this is shortly.)
- Such methods can work if you happen to know the answer, but in more complicated cases, where e.g. gravitational terms, or torsion are present, the multivector derivative STA method just gone through provides a fully safe route, where you will get the right answer, even with the new fields present

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CONSERVATION LAWS IN FIELD THEORY

- Want to obtain a version of Noether's theorem appropriate for field theory
- For simplicity we assume that only one field is present. The results are easily

extended to the case of more fields by summing over all of the fields present

- Suppose that $\psi'(x)$ is a new field obtained from $\psi(x)$ by a scalar-parameterised transformation of the form

$$\psi'(x) = f(\psi(x), \alpha),$$

with $\alpha = 0$ corresponding to the identity

- We define

$$\delta\psi = \left.\frac{\partial\psi'}{\partial\alpha}\right|_{\alpha=0}$$

- With $\mathcal{L}'$ denoting the original Lagrangian evaluated on the transformed fields we find that

$$\begin{aligned}\left.\frac{\partial\mathcal{L}'}{\partial\alpha}\right|_{\alpha=0} &= (\delta\psi)*\frac{\partial\mathcal{L}}{\partial\psi} + \partial_\mu(\delta\psi)*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} \\ &= \frac{\partial}{\partial x^\mu}\left((\delta\psi)*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right)\end{aligned}$$

where we used the EL equations

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right)$$

to substitute for the $\frac{\partial \mathcal{L}}{\partial \psi}$ in the first line and notice that this then gives the total derivative shown in the second line

- This equation relates the change in the Lagrangian to the divergence of the current J, where

$$J = \gamma_\mu \, (\delta\psi) * \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}$$

- We can write this in frame-independent form as

$$\boxed{J = \partial_a \left(\delta\psi * \partial_{\psi,a} \mathcal{L}\right)}$$

and note again that this is generally applicable — e.g. ψ could be a spinor (Dirac theory) or a vector (Electromagnetism, where the vector potential A is the quantity we vary in the Lagrangian)

- If the transformation is a symmetry of the system then $\mathcal{L}'$ is independent of α. In this case we immediately establish that the conjugate current is conserved, that is

$$\nabla \cdot J = 0$$

- Symmetries of a field Lagrangian therefore give rise to conserved currents. These in turn define Lorentz-invariant constants via,

$$Q = \int d^3x \, J^0,$$

where $J^0 = J \cdot \gamma^0$ is the density measured in the γ_0 frame.

- The fact that this is constant follows from

$$\frac{dQ}{dt} = \int d^3x \, \frac{\partial J^0}{\partial t} = \int d^3x \, \boldsymbol{\nabla} \cdot \boldsymbol{J} = \oint dS \, \boldsymbol{n} \cdot \boldsymbol{J} = 0$$

where we assume that the current $\boldsymbol{J}$ falls off sufficiently fast at infinity. The value of Q is constant, and independent of the spatial hypersurface used to define the integral

- Note that if $\mathcal{L}'$ is not independent of α, we can still get useful results from

$$\boxed{\left.\frac{\partial \mathcal{L}'}{\partial \alpha}\right|_{\alpha=0} = \nabla \cdot J} \tag{26}$$

which applies in the general case

Dirac Theory

To illustrate this approach, let us look at this within Dirac theory

- There are two classes of symmetry, according to whether or not the position vector x is transformed
- Here we will consider position-independent transformations of the spinor ψ — you could look at Section 13.3.1 of Doran & Lasenby for spacetime transformations, which are also very interesting, and where certain spacetime monogenics make a surprising appearance

- So the transformations we study at this point are of the type

$$\psi' = \psi e^{\alpha M}, \tag{27}$$

where M is a general multivector and α and M are independent of position

- Operations on the right of ψ arise naturally in the STA formulation of Dirac theory, and should be thought of as generalised gauge transformations

- In the standard Dirac theory with column spinors, however, transformations like (27) cannot be written down simply, and many of the results presented here are much harder to derive

We have that

$$\delta\psi = \left.\frac{\partial\psi'}{\partial\alpha}\right|_{\alpha=0} = \psi M$$

and since

$$\partial_{\psi,a}\mathcal{L} = I\gamma_3\,\tilde{\psi}\,a$$

for the Dirac Lagrangian, we find the current J is

$$J = \partial_a \left(\delta\psi * \partial_{\psi,a} \mathcal{L} \right) = \partial_a \langle \psi M I \gamma_3 \tilde{\psi} a \rangle = \langle \psi M I \gamma_3 \tilde{\psi} \rangle_1$$

