Unified Mathematics (Uni-Math) with Geometric Algebra (GA)

David Hestenes
Arizona State University

> *"For geometry, you know, is the gateway to science, and that gate is so low and small that you can enter only as a little child."*
>
> *William Kingdon Clifford*

Santalo 2016

Purpose of this Talk

To demonstrate how **geometric algebra** unifies and simplifies

- geometry, algebra and trigonometry at the elementary level,
- thereby simplifying and facilitating mathematical applications to physics and engineering at the most advanced levels.

References

- Introductory survey: *Oersted Medal Lecture* 2002 (AJP) <http://modelingnts.la.asu.edu>
- Most thorough treatment of GA fundamentals: *New Foundations for Classical Mechanics* (Springer)
- Interactive presentation for high school: *GA Primer* <http://geocalc.clas.asu.edu/GAPrimer/>

To Enter the Gate to Geometric Algebra

- You must relearn ***how to multiply vectors***
- Learn how ***vector multiplication is designed for optimal encoding of geometric structure.***

Basic geometric-algebraic objects (H. Grassmann, 1844)

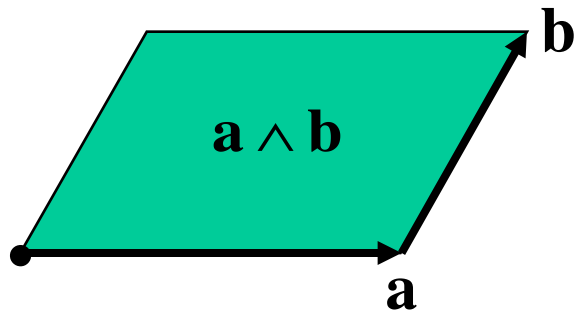
Geometric object ⇔ Algebraic object

Point • α — *scalar* (0-vector) α

Directed line segment

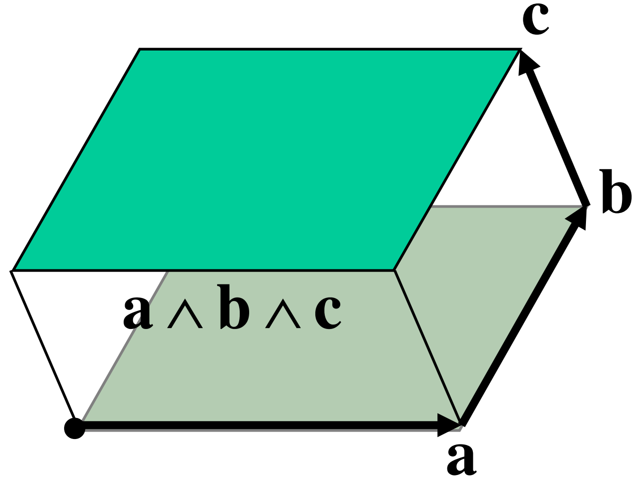
vector (1-vector) **a**

Directed plane segment



bivector (2-vector) $\mathbf{a} \wedge \mathbf{b}$

Directed volume



trivector (3-vector)

$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$

pseudoscalar in 3D

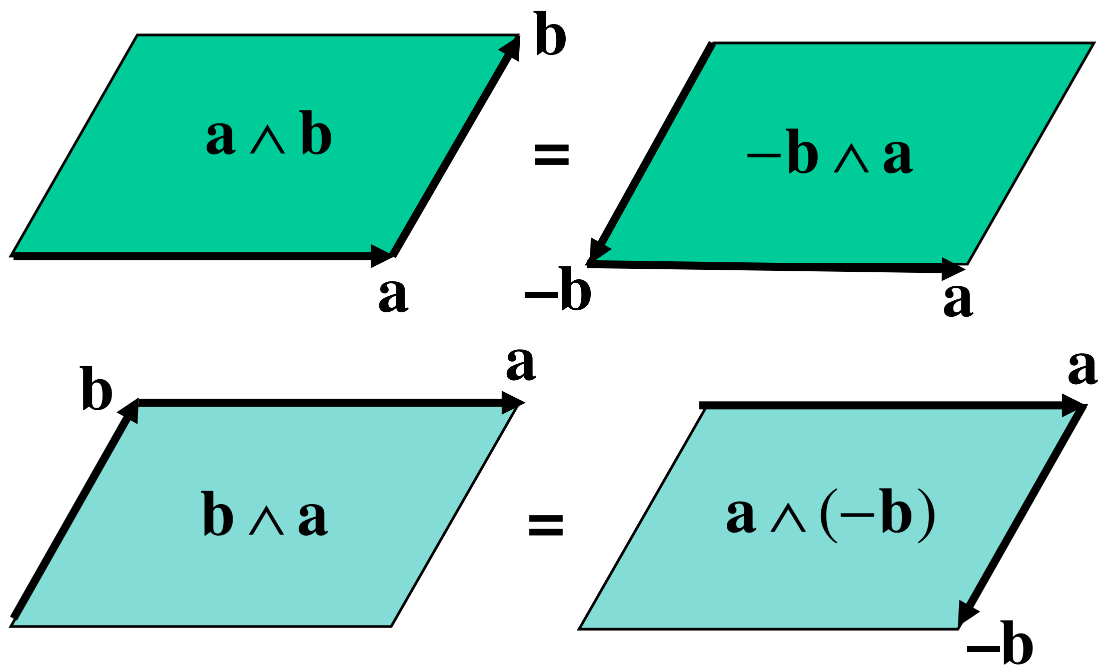
Orientation & antisymmetry of the outer product $\mathbf{a} \wedge \mathbf{b}$

Anticommutativity

Parallelogram rule for multiplication

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

$$\mathbf{b} \wedge \mathbf{a} = \mathbf{a} \wedge (-\mathbf{b})$$



Orientation (–) of vectors determines orientation of products:

$$\mathbf{b} \wedge (-\mathbf{a}) = (-\mathbf{b}) \wedge \mathbf{a} = -(\mathbf{b} \wedge \mathbf{a}) = -\mathbf{b} \wedge \mathbf{a} = \mathbf{a} \wedge \mathbf{b}$$

What we have established so far:

Geometry is built out of *basic* *geometric objects* with dimensions 0, 1, 2, 3, . . . , namely:
point, *line segment*, *plane segment*, *space segment*, . . .

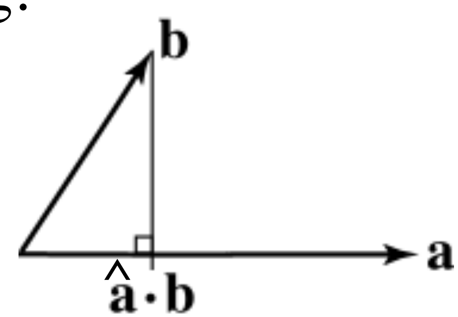
Basic geometric objects are represented by algebraic objects with grades 0, 1, 2, 3, . . . , namely:
scalar, vector, bivector, trivector (pseudoscalar), . . .
[0-vector, 1-vector, 2-vector, 3-vector, . . . (*k-vectors*)]

The *outer product* (wedge product) enables us to build *k-vectors* out of vectors, as in
$\mathbf{a},\ \mathbf{a} \wedge \mathbf{b},\ \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}, \ldots$

To represent geometric concepts of *magnitude and direction*, we need to extend the rules for combining k-vectors.

Assume familiarity with *vector addition & scalar multiplication!*

Geometric algebra = Clifford algebra (1878)
with geometric meaning!

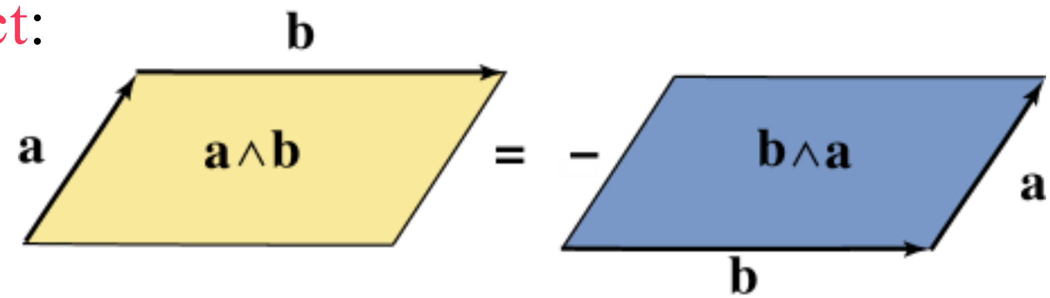


symmetric inner product (scalar-valued)

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$$

antisymmetric outer product:

$$\mathbf{a}\wedge\mathbf{b}=-\mathbf{b}\wedge\mathbf{a}$$



Combine
to form a single geometric product: $\boxed{\mathbf{ab}=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}}$

Theorem:

<u>Collinear vectors commute</u>: $\mathbf{a}\wedge\mathbf{b}=0 \quad\Leftrightarrow\quad \mathbf{ab}=\mathbf{ba}$

<u>Orthogonal vectors anticommute</u>: $\mathbf{a}\cdot\mathbf{b}=0 \quad\Leftrightarrow\quad \mathbf{ab}=-\mathbf{ba}$

Understanding the import of this formula:

$$\boxed{\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}}$$

is the single most important step in unifying the mathematical language of physics.

This formula integrates the concepts of

- vector
- complex number
- quaternion
- spinor
- Lorentz transformation

And much more!

We consider first how it integrates vectors and complex numbers into a powerful tool for 2D physics.

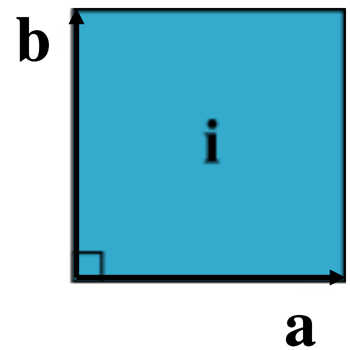
Consider the important special case of a unit bivector **i**

It has *two kinds of geometric interpretation!*

I. Object interpretation as an oriented area (additive)

Can construct **i** from a pair of orthogonal unit vectors:

$$\mathbf{i} = \mathbf{a} \wedge \mathbf{b} = \mathbf{ab} = -\mathbf{ba} \quad \Rightarrow \quad \boxed{\mathbf{i}^2 = -1}$$

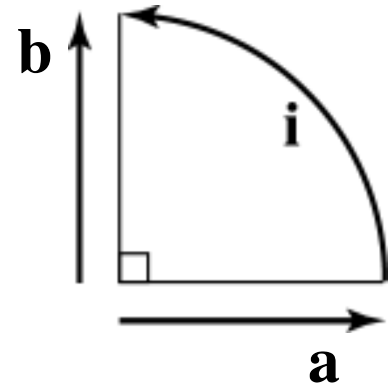


So

- **i** ≈ *oriented unit area* for a plane

II. Operator interpretation as rotation by 90° (multiplicative)

depicted as a directed arc



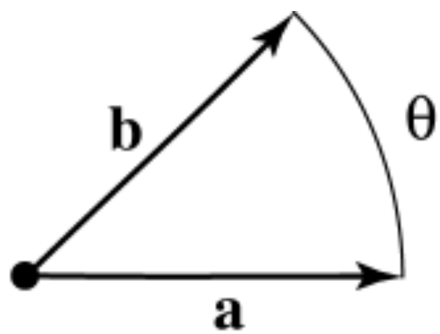
So

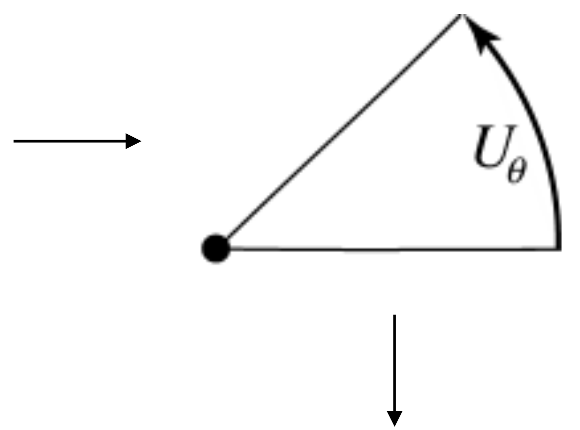
- **i** ≈ *rotation by a right angle*: $\boxed{\mathbf{ai} = \mathbf{b}}$

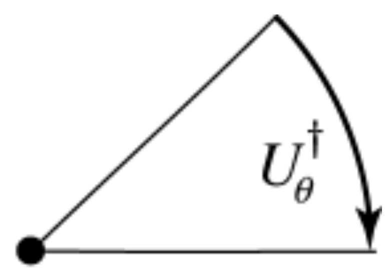
The operator interpretation of $\mathbf{i}$ generalizes to the concept of
Rotor U_θ, the entity produced by the geometric product $\mathbf{ab}$ of unit vectors with relative angle θ.

Rotor U_θ is depicted as a directed arc on the unit circle.

$$\mathbf{a}^2 = \mathbf{b}^2 = 1 \qquad\qquad \mathbf{ab} = U_\theta$$







Reversion: $\quad U_\theta^\dagger = \mathbf{ba}$

Defining sine and cosine functions from products of unit vectors

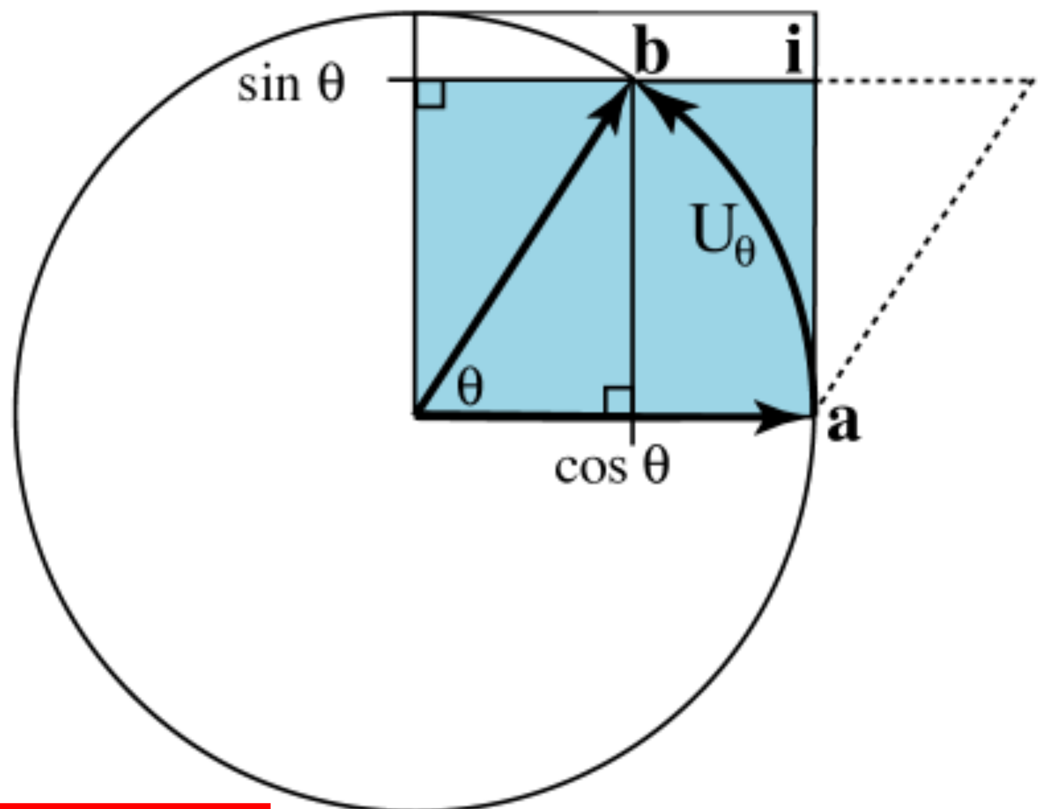
$$\mathbf{a}^2 = \mathbf{b}^2 = 1$$

$\mathbf{i}$ = unit bivector

$$\mathbf{i}^2 = -1$$

$$\mathbf{a} \cdot \mathbf{b} \equiv \cos\theta$$

$$\mathbf{a} \wedge \mathbf{b} \equiv \mathbf{i}\sin\theta$$

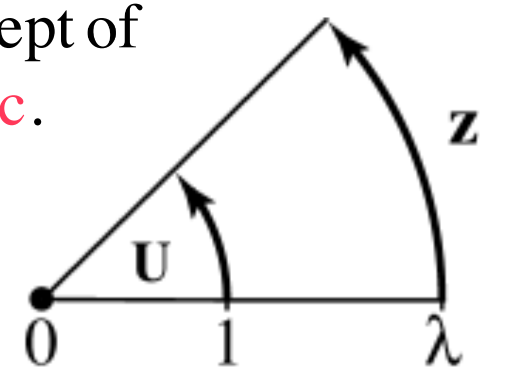


Rotor:

$$U_\theta = \mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$
$$= \cos\theta + \mathbf{i}\sin\theta = e^{\mathbf{i}\theta}$$

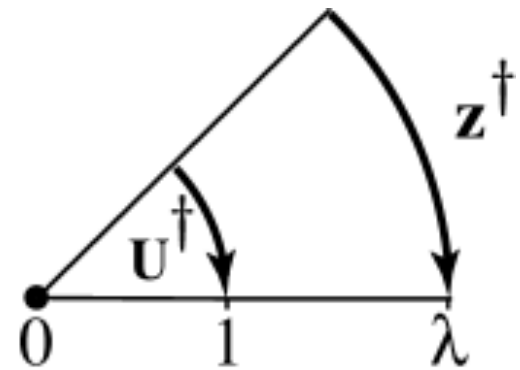
The concept of rotor generalizes to the concept of complex number interpreted as a directed arc.

$$z = \lambda U = \lambda e^{\mathbf{i}\theta} = \mathbf{ab}$$



Reversion = complex conjugation

$$z^{\dagger} = \lambda U_{\theta}^{\dagger} = \lambda e^{-\mathbf{i}\theta} = \mathbf{ba}$$



Modulus

$$zz^{\dagger} = \lambda^2 = (\mathbf{ab})(\mathbf{ba}) = \mathbf{a}^2\mathbf{b}^2 = |z|^2$$

$$|z| = \lambda = |\mathbf{a}||\mathbf{b}|$$

$$z = \operatorname{Re} z + \mathbf{i} \operatorname{Im} z = \mathbf{ab}$$

$$\operatorname{Re} z = \frac{1}{2}(z + z^{\dagger}) = \mathbf{a} \cdot \mathbf{b}$$

$$\mathbf{i} \operatorname{Im} z = \frac{1}{2}(z - z^{\dagger}) = \mathbf{a} \wedge \mathbf{b}$$

This represention of complex numbers in a real GA is a special case of spinors for 3D.

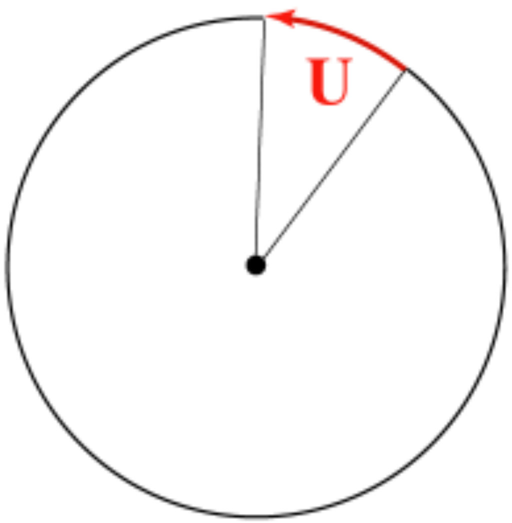
- Our development of GA to this point is sufficient to formulate and solve any problem in 2D physics without resorting to coordinates.
- Of course, like any powerful tool, it takes some skill to apply it effectively.
- For example, every physicist knows that skillful use of complex numbers avoids decomposing them into real and imaginary parts whenever possible.
- Likewise, skillful use of the geometric product avoids decomposing it into inner and outer products.
- In the next portion of this lecture I demonstrate how rotor algebra facilitates the treatment of 2D rotations and mechanics.
- In particular, note the one-to-one correspondence between algebraic operations and geometric depictions!

Properties of rotors

Rotor equivalence of directed arcs

is like

Vector equivalence of directed line segments



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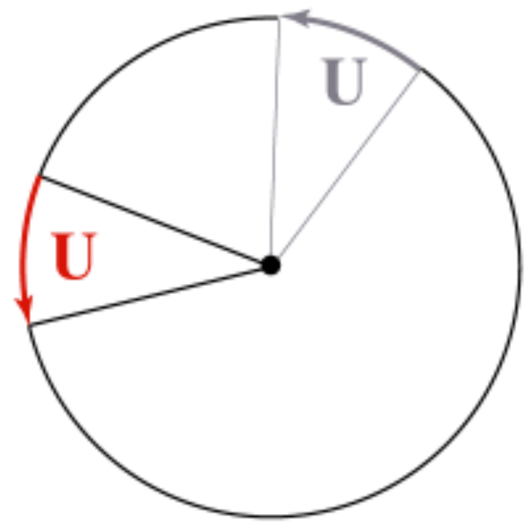

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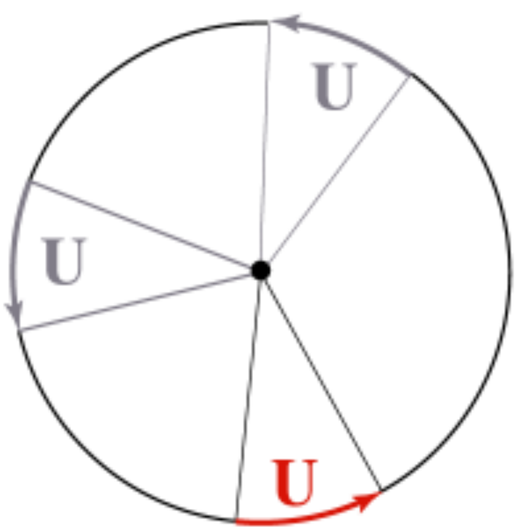
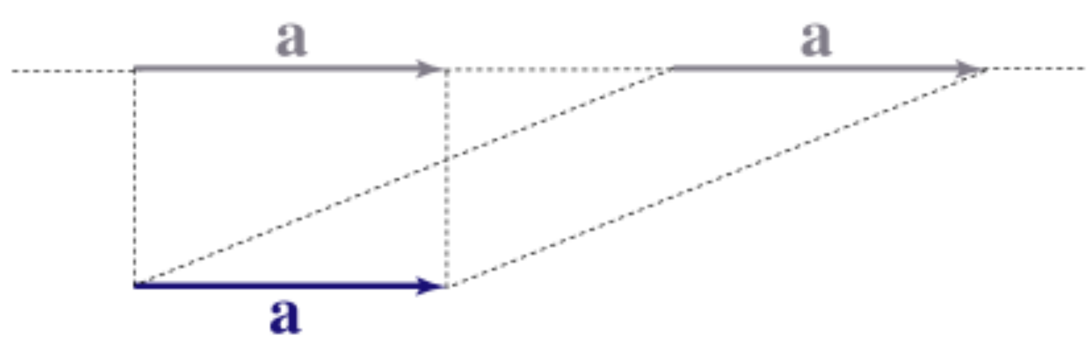
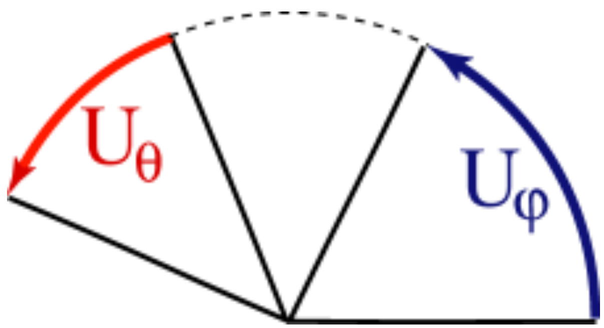
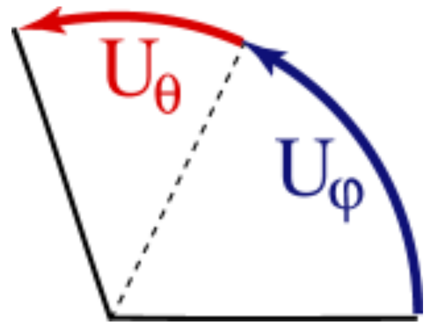


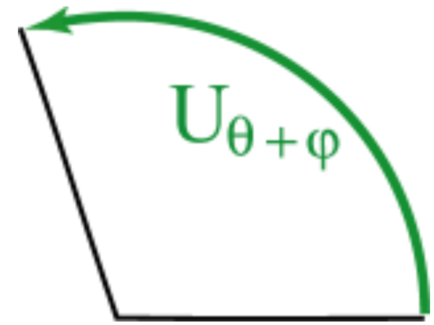


Properties of rotors

Product of rotors $\Leftrightarrow$ Addition of arcs



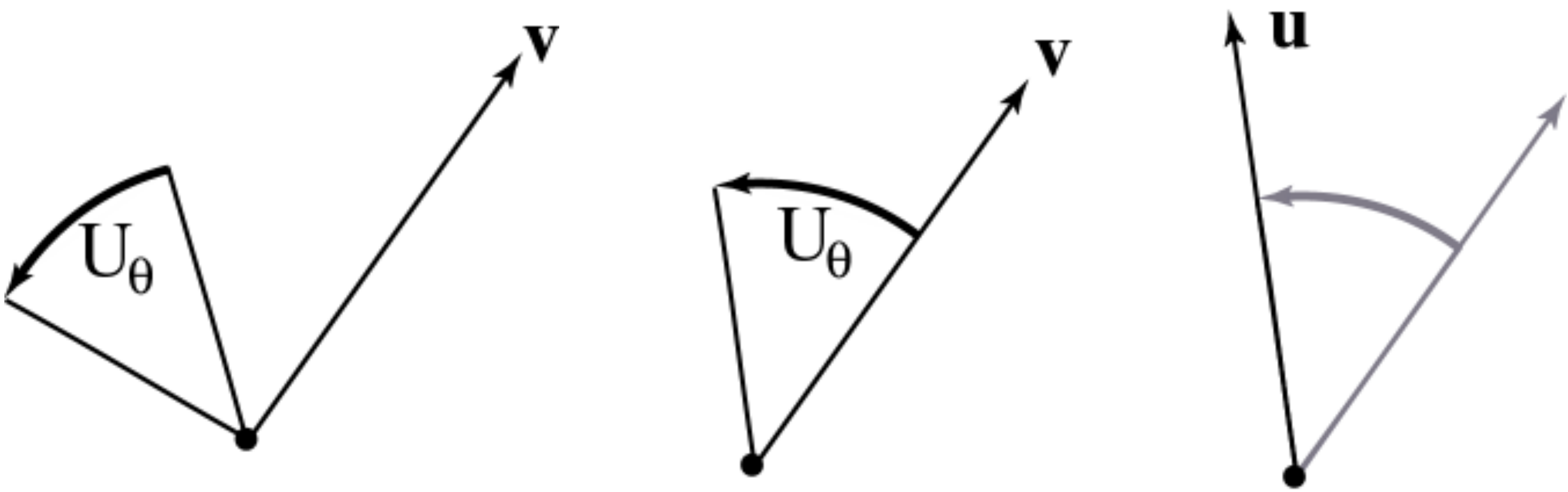




U_θ, U_φ $\qquad$ $U_\theta U_\varphi = U_{\theta+\varphi}$

Properties of rotors

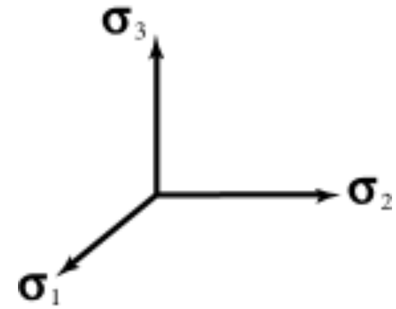
Rotor-vector product = vector



U_θ, $\mathbf{v}$ $\qquad\qquad$ $U_\theta\mathbf{v} \quad = \quad \mathbf{u}$

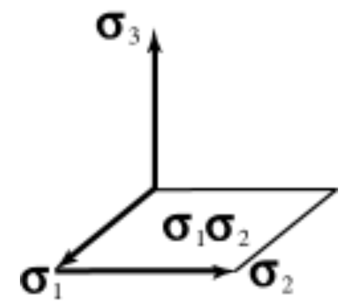
Basis for $\mathcal{R}_3 = \mathcal{G}(\mathcal{R}^3)$

Generated by orthonormal frame $\{\boldsymbol{\sigma}_k\}$



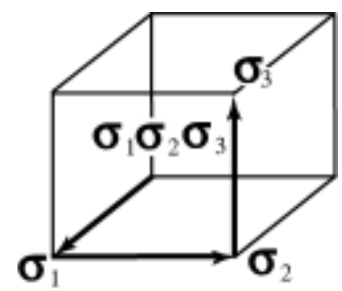
Scalars (0-vectors} $1 = \boldsymbol{\sigma}_1^{\;2} = \boldsymbol{\sigma}_2^{\;2} = \boldsymbol{\sigma}_3^{\;2}$

Vectors (1-vectors) $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$



Bivectors (2-vectors)

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = i\boldsymbol{\sigma}_3$$
$$\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = i\boldsymbol{\sigma}_1$$
$$\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 = i\boldsymbol{\sigma}_2$$



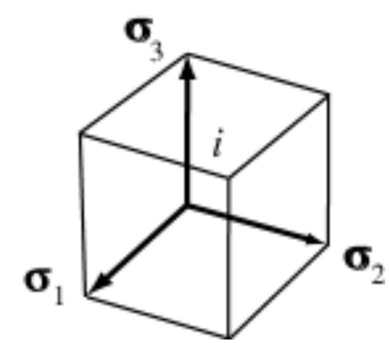
Pseudoscalar (3-vector): $i = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$

Expanded form for any *multivector M* in $\mathcal{R}_3$

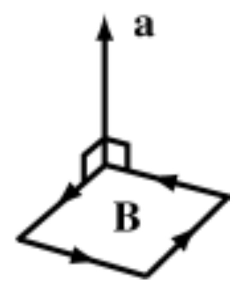
$$M = \alpha 1 + \textstyle\sum_k a_k \boldsymbol{\sigma}_k + \sum_k b_k i \boldsymbol{\sigma}_k + \beta i = \alpha + \mathbf{a} + i\mathbf{b} + i\beta$$

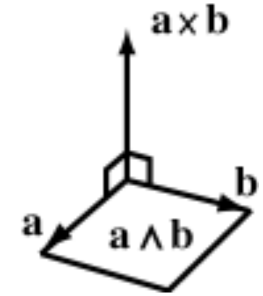
$\mathcal{R}_3$, the Geometric Algebra for Euclidean 3-space, embraces and generalizes vector algebra

The *unit right-handed pseudoscalar* in $\mathcal{R}_3$



- Special symbol $\quad i = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3$
- Basic properties $\quad i^2 = -1$
  $i\mathbf{a} = \mathbf{a}\,i$
- Duality $\quad i\mathbf{a} = \mathbf{B} =$ bivector





- The vector cross product $\mathbf{a}\times\mathbf{b}$ is defined as the *dual* of the outer product

$$i\,(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\wedge\mathbf{b}$$

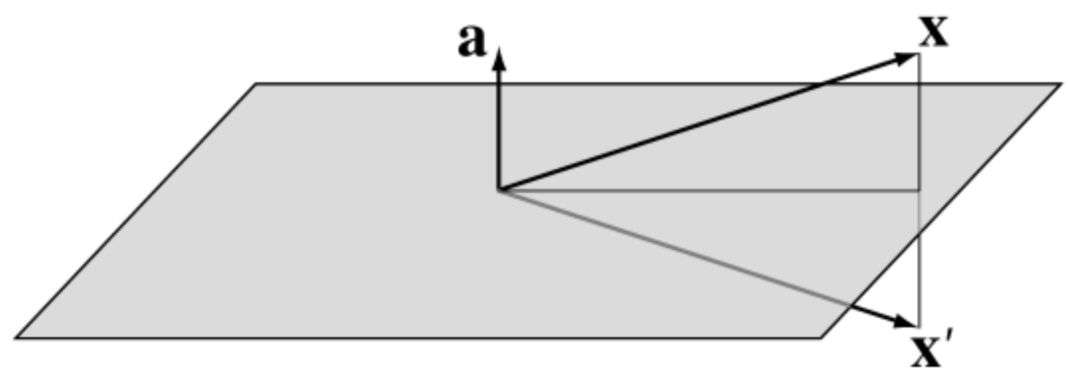
$$(\mathbf{a}\times\mathbf{b}) = -i(\mathbf{a}\wedge\mathbf{b})$$

- Relation to the geometric product

$$\boxed{\begin{aligned}\mathbf{a}\mathbf{b} &= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{b} + i(\mathbf{a}\times\mathbf{b})\end{aligned}}$$

Reflection in a plane with normal $\mathbf{a}$

Canonical form: $$\boxed{\mathbf{x}' = -\mathbf{a}\mathbf{x}\mathbf{a}^{-1}}$$ $$\mathbf{x}' = -\mathbf{a}\mathbf{x}\mathbf{a} \quad \text{if} \quad \mathbf{a}^2 = 1$$



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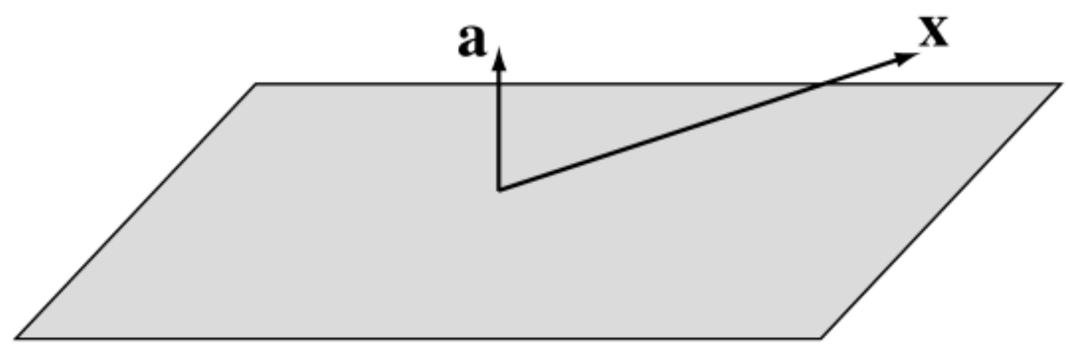

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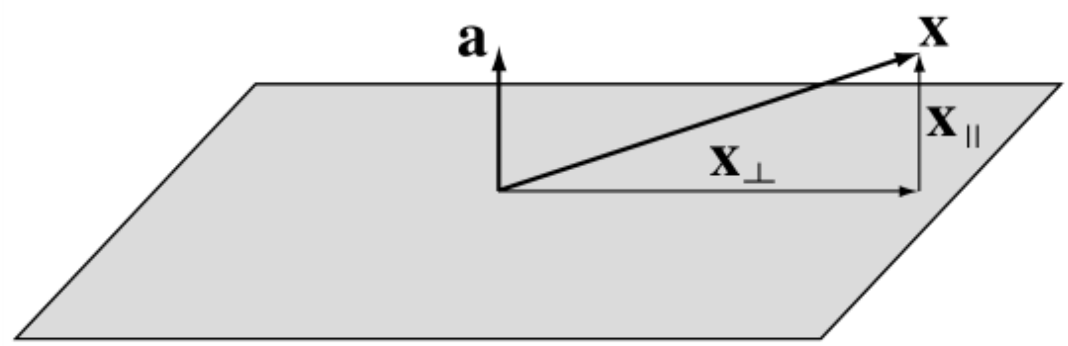
Proof:

$$
\begin{aligned}
-\mathbf{a}\mathbf{x}\mathbf{a}^{-1} &= -\mathbf{a}(\mathbf{x}_{\parallel} + \mathbf{x}_{\perp})\mathbf{a}^{-1} \\
&= -\mathbf{a}\mathbf{x}_{\parallel}\mathbf{a}^{-1} - \mathbf{a}\mathbf{x}_{\perp}\mathbf{a}^{-1}
\end{aligned}
$$



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\end{aligned}
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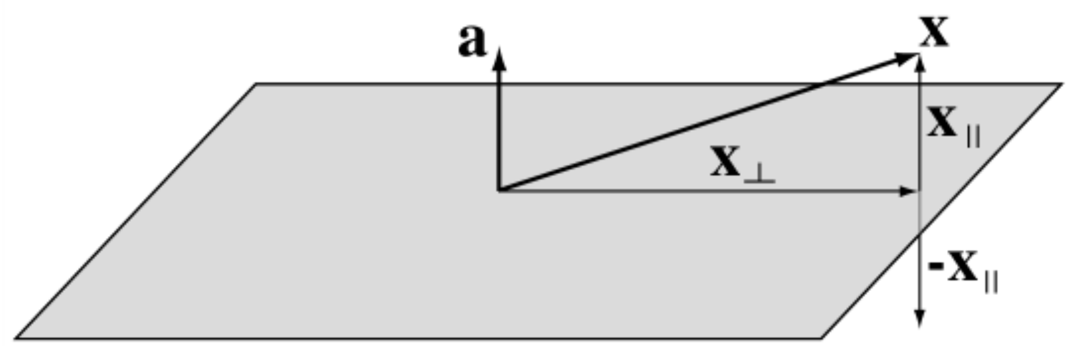

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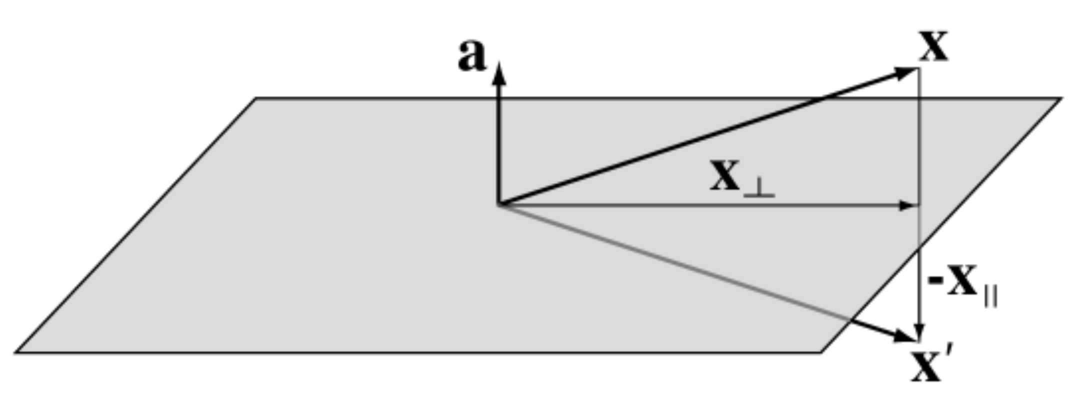

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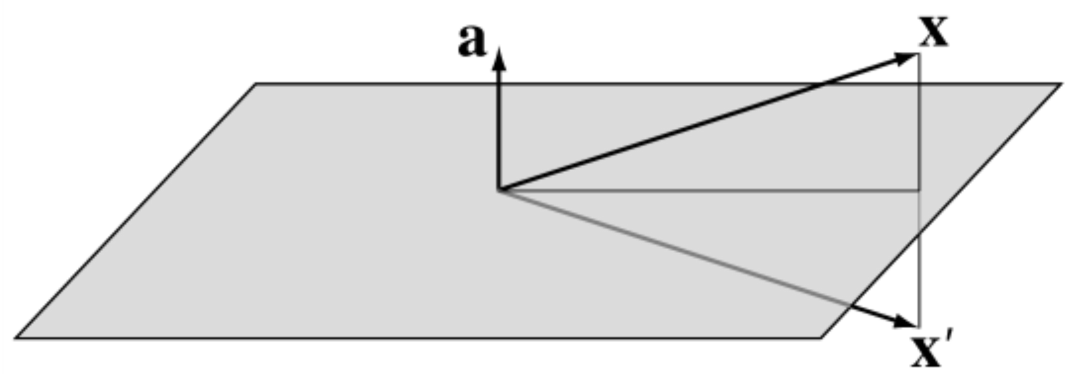
&= -\mathbf{x}_\parallel\mathbf{a}\mathbf{a}^{-1} + \mathbf{x}_\perp\mathbf{a}\mathbf{a}^{-1} \\
&= -\mathbf{x}_\parallel + \mathbf{x}_\perp \\
&= \mathbf{x}'
\end{aligned}
$$

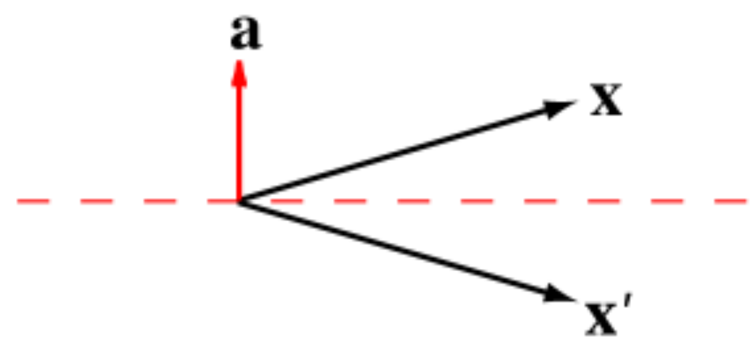


Rotation as double reflection
represented by *rotor*: $\boxed{U = \mathbf{ba} = e^{\frac{1}{2}\mathbf{i}\theta}}$

$$\mathbf{a}^2 = \mathbf{b}^2 = 1 \qquad\qquad U^\dagger = \mathbf{ab} = e^{-\frac{1}{2}\mathbf{i}\theta}$$

Proof:

$$\mathbf{x}' = -\mathbf{axa}$$



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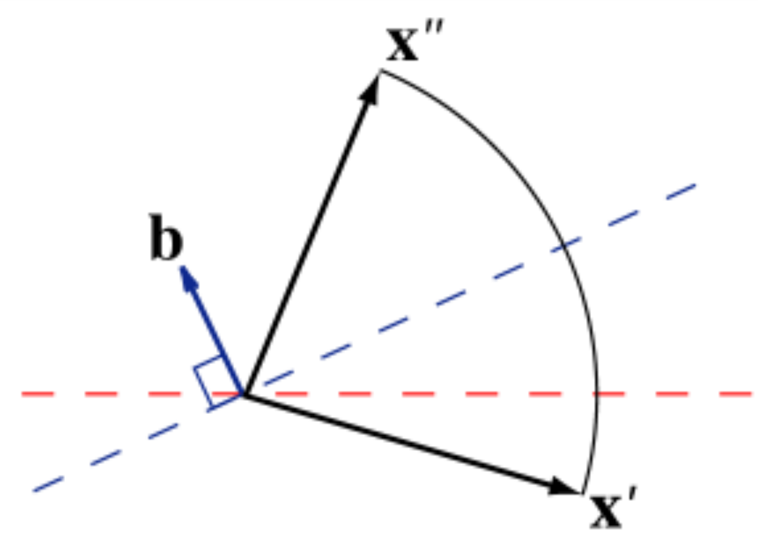

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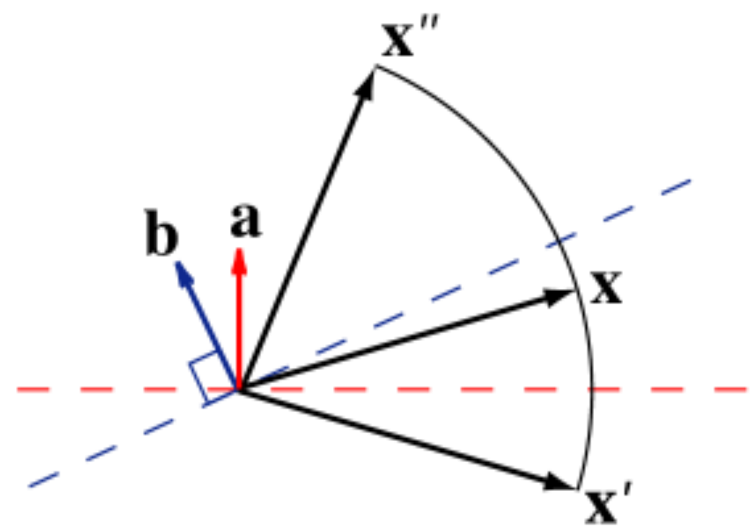
$$= -\mathbf{b}(-\mathbf{axa})\mathbf{b}$$



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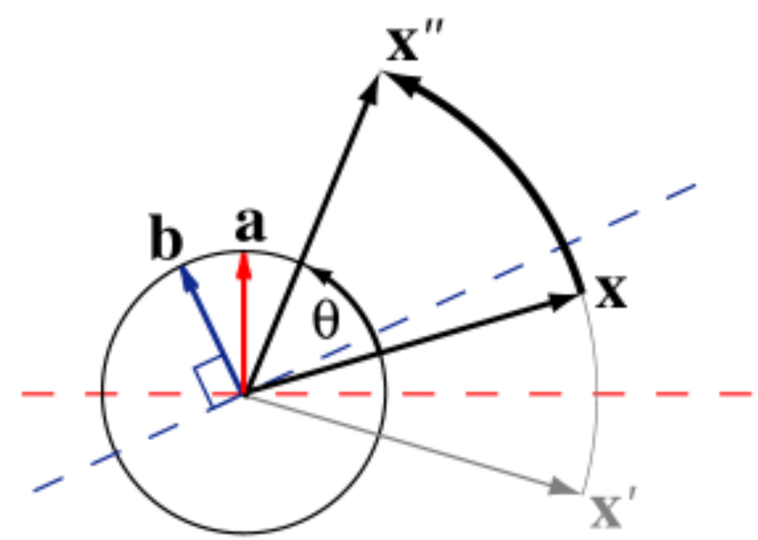
Proof:

$$\mathbf{x}' = -\mathbf{axa}$$

$$\begin{aligned}\mathbf{x}'' &= -\mathbf{bx}'\mathbf{b}\\ &= -\mathbf{b}(-\mathbf{axa})\mathbf{b}\\ &= (\mathbf{ba})\mathbf{x}(\mathbf{ab})\end{aligned}$$



Rotation as double reflection represented by *rotor*:

$$\boxed{U = \mathbf{ba} = e^{\frac{1}{2}\mathbf{i}\theta}}$$

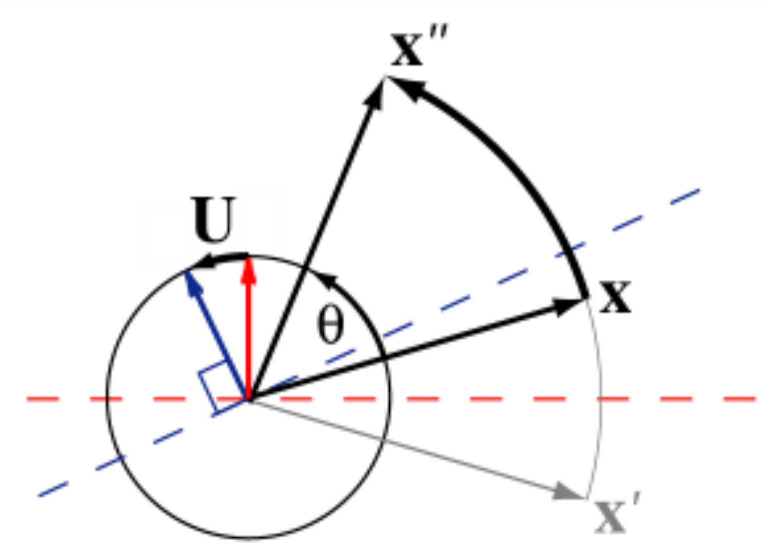
$$\mathbf{a}^2 = \mathbf{b}^2 = 1$$

$$U^\dagger = \mathbf{ab} = e^{-\frac{1}{2}\mathbf{i}\theta}$$

Proof:

$$\mathbf{x}' = -\mathbf{axa}$$

$$\begin{aligned}\mathbf{x}'' &= -\mathbf{bx}'\mathbf{b} \\ &= -\mathbf{b}(-\mathbf{axa})\mathbf{b} \\ &= (\mathbf{ba})\mathbf{x}(\mathbf{ab}) \\ &= U\mathbf{x}U^\dagger\end{aligned}$$



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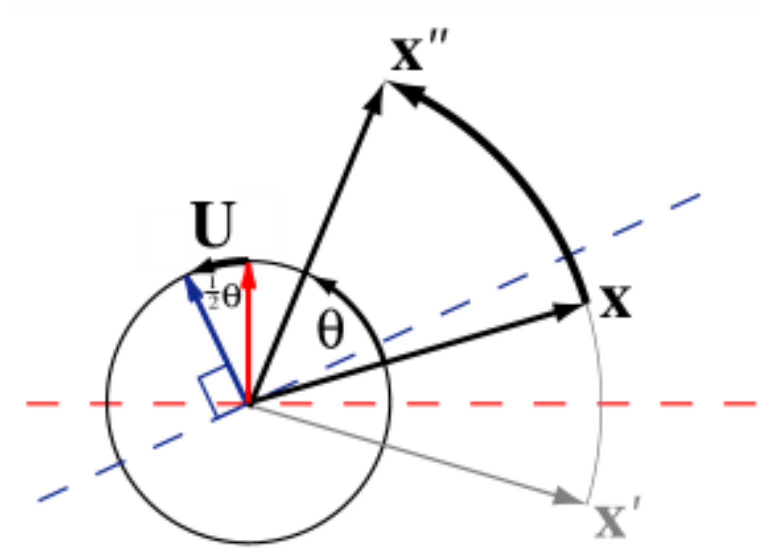


U represents rotation through twice the angle between $\mathbf{a}$ and $\mathbf{b}$.

Summary: Orthogonal transformations in Euclidean space

Orthogonal transformation $\underline{U}:\mathbf{x}\rightarrow\mathbf{x}'=\underline{U}\,\mathbf{x}$

Defining property : $\mathbf{x}'^2=\mathbf{x}^2$

Canonical form: $\boxed{\underline{U}\mathbf{x}=\varepsilon_U U\mathbf{x}U^\dagger}$

Unimodular versor: $UU^\dagger=1$

Versor parity: $\varepsilon_U=-1$ if U odd (reflection)

$\varepsilon_U=1$ if U even (rotation)

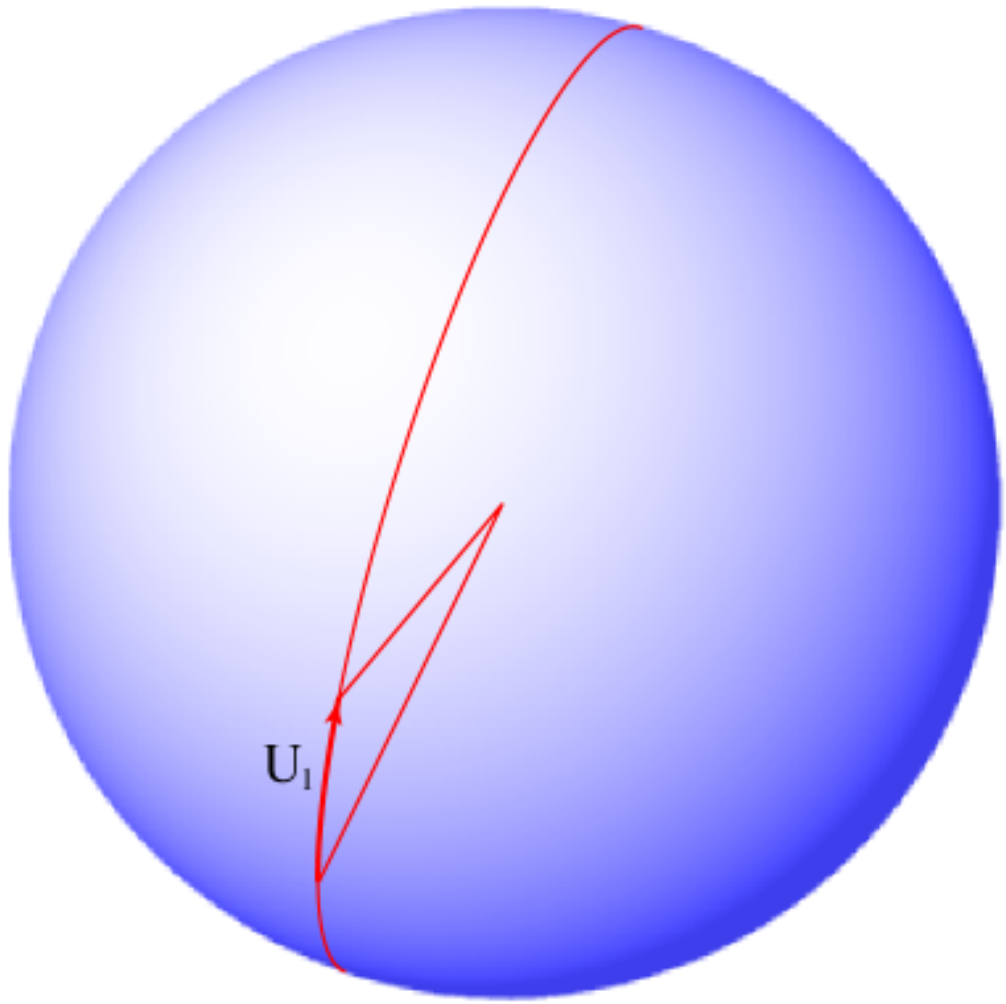
Main advantage:

Composition of transformations: $\underline{U}_2\underline{U}_1=\underline{U}_3$

Reduced to versor products: $\boxed{U_2U_1=U_3}$

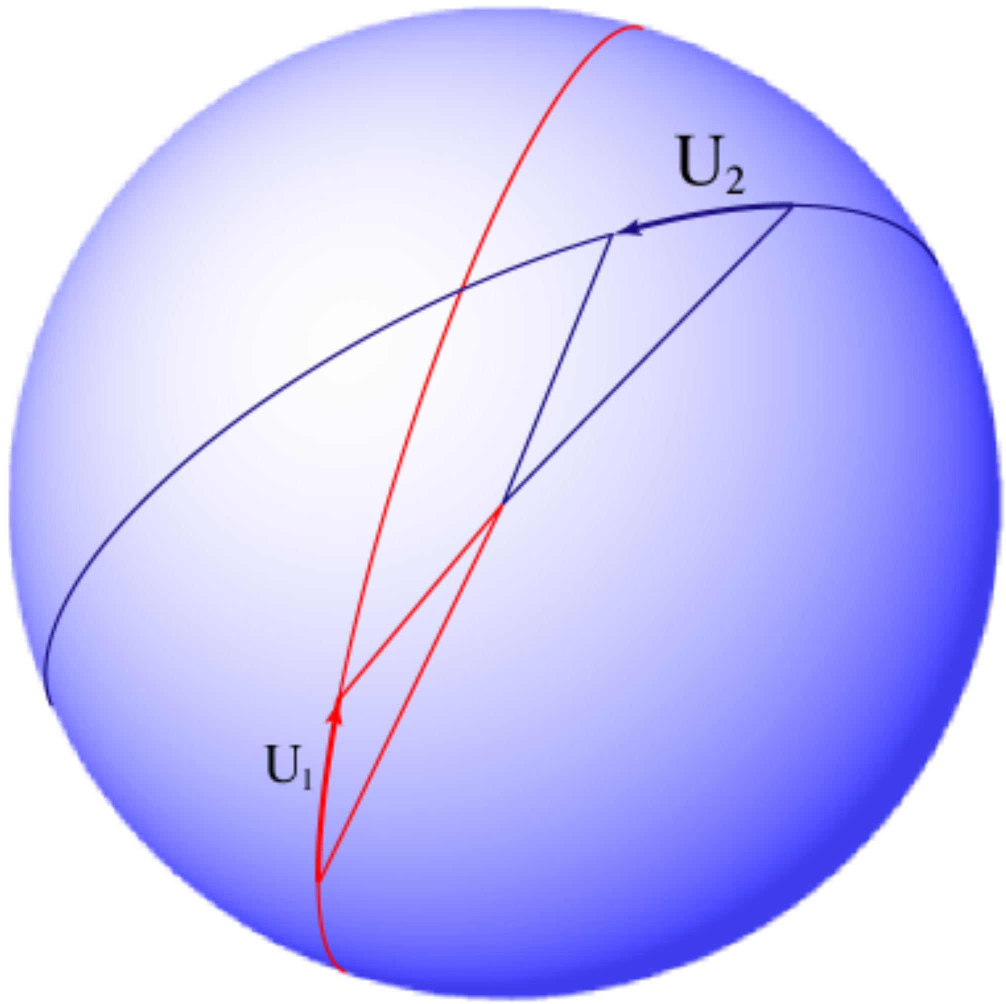
Rotor products $\Leftrightarrow$ composition of rotations in 3D

U_1



Rotor products $\Leftrightarrow$ composition of rotations in 3D

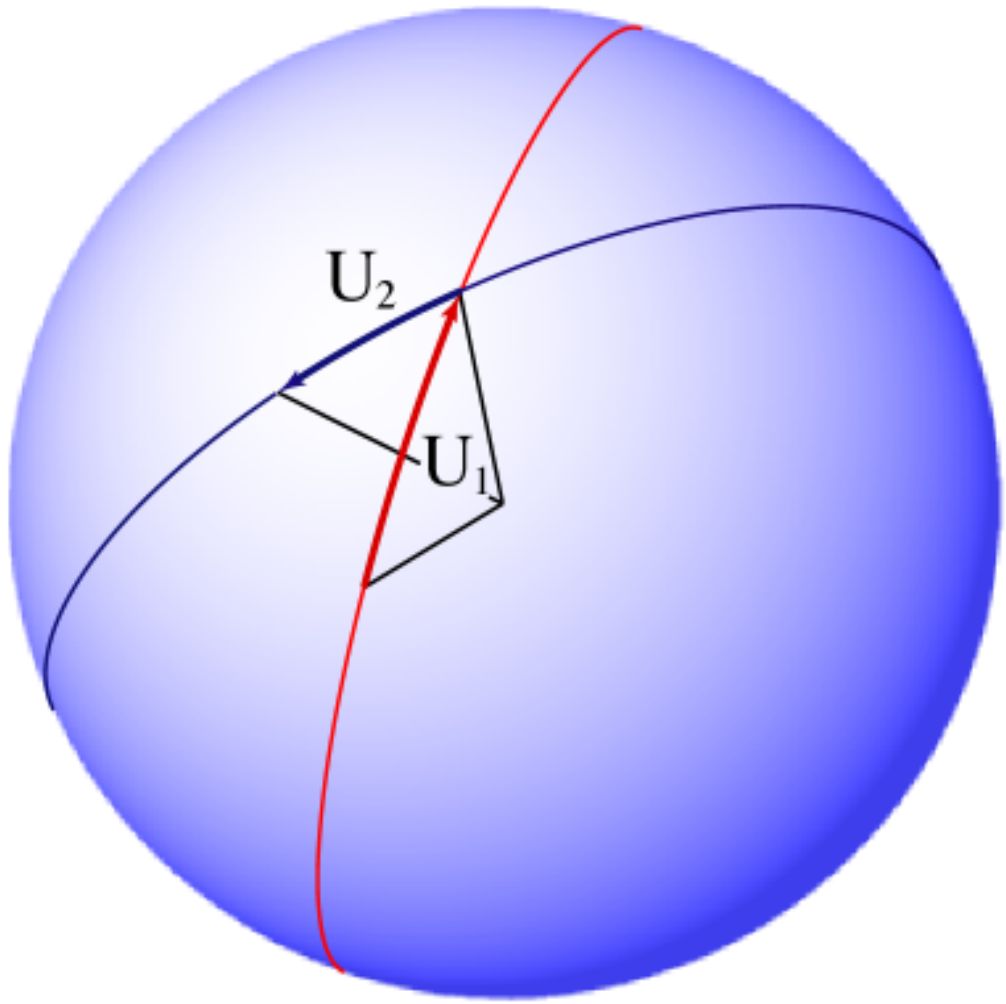
U_1 , U_2



Rotor products $\Leftrightarrow$ composition of rotations in 3D

U_1 , U_2

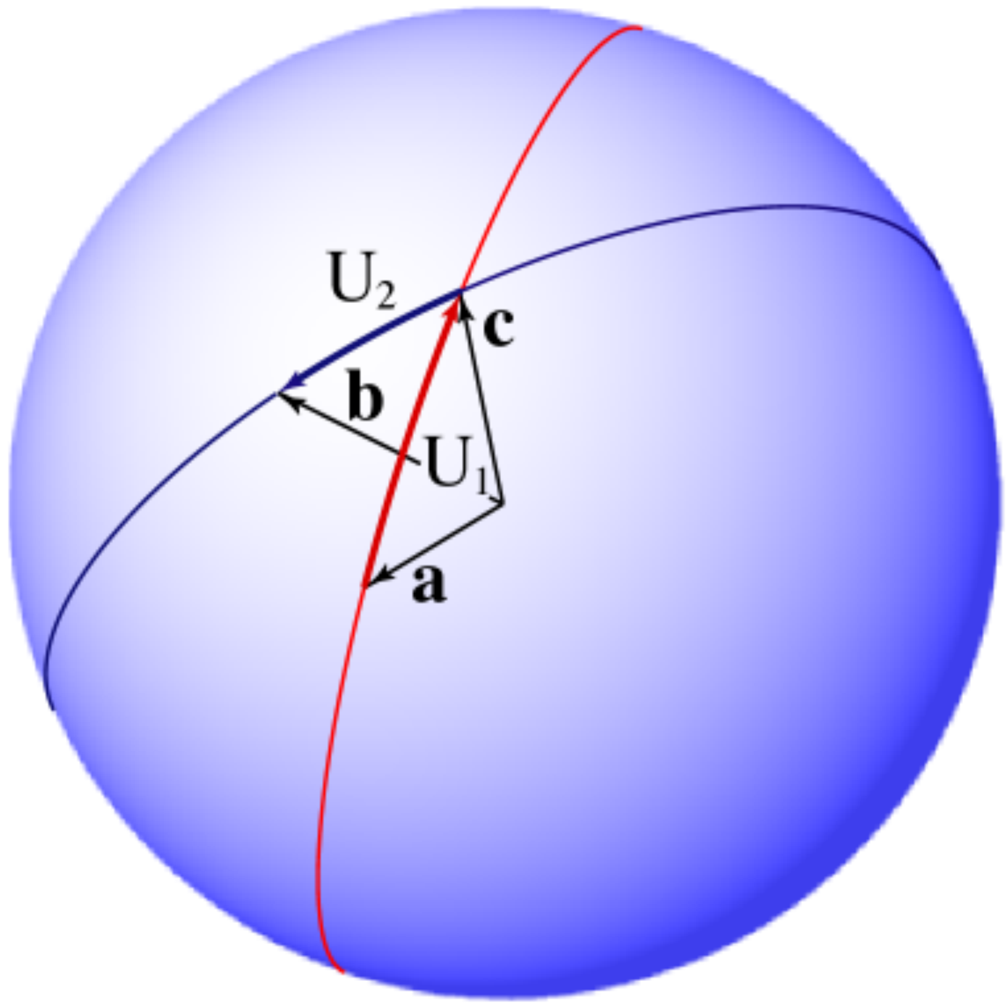
U_2U_1



Rotor products $\Leftrightarrow$ composition of rotations in 3D

U_1 , U_2

$U_2U_1 = (\mathbf{bc})(\mathbf{ca})$



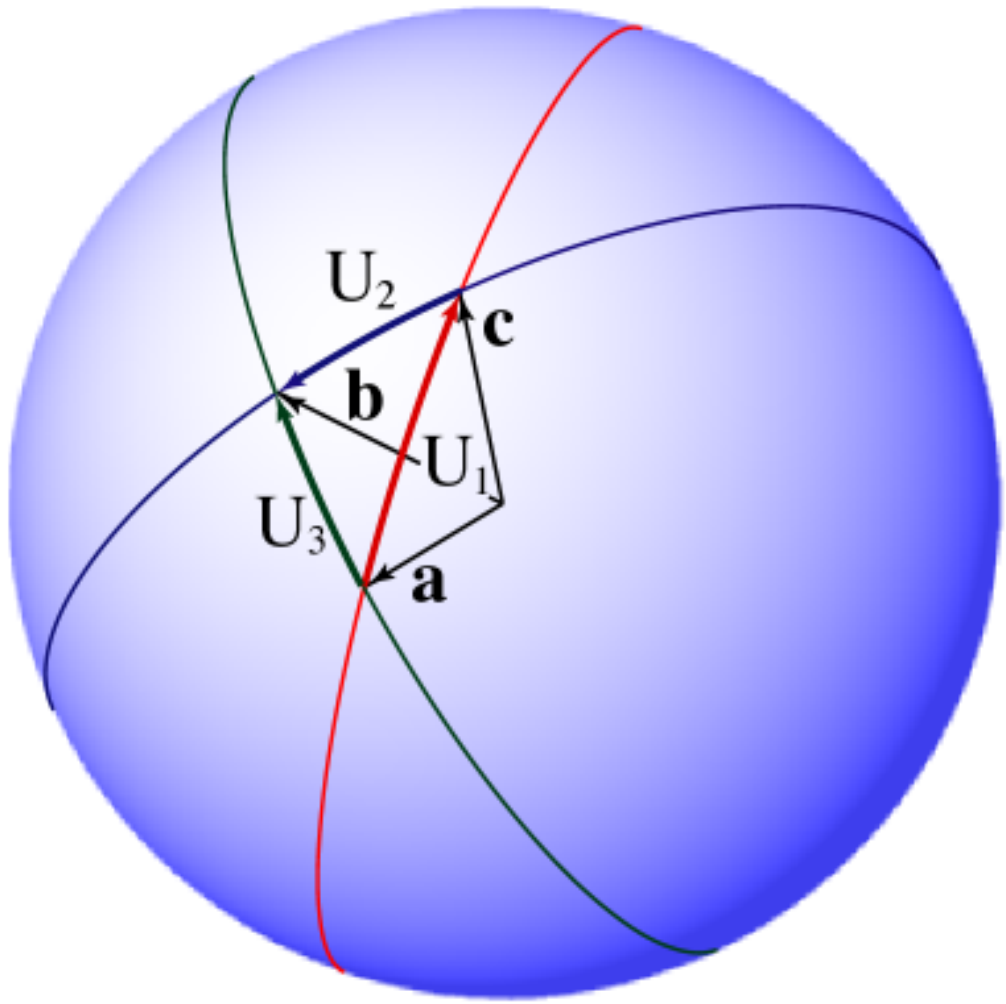
Rotor products ⇔ composition of rotations in 3D

U_1 , U_2

$U_2U_1 = (\mathbf{bc})(\mathbf{ca})$

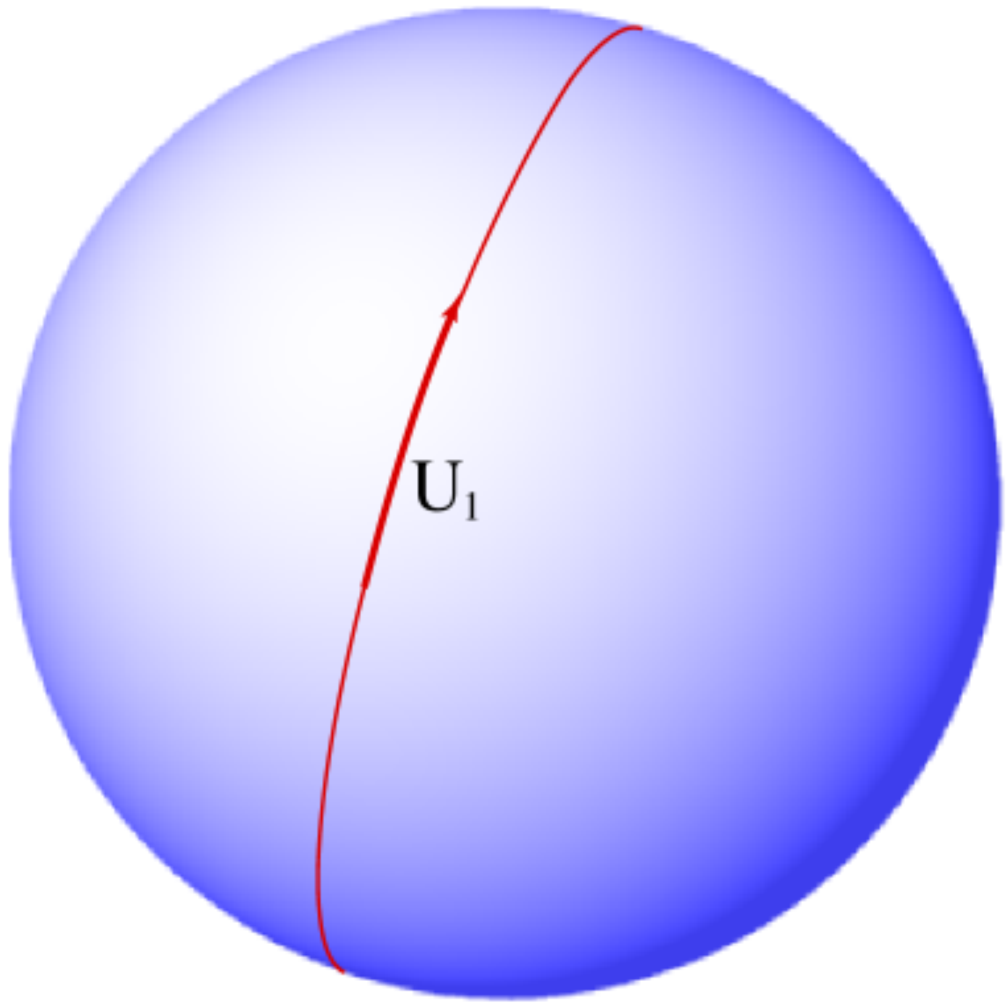
$= \mathbf{ba} = U_3$

$$\boxed{U_2 U_1 = U_3}$$



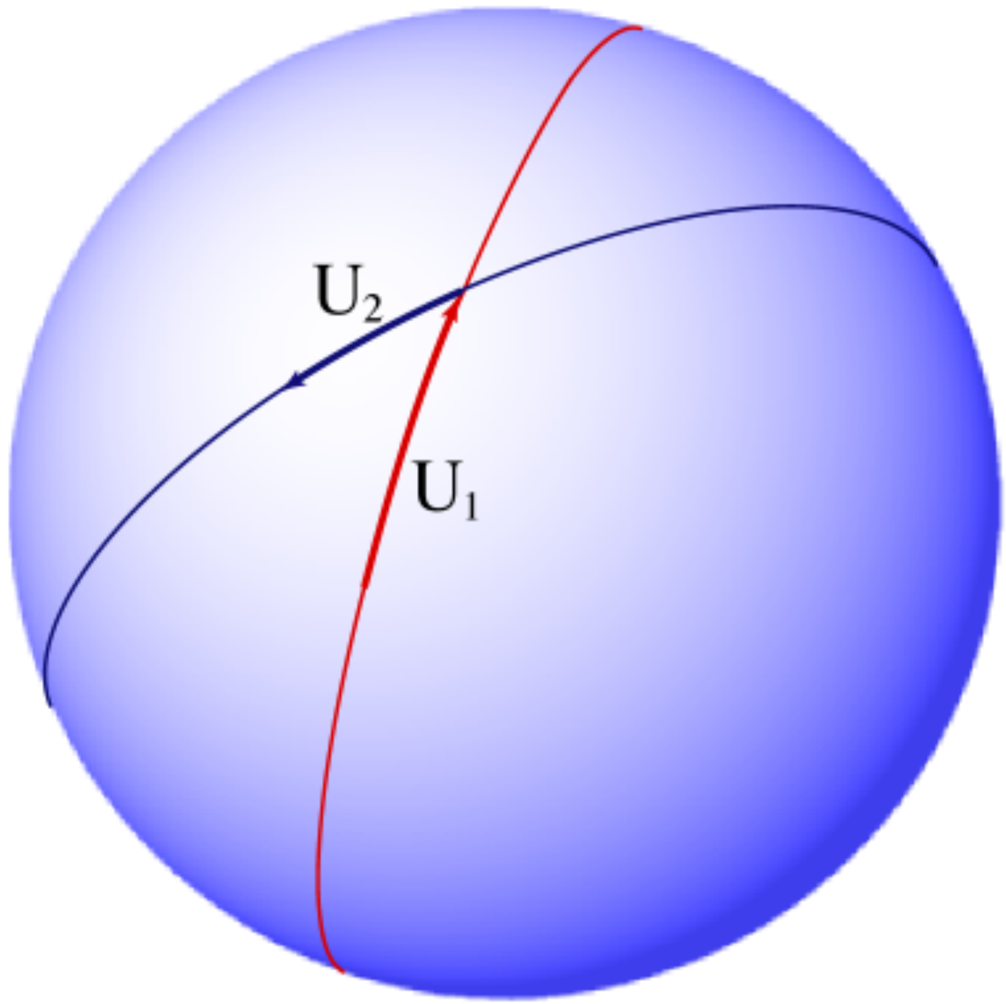
Noncommutativity of Rotations

U_1



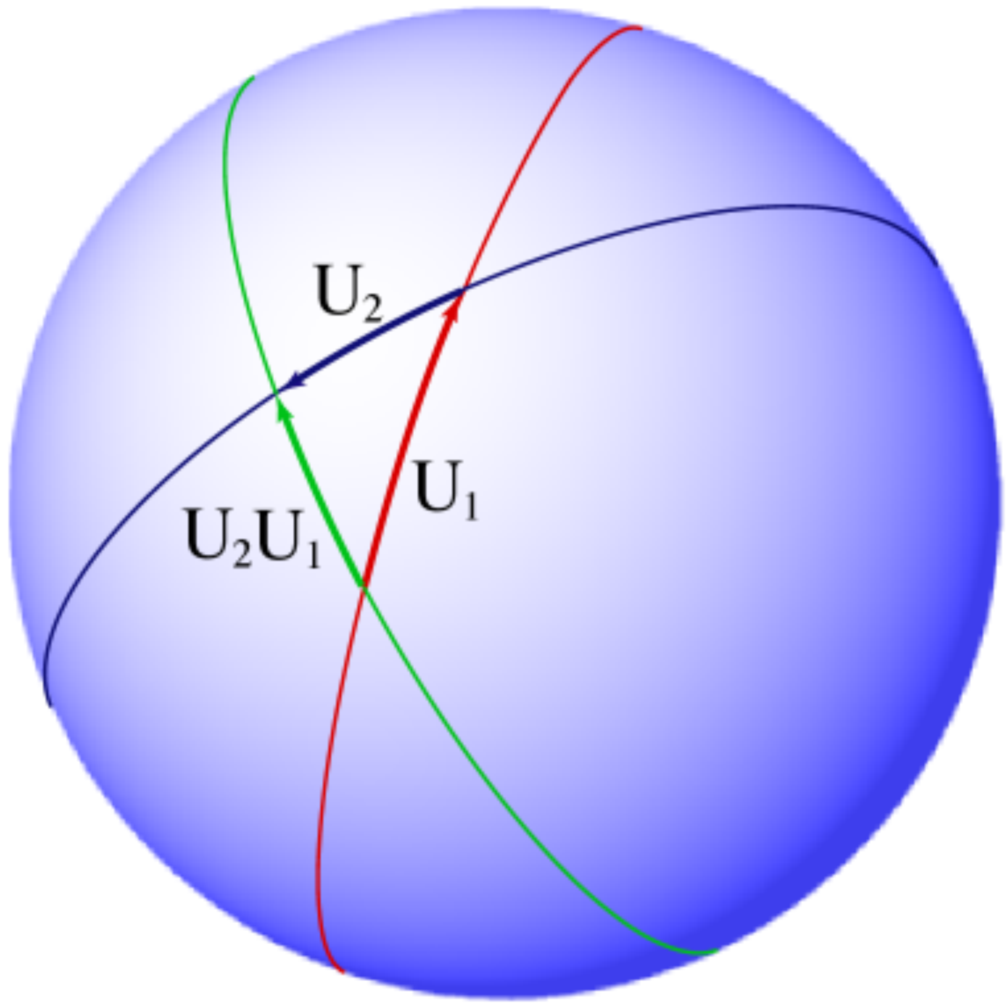
Noncommutativity of Rotations

$U_2\ (U_1)$



Noncommutativity of Rotations

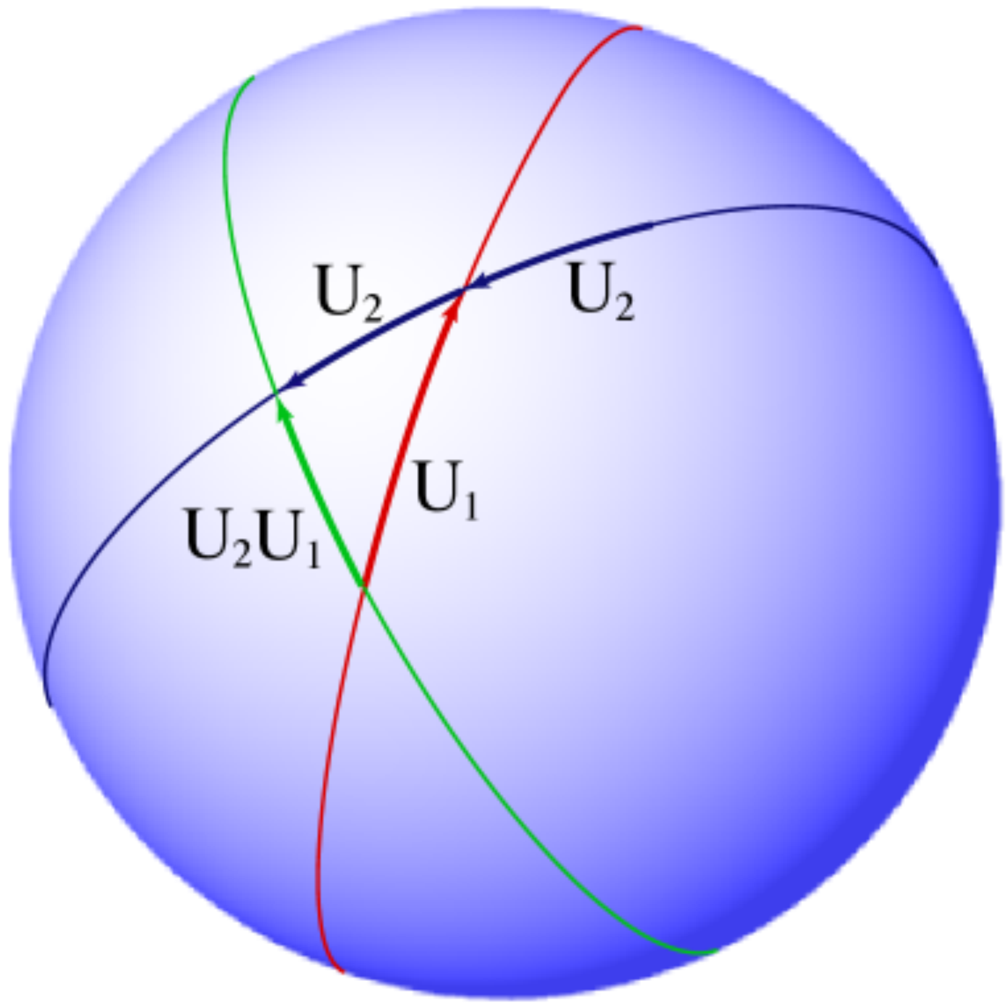
$U_2\ (U_1)\ =\ U_2U_1$



Noncommutativity of Rotations

$U_2\,(U_1) \;=\; U_2U_1$

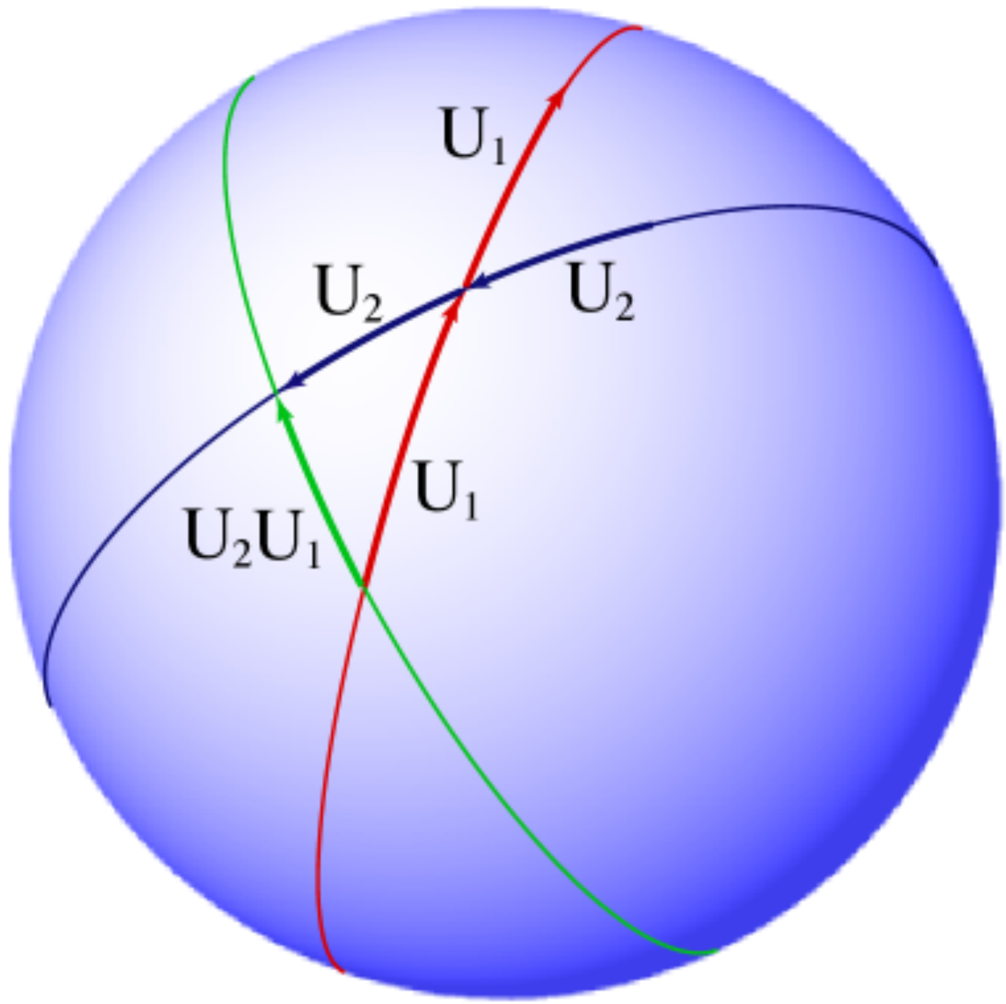
U_2



Noncommutativity of Rotations

$U_2\,(U_1) \;=\; U_2U_1$

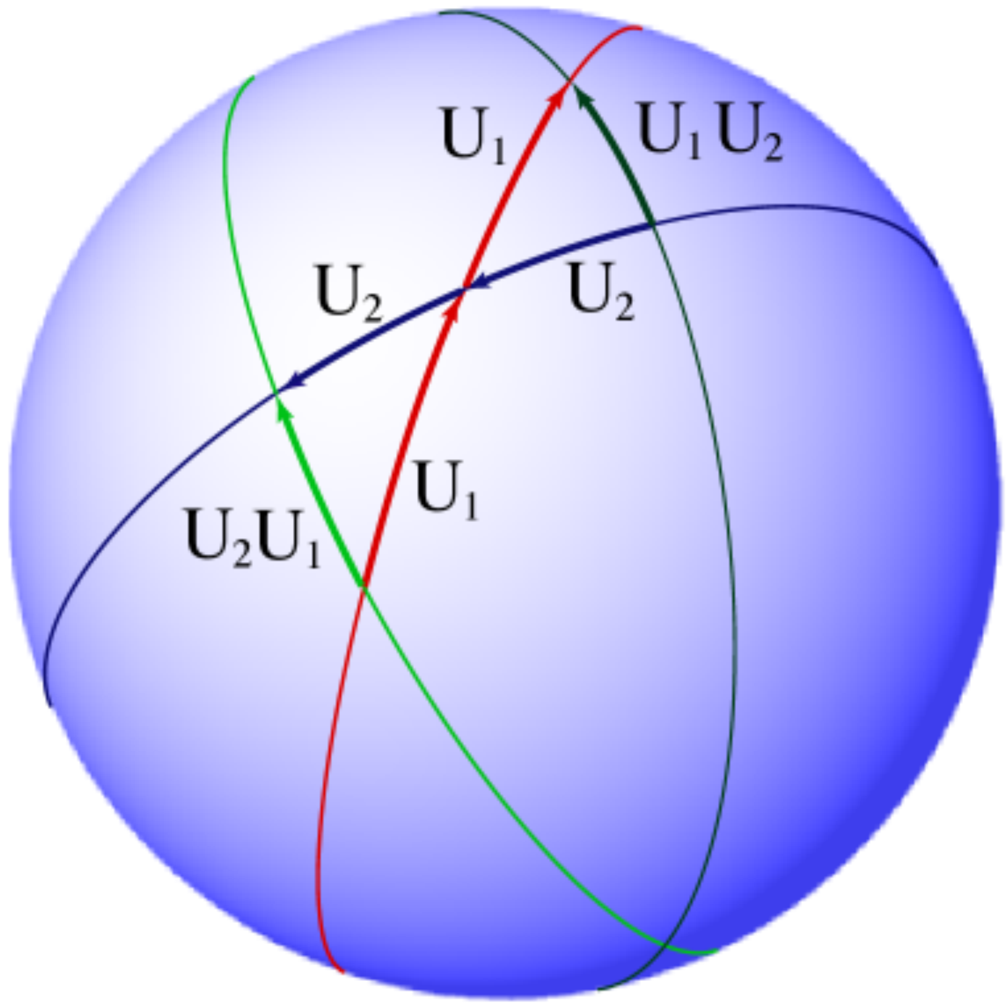
$U_1\,(U_2)$



Noncommutativity of Rotations

$U_2\,(U_1) \;=\; U_2 U_1$

$U_1\,(U_2) \;=\; U_1 U_2$



What have we learned so far?

- Rules for multiplying vectors that apply to vector spaces of any dimension.
- Geometric meaning of the geometric product and its component parts in

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$$

- Integration of complex numbers with vectors, and interpretation as directed arcs.
- How rotor algebra clarifies and facilitates the treatment of rotations in 2D and 3D.

Point symmetry groups of molecules & crystals

- Increasing in importance as we enter the age of nanoscience and molecular biology

- GA makes point groups accessible to students early in the curriculum at no academic cost

- Each finite symmetry group is generated multiplicatively by 3 vectors in GA

Symmetries of the Cube

Generators: **a, b, c**

$$\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = 1$$

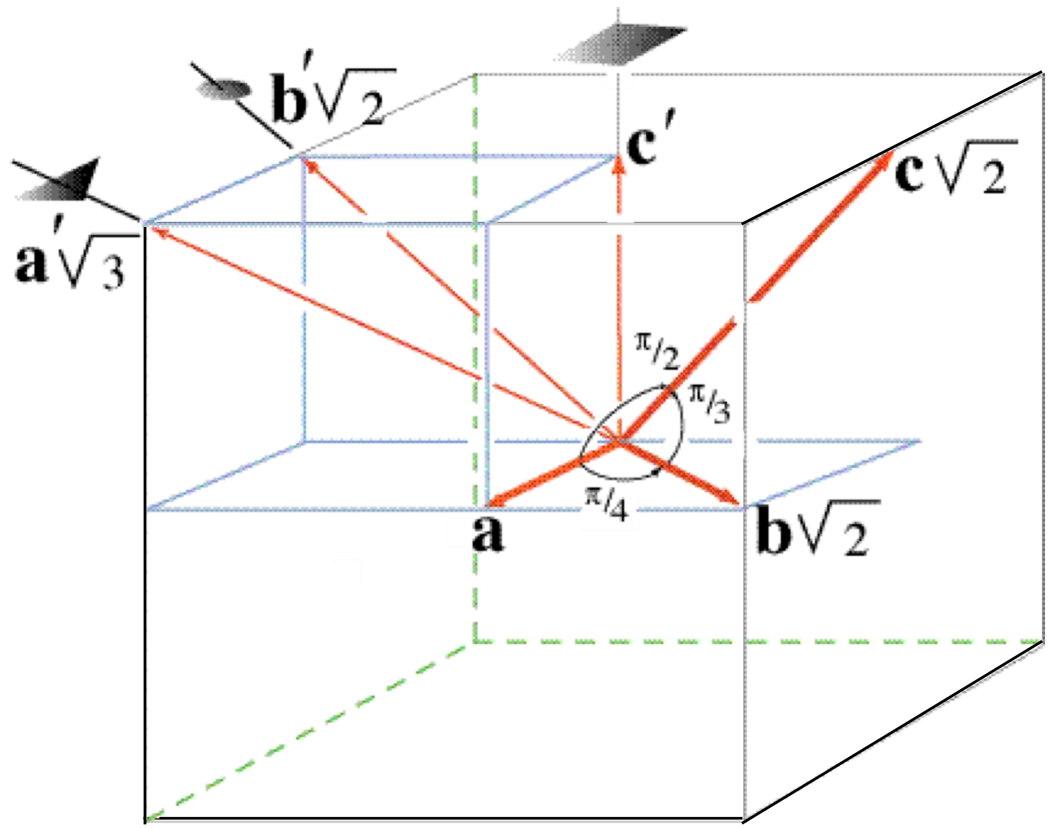
Relations:

$$(\mathbf{ca})^2 = -1$$

$$(\mathbf{bc})^3 = -1$$

$$(\mathbf{ab})^4 = -1$$

Symbol: $\{4, 3, 2\}$

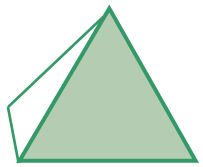
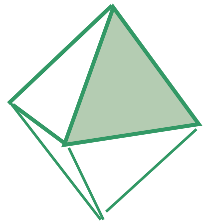


32 Lattice Point Groups

Generators: **a, b, c** $\qquad \mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = 1$

$\boxed{\text{Relations:} \quad (\mathbf{ab})^p = (\mathbf{bc})^q = (\mathbf{ca})^r = -1}$ $\qquad$ Roots of -1

Crystallographic restriction: $\quad r = 2, \quad q \le p = 1,2,3,4,6$

Groups	$\{p, q, r\}$	
Tetrahedral group	$\{3,3,3\}$	
Octahedral group	$\{4,3,2\}$	
Icosahedral group	$\{5,3,2\}$	

230 distinct 3D Space Groups

- Generated by reflections in 5D Minkowski space
- wherein Euclidean points are represented by null vectors
- the optimal representation for 3D Euclidean space

D. Hestenes & J. Holt, The Crystallographic Space Groups in Geometric Algebra, *Journal of Mathematical Physics . 48*, 023514 (2007)

Echard Hitzer & Christian Perwass

http://www.spacegroup.info

- Interactive Visualization of the 32 3D Point Groups
- Interactive Visualization of the 17 2D Space Groups
- Space Group Visualizer for the 230 3D Space Groups
- Contacts with International Union of Crystallography (IUCr)

Towards Official Adoption of The Space Group Visualizer Software

- Great potential for molecular modeling and diffraction theory!

Summary for rotations in 2D, 3D and beyond

Thm. I: Every rotation can be expressed in the canonical form:

$$\mathbf{x} \to \quad \boxed{\mathbf{x}' = U\mathbf{x}U^{\dagger}} \qquad \text{where } UU^{\dagger} = 1 \text{ and } U \text{ is even}$$

Note: $(\mathbf{x}')^2 = U\mathbf{x}U^{\dagger}U\mathbf{x}U^{\dagger} = U\mathbf{x}^2U^{\dagger} = UU^{\dagger}\mathbf{x}^2 = \mathbf{x}^2$

Thm. II: Every rotation in 3D can be expressed as product of two reflections:

$$\left.\begin{aligned} U &= \mathbf{ba} \\ U^{\dagger} &= \mathbf{ab} \end{aligned}\right\} \qquad UU^{\dagger} = \mathbf{baab} = \mathbf{a}^2 = 1$$

Generalizations:

III. Thm I applies to **Lorentz transformations of spacetime**

IV. Cartan-Dieudonné Thm (Lipschitz, 1880): Every orthogonal transformation can be represented in the form: $\boxed{U = \mathbf{a}_n \ldots \mathbf{a}_2\mathbf{a}_1}$

Advantages over matrix form for rotations:

— coordinate-free

— composition of rotations: $\boxed{U_2U_1 = U_3}$

— parametrizations (see NFCM)

Rotor vs. matrix representations for rotations

Rotation $\underline{U}: \boldsymbol{\sigma}_k \rightarrow \mathbf{e}_k = \underline{U}(\boldsymbol{\sigma}_k)$

Matrix representation: $\mathbf{e}_k = \alpha_{kj}\boldsymbol{\sigma}_j \qquad \alpha_{kj} = \boldsymbol{\sigma}_j \cdot \underline{U}(\boldsymbol{\sigma}_k)$

Rotor representation: $\mathbf{e}_k = U\boldsymbol{\sigma}_k U^\dagger$

Matrix from rotor: $\alpha_{kj} = \mathbf{e}_k \cdot \boldsymbol{\sigma}_j = \left\langle U\boldsymbol{\sigma}_k U^\dagger \boldsymbol{\sigma}_j \right\rangle$

Rotor from matrix (NFCM, p. 286)

Result: Form $\psi = 1 + \mathbf{e}_k\boldsymbol{\sigma}_k = 1 + \alpha_{kj}\boldsymbol{\sigma}_k\boldsymbol{\sigma}_j$

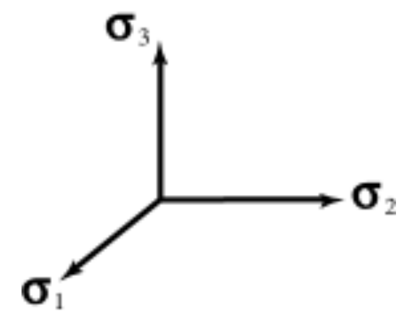
Normalize to: $$\boxed{U = \frac{\psi}{\left(\psi\psi^\dagger\right)^{\frac{1}{2}}}}$$

Establishes $\alpha_{kj} \quad \leftrightarrow \quad U$

But it is invariably simpler to use rotors
without reference to matrices!

Rotational Kinematics

Time dependent rotor $U = U(t)$



$\Rightarrow$ Rotating frame: $\boxed{\mathbf{e}_k(t) = U\boldsymbol{\sigma}_k U^\dagger}$

Rotor eqn. of motion: $\boxed{\dfrac{dU}{dt} = \dfrac{1}{2}\boldsymbol{\Omega} U}$

$\boldsymbol{\Omega} = \boldsymbol{\Omega}(t) = -i\boldsymbol{\omega}$ = rotational velocity (bivector) from dynamics

$\Rightarrow$ Frame eqn. of motion: $\dfrac{d\mathbf{e}_k}{dt} = \boldsymbol{\Omega}\cdot\mathbf{e}_k = \boldsymbol{\omega}\times\mathbf{e}_k$

- Rotor eqn. is easier to solve than vector or 3×3 matrix eqns.
- Quaternions used in aerospace industry
- Rigid body solutions in NFCM, Chap.13

Proofs: $UU^\dagger = 1 \quad \Rightarrow \quad \boldsymbol{\Omega} = \text{bivector}$

$$\frac{dU^\dagger}{dt} = -\frac{1}{2}U^\dagger\boldsymbol{\Omega} \quad \Rightarrow \quad \frac{d\mathbf{e}_k}{dt} = \frac{dU}{dt}\boldsymbol{\sigma}_k U^\dagger + U\boldsymbol{\sigma}_k\frac{dU^\dagger}{dt}$$

Classical model of spin:

$$\frac{d\mathbf{s}}{dt} = \boldsymbol{\mu} \times \mathbf{B} = (-\gamma \mathbf{B}) \times \mathbf{s} \qquad (\gamma = \text{gyromagnetic ratio})$$

$$\hat{\mathbf{s}} = U \boldsymbol{\sigma}_3 U^\dagger \qquad \Longrightarrow \qquad \boxed{\frac{dU}{dt} = \tfrac{1}{2} i\gamma \mathbf{B} U} \qquad \boldsymbol{\Omega} = i\gamma \mathbf{B}$$

$$\omega = |\omega|$$

Magnetic resonance: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}_0 e^{i\boldsymbol{\omega} t}$

$$\boldsymbol{\omega} \wedge \mathbf{B}_0 = 0$$

$$\boldsymbol{\omega} \cdot \mathbf{b}_0 = 0 = \mathbf{B}_0 \cdot \mathbf{b}_0$$

Solution: $U = e^{\frac{1}{2} i\gamma \mathbf{B}' t} e^{-\frac{1}{2} i\boldsymbol{\omega} t}$ $\qquad \mathbf{B}' = \mathbf{B}_o + \frac{1}{\gamma}\boldsymbol{\omega} + \mathbf{b}_0$

Resonance at $\boldsymbol{\omega} = -\gamma \mathbf{B}_0$ $\qquad U = e^{\frac{1}{2} i\gamma \mathbf{b}_0 t} e^{-\frac{1}{2} i\gamma \mathbf{B}_0 t}$

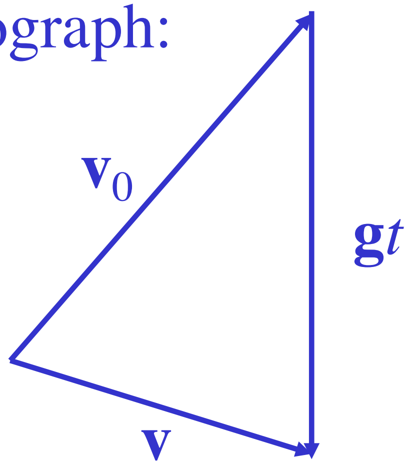
- Produces spiraling spin reversal in time: $T = \dfrac{2\pi}{b_0 \gamma}$
- Can be tuned to γ for different materials

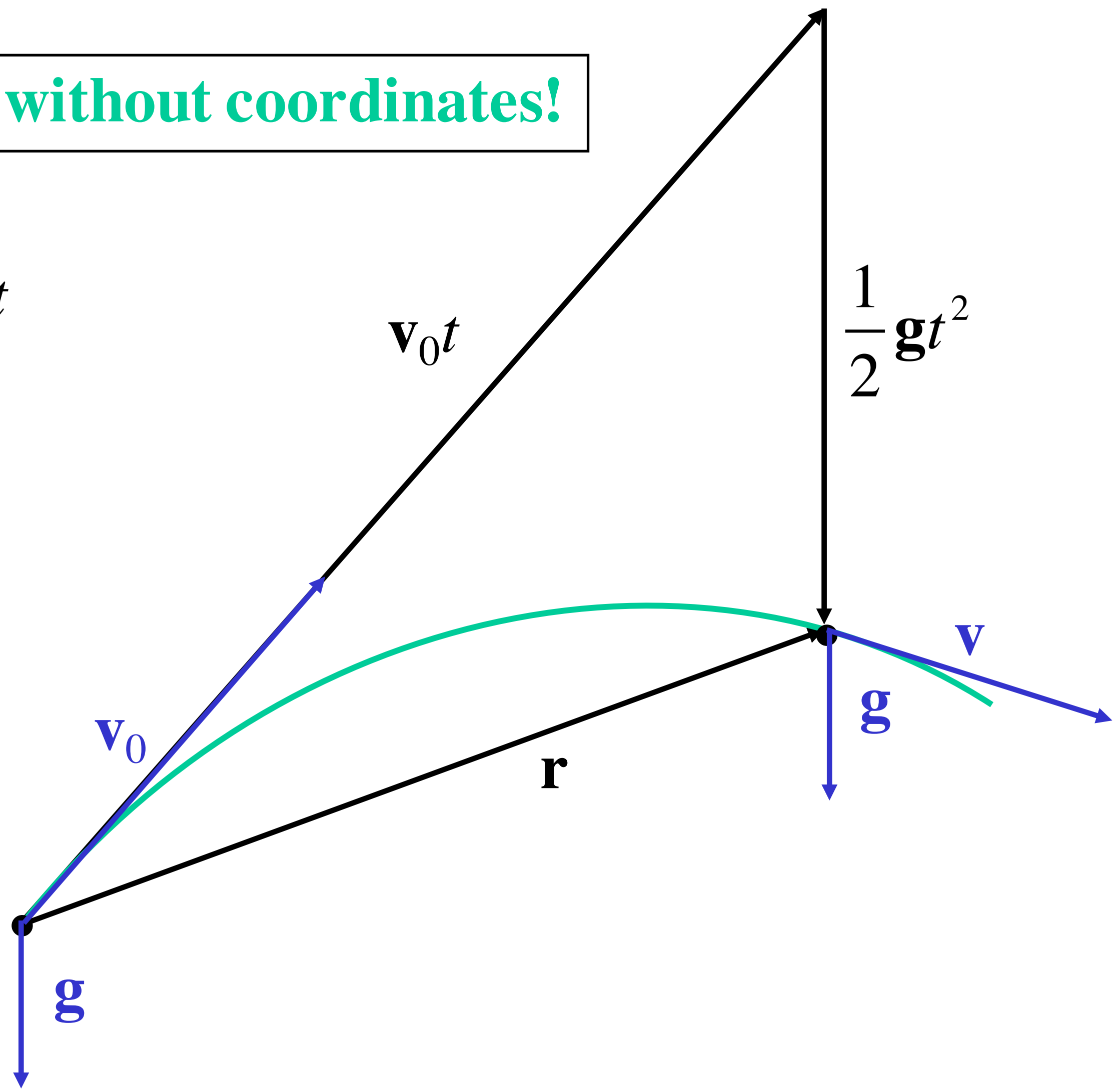
[Ref. NFCM. p. 473]

Constant Acceleration without coordinates!

$$\frac{d\mathbf{v}}{dt} = \mathbf{g} \quad \Rightarrow \quad \mathbf{v} = \mathbf{v}_0 + \mathbf{g}t$$

hodograph:



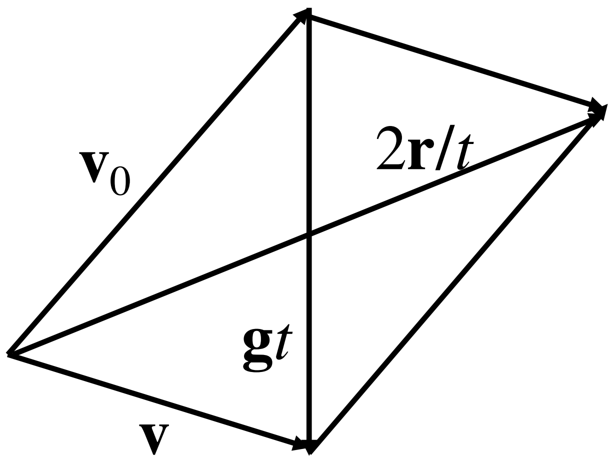


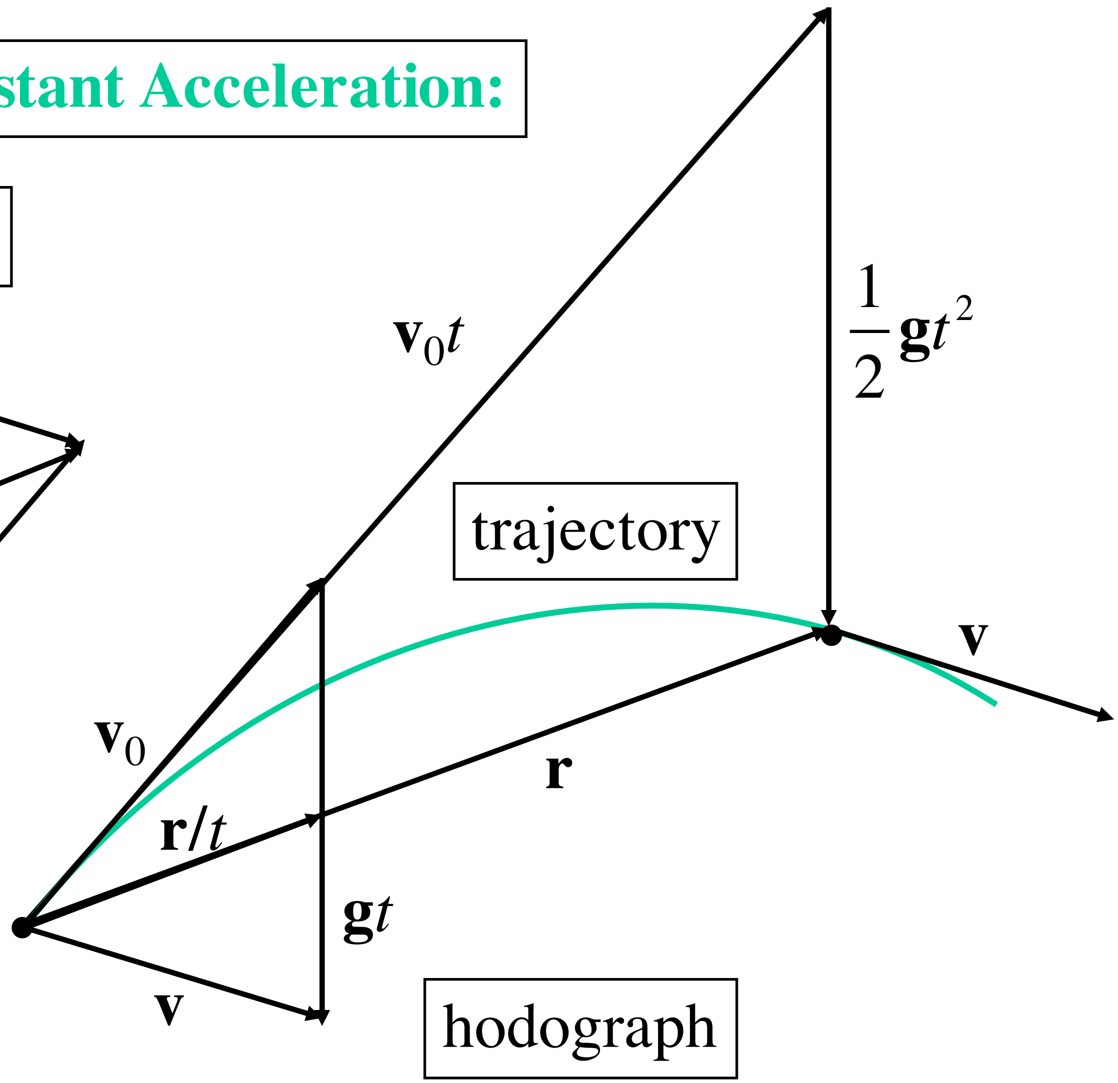
trajectory:

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad \Rightarrow \quad \mathbf{r} = \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2$$

Constant Acceleration:

Vector algebraic model





$$\mathbf{v} - \mathbf{v}_0 = \mathbf{g}t$$

$$\frac{1}{2}\left(\mathbf{v} + \mathbf{v}_0\right) = \frac{\mathbf{r}}{t} \equiv \overline{\mathbf{v}}$$

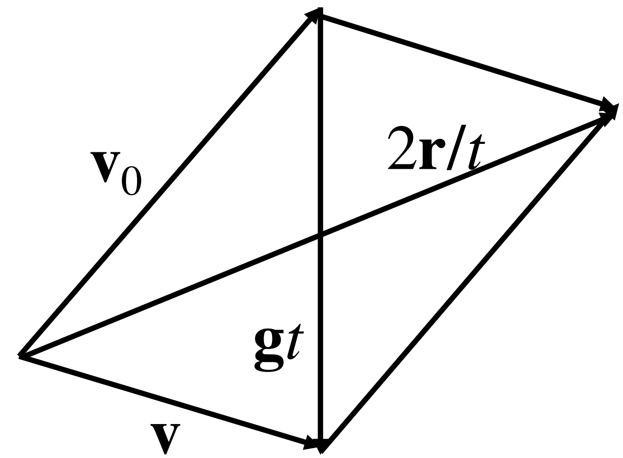
Reduces all projectile problems to solving a parallelogram!

Solving a parallelogram with Geometric Algebra

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{g}t$$

$$\mathbf{v} + \mathbf{v}_0 = \frac{2\mathbf{r}}{t}$$

$\Leftrightarrow$



Problem: Determine

(a) the range r of a target sighted in a direction $\hat{\mathbf{r}}$ that has been hit by a projectile launched with velocity $\mathbf{v}_0$;

(b) launching angle for maximum range;

(c) time of flight

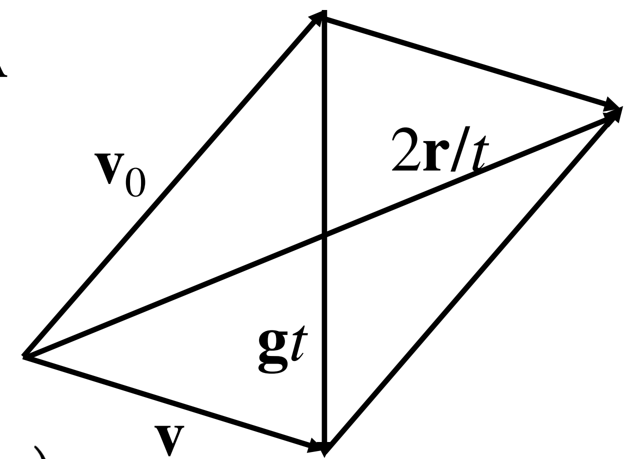
General case: Elevated target.

- Complicated solution with rectangular coordinates in AJP.
- Much simpler GA solution in NFCM.

Solving a parallelogram with GA

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{g}t \qquad \mathbf{v} + \mathbf{v}_0 = \frac{2\mathbf{r}}{t}$$

$$\Rightarrow \quad (\mathbf{v} - \mathbf{v}_0)(\mathbf{v} + \mathbf{v}_0) = 2\mathbf{r}\mathbf{g}$$



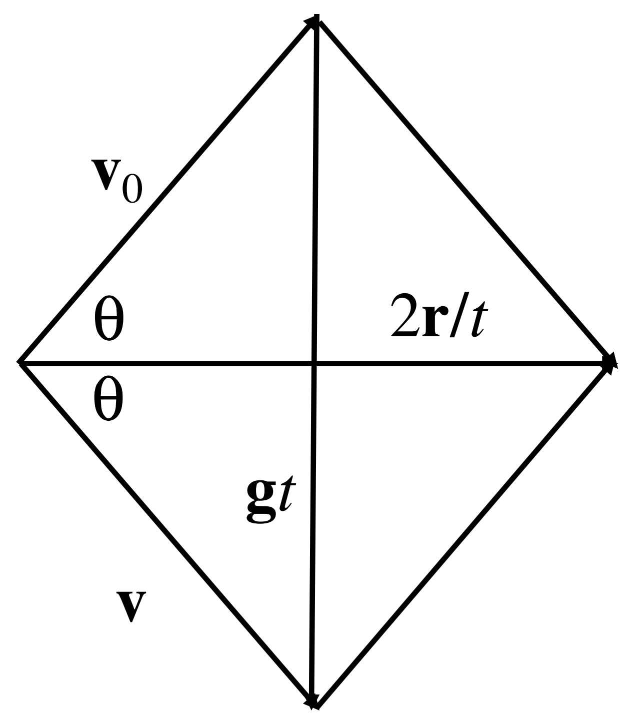
$$v^2 - {v_0}^2 + \underbrace{\mathbf{v}\mathbf{v}_0 - \mathbf{v}_0\mathbf{v}}_{2\mathbf{v}\wedge\mathbf{v}_0} = 2(\mathbf{r}\cdot\mathbf{g} + \mathbf{r}\wedge\mathbf{g})$$

$$v^2 - {v_0}^2 = 2\mathbf{r}\cdot\mathbf{g} \qquad \mathbf{v}\wedge\mathbf{v}_0 = 2\mathbf{r}\wedge\mathbf{g}$$

$\hat{\mathbf{r}}$ horizontal $\Rightarrow \quad \mathbf{r}\cdot\mathbf{g} = 0 \quad \Rightarrow \quad v^2 = {v_0}^2$

$\mathbf{r} = r\hat{\mathbf{r}}$ $\qquad |\mathbf{v}\wedge\mathbf{v}_0| = 2|\mathbf{r}\wedge\mathbf{g}|$

$\mathbf{g} = g\hat{\mathbf{g}}$ $\qquad = v_0^2 \sin 2\theta = 2rg$



A challenge to the math-science community!

Critically examine the following claims:

- GA provides a unified mathematical language that is conceptually and computationally superior to alternative math systems in every application domain.
- GA can enhance student understanding and accelerate student learning.
- GA is ready to incorporate into the curriculum.
- GA provides new insight into the structure and interpretation of quantum mechanics and relativity theory.
- Research on the design and use of mathematical software is equally important for instruction and for applications.

A proposal for GA in the curriculum

Unification and simplification of the high school math-science curriculum with Geometric Algebra should be centered on geometry because:

- *Geometry is the foundation for mathematical modeling* in physics and engineering and for the science of measurement in the real world.
- The computationally and conceptually superior methods of analytic geometry with GA facilitate real world applications.
- Reformulated Euclidean geometry with vector methods emphasizes the natural connection to kinematics and rigid body motions.

The effect will be to simplify theorems and proofs, and vastly increase applicability of mathematics to physics and engineering.

Whether or not the high school geometry course can be reformed in practice, the course content deserves to be reformed to make it more useful in physics and engineering applications.

Reform of the high school math-science curriculum can be greatly deepened and accelerated by introducing GA modeling software that is equally attractive to math and science teachers!

References

- Introductory survey: *Oersted Medal Lecture* 2002 (AJP)
  <http://modelingnts.la.asu.edu>

- Most thorough treatment of GA fundamentals:
  New Foundations for Classical Mechanics (Springer)

- Interactive presentation for high school:
  GA Primer <http://geocalc.clas.asu.edu/GAPrimer/>

Geometric Algebra (GA) software for modeling & simulation – to unify the *math-art-science-technology* (MAST) curriculum