- Applying (26) to (27), we thus have

$$\nabla \cdot \langle \psi M I \gamma_3 \tilde{\psi} \rangle_1 = \partial_\alpha \mathcal{L}' |_{\alpha=0} \tag{28}$$

which is a result we shall exploit by substituting various quantities for M

- If M is odd, the equation yields no information, since both sides vanish identically
- The first even M we consider is a scalar, λ, so that $\langle \psi M I \gamma_3 \tilde{\psi} \rangle_1$ is zero. It follows that

$$\partial_\alpha \left(e^{2\alpha\lambda} \mathcal{L} \right) \Big|_{\alpha=0} = 0$$
$$\Rightarrow \mathcal{L} = 0,$$

so that, when the equations of motion are satisfied, the Dirac Lagrangian vanishes (which we had already noticed above, but now know the reason)

- We next consider a duality transformation. Setting $M = I$, in equation (28) gives

$$
\begin{aligned}
-\nabla\cdot(\rho s) &= -m\partial_\alpha \langle e^{2I\alpha} \rho e^{I\beta} \rangle\big|_{\alpha=0} \\
\Rightarrow \nabla\cdot(\rho s) &= -2m\rho \sin\beta
\end{aligned}
\tag{29}
$$

where $\psi\tilde{\psi} = \rho e^{I\beta}$ and the spin current ρs is defined as $\psi\gamma_3\tilde{\psi}$

- The role of the β-parameter in the Dirac equation remains unclear, although (29) relates it to non-conservation of the spin current

- Equation (29) was already known (by David!) when we obtained it using the multivector approach, but it didn't seem to have been pointed out before that the spin current is the conjugate current to duality rotations

- In conventional versions, these would be called 'axial rotations', with the role of I taken by γ_5

- However, in our approach, these rotations are identical to duality transformations for the electromagnetic field (see shortly) — another unification provided by geometric algebra

- The duality transformation $e^{I\alpha}$ is also the continuous analogue of discrete mass conjugation symmetry, since $\psi \mapsto \psi I$ changes the sign of the mass term in $\mathcal{L}$. Hence we expect that the conjugate current, ρs, is conserved for massless particles.

- Finally, taking M to be an arbitrary bivector B yields

$$\begin{aligned} \nabla\cdot(\psi B\cdot(I\gamma_3)\tilde{\psi}) &= 2\langle\nabla\psi IB\cdot\gamma_3\tilde{\psi} - eA\psi B\cdot\gamma_0\tilde{\psi}\rangle \\ &= 2\left\langle eA\psi(\sigma_3 B\sigma_3 - B)\gamma_0\tilde{\psi}\right\rangle, \end{aligned} \tag{30}$$

where we have used the equations of motion

- Both sides of (30) vanish for $B = I\sigma_1, I\sigma_2$ and σ_3, with useful equations arising on taking $B = \sigma_1, \sigma_2$ and $I\sigma_3$

- The last of these, $B = I\sigma_3$, corresponds to the usual $U(1)$ gauge transformation of the spinor field, and gives

$$\nabla\cdot(\rho v) = 0, \tag{31}$$

where $\rho v = \psi\gamma_0\tilde{\psi}$ is the current conjugate to phase transformations, and is strictly conserved.

- If we multiply this by e we see that this can be interpreted as the charge current, and the 'Q' we would get for this as described above would be the electric charge in a region
- The remaining transformations, $e^{\alpha\sigma_1}$ and $e^{\alpha\sigma_2}$, give

$$\begin{aligned} \nabla\cdot(\rho e_1) &= 2e\rho A\cdot e_2 \\ \nabla\cdot(\rho e_2) &= -2e\rho A\cdot e_1, \end{aligned} \tag{32}$$

where $\rho e_\mu = \psi\gamma_\mu\tilde{\psi}$

- Although these equations had been found before (again by David!), the role of ρe_1 and ρe_2, as currents conjugate to right-sided $e^{\alpha\sigma_2}$ and $e^{\alpha\sigma_1}$ transformations, had not been noted
- Right multiplication by σ_1 and σ_2 provide continuous versions of charge

conjugation, since the transformation $\psi \mapsto \psi\sigma_1$ takes the Dirac equation (25) into

$$\nabla\psi i\sigma_3 + eA\psi = m\psi\gamma_0.$$

- It follows that the conjugate currents are conserved exactly if the external potential vanishes, or the particle has zero charge.
- Many of the results here were derived by David through an analysis of the local observables of the Dirac theory
- However, the Lagrangian approach simplifies things and reveals that many of the observables in the Dirac theory are conjugate to symmetries of the Lagrangian, and that these symmetries have natural geometric interpretations

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ELECTROMAGNETISM

- Worth also having a brief look at the Electromagnetic Lagrangian

- The dynamical field we vary is the EM 4-potential A, from which the Faraday tensor F is defined via $F = \nabla\wedge A$

- The EM Lagrangian, including a term for interaction with a fixed external current J is then

$$\mathcal{L} = \tfrac{1}{2} F\cdot F - A\cdot J$$

- Note $F = \boldsymbol{E} + I\boldsymbol{B}$, so $F\cdot F = \boldsymbol{E}^2 - \boldsymbol{B}^2$, and is clearly an invariant under Lorentz rotations $F \mapsto RF\tilde{R}$ for a spacetime rotor R

- So let us get the Euler-Lagrange equations for this Lagrangian using the above approach, with $\psi = A$. We have

$$\begin{aligned}
\mathcal{L} &= \tfrac{1}{2} (\nabla\wedge A)\cdot(\nabla\wedge A) - A\cdot J \\
&= \tfrac{1}{2} (\partial_b \wedge b\cdot\nabla A)\cdot(\nabla\wedge A) - A\cdot J \\
&= -\tfrac{1}{2}\langle b\cdot\nabla A\,(\partial_b\cdot(\nabla\wedge A))\rangle - A\cdot J
\end{aligned}$$

- Thus

$$\partial_{A,a}\mathcal{L} = -\tfrac{2}{2}\, a\cdot b\, \partial_b\cdot(\nabla\wedge A) = -a\cdot(\nabla\wedge A)$$

and $\partial_a\cdot\nabla$ of this is $-\nabla\cdot(\nabla\wedge A)$

- Overall then, the equations of motion are

$$-\nabla\cdot(\nabla\wedge A) = \partial_A\mathcal{L} = -J, \quad \text{i.e.} \quad \nabla\cdot F = J$$

- Combining this with the identity $\nabla\wedge F = \nabla\wedge\nabla\wedge A = 0$, we get

$$\boxed{\nabla F = J}$$

i.e. the expected Maxwell equations in STA form

- We can also look at the situation as regards Noether currents
- It turns out that in this case, it's the spacetime dependent transformations that are the interesting ones, and these are discussed e.g. in D&L, Chap 13
- This is not unexpected, since the principle transformation of interest for EM is

the gauge transformation

$$A \mapsto A + \nabla\phi(x)$$

where ϕ is a scalar function of position. Note $F = \nabla \wedge A$ is left invariant by this

- However, as a little check of our understanding of what we have been carrying out for the Dirac equation, let's try the analogue of $\psi \mapsto \psi e^{\alpha}$, to see if we expect the Lagrangian to vanish in the EM case
- We've already found

$$\partial_{A,a}\mathcal{L} = -a\cdot(\nabla \wedge A)$$

so quite generally, for a modified $A \mapsto A'$ we have

$$\begin{aligned} J_{\text{Noether}} &= \partial_a \left(\left.\frac{dA'}{d\alpha}\right|_{\alpha=0} \cdot(-a\cdot(\nabla \wedge A))\right) \\ &= \left.\frac{dA'}{d\alpha}\right|_{\alpha=0} \cdot F \end{aligned}$$

- Putting $A' = Ae^{\alpha}$ but leaving the external current fixed, we get

$$\left.\frac{\partial \mathcal{L}'}{\partial \alpha}\right|_{\alpha=0} = \frac{\partial}{\partial \alpha}\left[e^{2\alpha}\tfrac{1}{2}\left(\nabla\wedge A\right)\cdot\left(\nabla\wedge A\right) - e^{\alpha}A\cdot J\right]$$
$$= F\cdot F - A\cdot J$$

- Meanwhile

$$\nabla\cdot J_{\mathrm{Noether}} = \nabla\cdot\left(A\cdot F\right) = \left(\nabla\wedge A\right)\cdot F - A\cdot\left(\nabla\cdot F\right) = F\cdot F - A\cdot J$$

- So we have verified the generalised form of Noether's theorem, that

$$\left.\frac{\partial \mathcal{L}'}{\partial \alpha}\right|_{\alpha=0} = \nabla\cdot J_{\mathrm{Noether}}$$

  but we can see that due to the absence of scale invariance, for a fixed external current, there is no reason for the Lagrangian to vanish

The Stress-Energy Tensor

- Now a key thing one might do at this point, in a flat space context, is look at what the Noether current conjugate to translations is, and what translation invariance of a system leads to
- Similarly, one could look at what the Noether current conjugate to rotations is, and what rotational invariance of a system leads to
- Our multivector derivative approach is good for this, and leads to the canonical stress energy tensor and canonical angular momentum tensor, and their conservation, respectively
- However, the tensors one obtains in the way have some issues, which historically have caused problems
- E.g., for electromagnetism, the canonical SET is not gauge-invariant, and has to be repaired by adding in extra terms

- And e.g. it is sometimes not clear whether one should be taking the adjoint or non-adjoint of the tensor as being the SET (we were guilty of this originally for the Dirac SET), or whether one should symmetrise to make all SETs symmetric
- Similar problems with angular momentum tensors
- However, all this can be cured in a gravitational context
- Turns out that there, there is a route which always works, and avoids all the above problems!
- So what we will do, is embark on our study of gravity now, and return to stress-energy tensors once we have got some of the gravitational theory established
- We start gravity by looking at: