Geometric Calculus & Differential Forms

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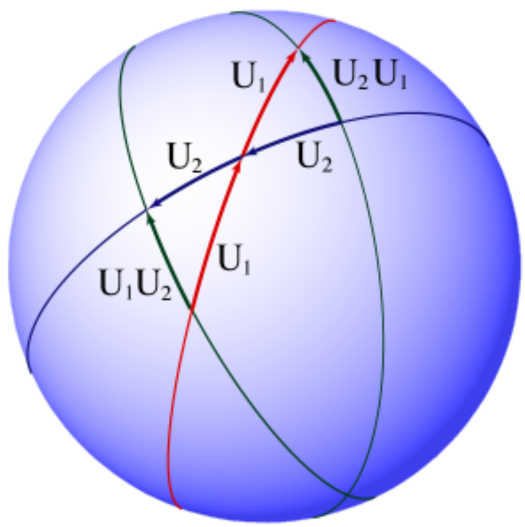
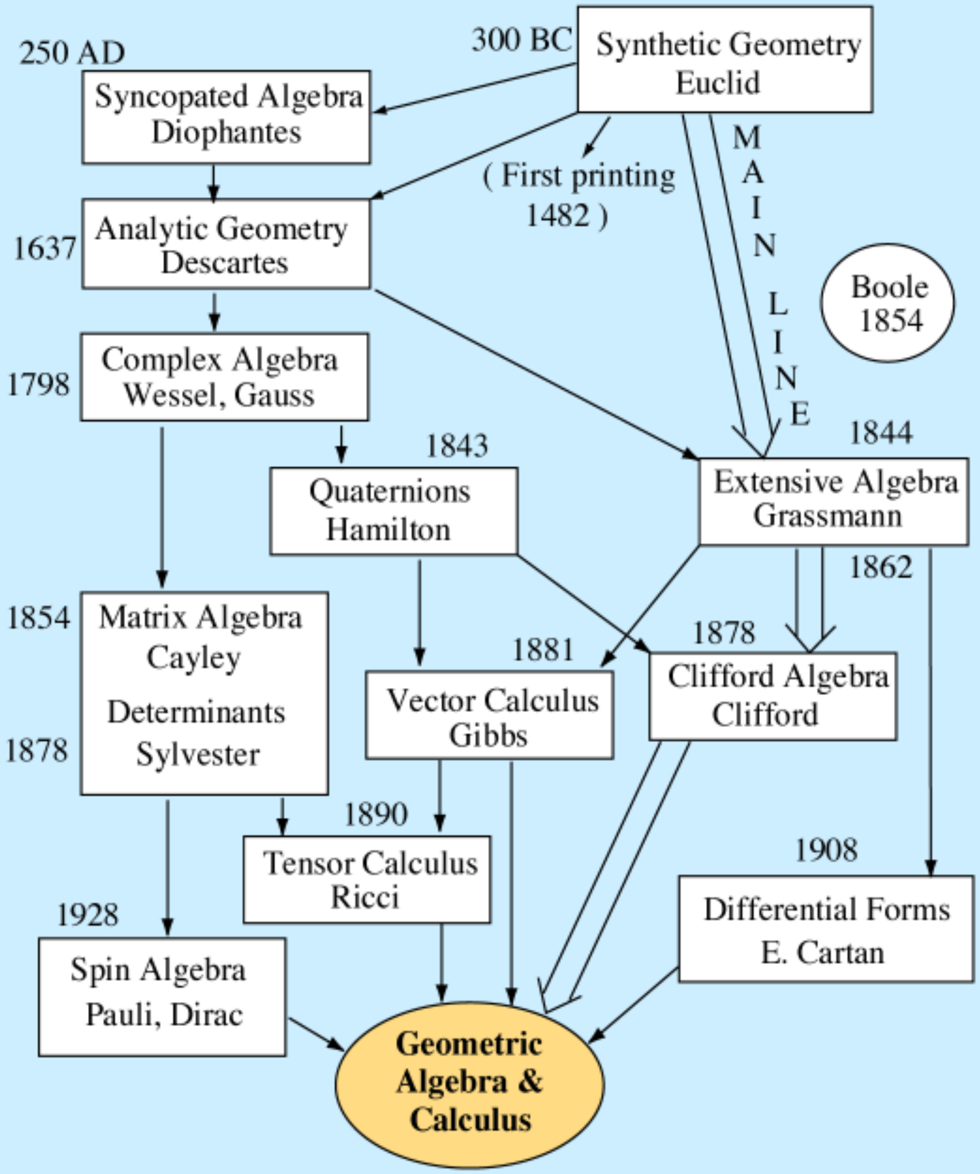


Santalo: 2016

Family Tree for Geometric Calculus



What is Geometric Algebra?

First answer: a universal number system for all of mathematics!

An extension of the real number system to incorporate the geometric concepts of direction, dimension and orientation

Can define by introduce anticommuting units (vectors):

(Grassmann) $e_j e_k = -e_k e_j$ for $j \neq k$ $j, k = -m, \ldots, -1, 1, 2, \ldots n$

(Clifford) $e_k^2 = \pm 1$ *signature* = sign of index k

associative and distributive rules

Arithmetic constructs: *vector space* $\mathcal{R}^{n,m}$ *algebra* $\mathcal{R}_{n,m}$

(Dirac, Jordan) *grandmother algebra* $\mathcal{R}_{\infty,\infty}$ (quantum field theory)

Naming the numbers: Clifford numbers or la rue de Bourbaki?

Clifford followed Grassmann in selecting *descriptive names*:

Directed numbers or multivectors: vectors, bivectors,…
versors, rotors, spinors

What is Geometric Algebra?
Second answer: a universal geometric language!

Geometric interpretation elevates the mathematics of $\mathcal{R}_{n,m}$
from mere arithmetic to the status of a language!!

Hermann Grassmann's contributions:

- Concepts of *vector* and *k-vector* with geometric interpretations
- System of *universal operations* on *k*-vectors
  - Progressive (outer) product (step raising)
  - Regressive product (step lowering)
  - Inner product
  - Duality
- System of identities among operations (repeatedly rediscovered in various forms)
- Abstraction of algebraic form from geometric interpretation
- Unsuccessful algebra of points $\longrightarrow$ (Conformal GA)

William Kingdon Clifford — intellectual exemplar

Deeply appreciated and freely acknowledged work of others:
Grassmann, Hamilton, Riemann

Modestly assimilated it into his own work

A model of self-confidence without arrogance

Clifford's contribution to Geometric Algebra:

- Essentially completed Grassmann's number system
- Reduced all of Grassmann's operations to a single *geometric product*
- Combined k-vectors into *multivectors* of mixed step (grade).

Overlooked the significance of mixed signature and null vectors
— opportunity to incorporate his biquaternions into GA

Subsequently, Clifford algebra was developed abstractly
with little reference to its geometric roots

The grammar of Geometric Algebra:

An arithmetic of directed numbers encoding the geometric concepts of magnitude, direction, sense & dimension

- To define the grammar, begin with **signature: positive, negative**

$$\mathcal{R}^{p,q} \equiv \text{Real vector space of dimension } n = p + q$$

Vector addition and scalar multiplication do not fully encode the geometric content of the vector concept.

- To encode the geometric concept of relative direction, we define an associative geometric product **ab** of vectors **a**, **b**, . . . with

$$\mathbf{a}^2 = \varepsilon_{\mathbf{a}} |\mathbf{a}|^2 \quad \text{where} \quad \varepsilon_{\mathbf{a}} = 1, 0, -1$$ is the **signature** of **a**

- With the geometric product the vector space $\mathcal{R}^{p,q}$ generates the

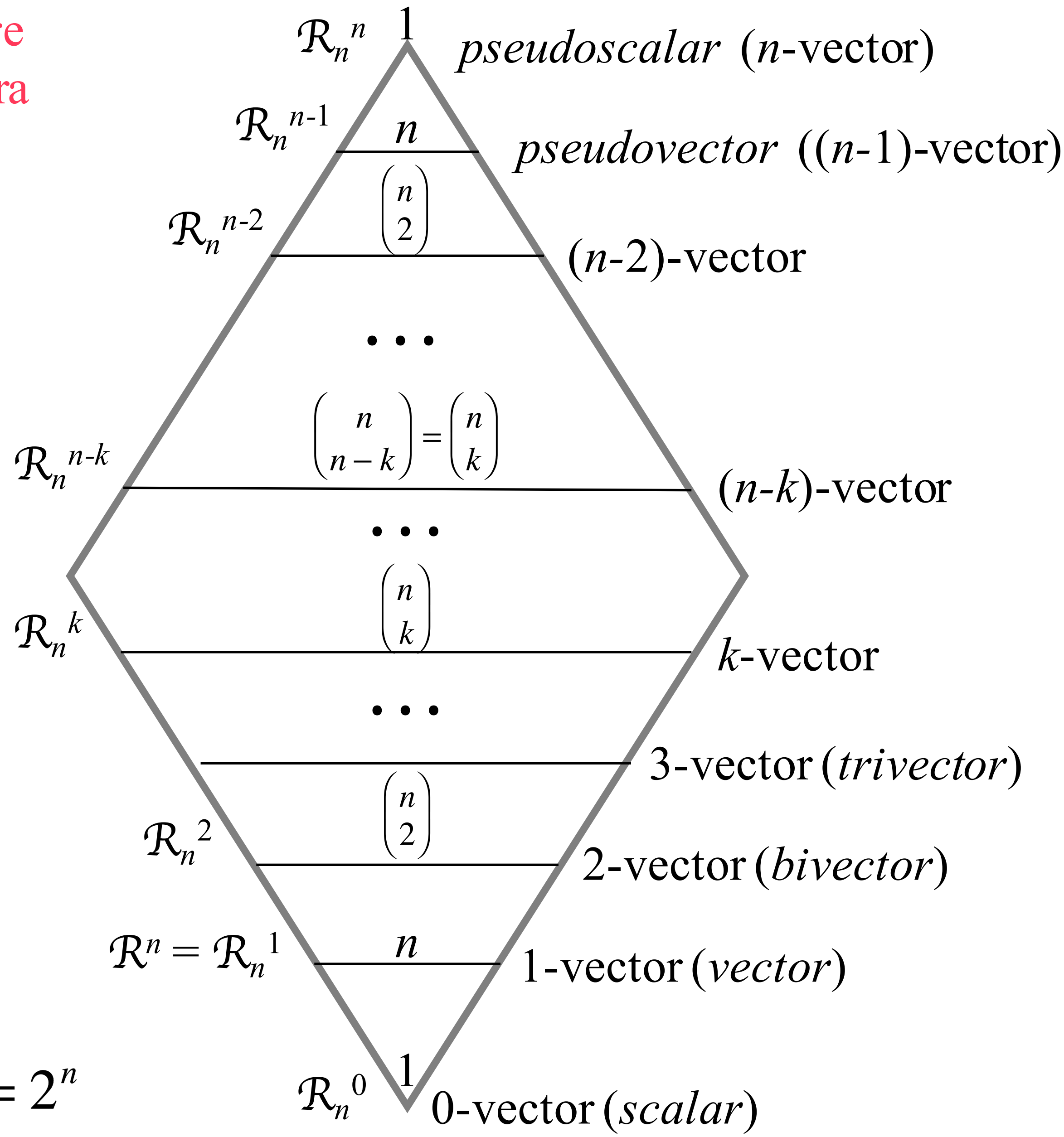
Geometric Algebra $$\boxed{\mathcal{R}_{p,q} = \mathcal{G}(\mathcal{R}^{p,q}) = \sum_{k=0}^{n} \mathcal{R}^{k}_{p,q}}$$ **space of *k*–vectors**

A linear space with $$\dim \mathcal{R}_{p,q} = \sum_{k=0}^{n} \dim \mathcal{R}^{k}_{p,q} = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Linear space structure
Of Geometric Algebra

$$\mathcal{R}_n = \sum_{k=0}^{n} \mathcal{R}_n^k$$

Duality: $\updownarrow$



$$\dim \mathcal{R}_n = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Quadratic forms vs. *contractions*

Claim: Linear forms on a vector space can be represented by inner products in a geometric algebra without assuming a metric.

$\mathcal{V}^n$ a real vector space spanned by $\{w_i\}$

Dual space $\mathcal{V}^{*n}$ of linear forms spanned by $\{w_j^*\}$

And defined by $w_i^*(w_j) = \frac{1}{2}\delta_{ij}$ or $\boxed{w_i^* \cdot w_j = \frac{1}{2}\delta_{ij}}$

The associative outer product generates the Grassmann algebra: $w_i \wedge w_j = -w_j \wedge w_i$

$$\Lambda_n = \Lambda_n^0 + \Lambda_n^1 + \ldots + \Lambda_n^n = \sum_{k=0}^{n} \Lambda_n^k \qquad \Lambda_n^0 = \mathcal{R}, \quad \Lambda_n^1 = \mathcal{V}^n$$

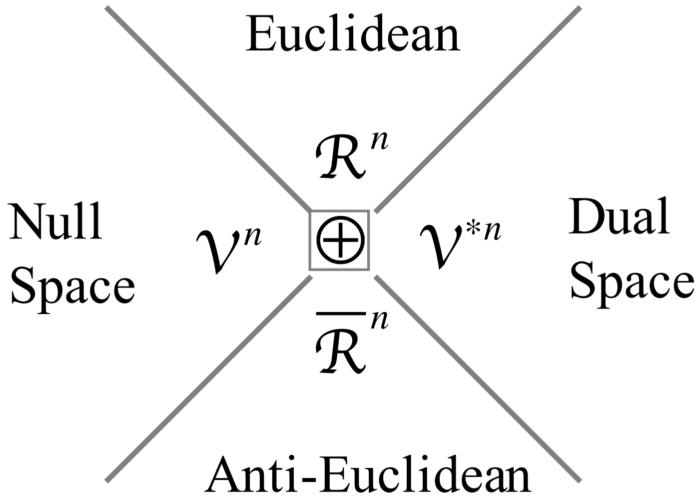
Likewise, the dual space generates the dual algebra $\Lambda_n^* = \sum_{k=0}^{n} \Lambda_n^{*k}$

Assume the null metric $w_i \cdot w_j = 0 = w_i^* \cdot w_j^*$ so

$\mathcal{V}^n$ has geometric product $w_i w_j = w_i \wedge w_j = -w_j w_i$

Now define $\boxed{w_i w_j^* + w_i^* w_j = \delta_{ij}}$ $\boxed{w_i^2 = 0 = w_i^{*2}}$

The algebra of fermion creation and annihilation operators!!



Mother algebra

$$\boxed{\mathcal{R}_{n,n} = \mathcal{G}(\mathcal{R}^{n,n})}$$

Covers all!

$$\boxed{\mathcal{R}^{n,n} = \mathcal{V}^n \oplus \mathcal{V}^{*n} = \mathcal{R}^n \oplus \overline{\mathcal{R}}^n}$$

$$\boxed{\mathcal{R}_{n,n} = \Lambda_n \otimes \Lambda_n^* = \mathcal{R}_n \otimes \overline{\mathcal{R}}_n}$$

Euclidean basis: $e_i = w_i + w_i^*$ spans $\mathcal{R}^n$

Anti-Euclidean basis: $\bar{e}_i = w_i - w_i^*$ spans $\mathcal{R}^n$

$$e_i \cdot e_j = \delta_{ij}, \quad e_i \cdot \bar{e}_j = 0, \quad \bar{e}_i \cdot \bar{e}_j = -\delta_{ij}$$

$\mathcal{R}_{p,q} \subset \mathcal{R}_{n,n}\ (p+q=n)$ All signatures in subalgebras

Ideal arena for linear algebra!

$\mathcal{R}_{\infty,\infty}$ All dimensions!

[Ref. Doran *et. al.*, *Lie Groups as Spin Groups*]

Universal Geometric Algebra

Real Vector Space: $\mathbb{V}^{r,s} = \{a, b, c, \ldots\}$ dimension $r+s = n$

Geometric product: $a^2 = \pm |a|^2$ nondegenerate signature $\{r, s\}$

generates Real GA: $\mathbb{G}^{r,s} = \mathbb{G}(\mathbb{V}^{r,s}) = \{A, G, M \ldots\} = \{\text{Multivectors}\}$

Inner product: $a \cdot b \equiv \frac{1}{2}(ab + ba)$ Outer product: $a \wedge b \equiv \frac{1}{2}(ab - ba)$

$\Rightarrow$ $\boxed{ab = a \cdot b + a \wedge b}$ $a \wedge A_k \equiv \frac{1}{2}\left(aA_k + (-1)^k A_k a\right)$

k-blade: $a_1 \wedge a_2 \wedge \ldots \wedge a_k = \langle a_1 a_2 \ldots a_k \rangle_k \equiv A_k$ $\Rightarrow$ k-vector

$$a \cdot (a_1 \wedge a_2 \wedge \ldots \wedge a_k) = \sum_{j=1}^{k} (-1)^{j+1} a \cdot a_j (a_1 \wedge \ldots \wedge \breve{a}_j \wedge \ldots \wedge a_k)$$

Graded algebra: $\mathbb{G}^{r,s} = \sum_{k=0}^{n} \mathbb{G}_k^{r,s} = \left\{ A = \sum_{k=0}^{n} \langle A \rangle_k \right\}$

Reverse: $(a_1 \wedge a_2 \wedge \ldots \wedge a_k)^{\sim} = a_k \wedge \ldots \wedge a_2 \wedge a_1$ $\tilde{A} = \sum_{k=0}^{n} \langle \tilde{A} \rangle_k = \sum_{k=0}^{n} (-1)^{k(k-1)/2} \langle A \rangle_k$

Unit pseudoscalar: $\mathrm{I} = \langle \mathrm{I} \rangle_n$ $\mathrm{I}\tilde{\mathrm{I}} = (-1)^s$ $a \wedge \mathrm{I} = 0$

Dual: $A^* \equiv A\mathrm{I}$ Thm: $a \cdot A^* = a \cdot (A\mathrm{I}) = (a \wedge A)\mathrm{I}$

Group Theory with Geometric Algebra

odd/even parity

Versor (of order *k*): $\boxed{G = n_k \ldots n_2 n_1}$ $\quad G^{-1} = n_1^{-1} n_2^{-1} \ldots n_k^{-1}$ $\quad G^{\#} = (-1)^k G$

Groups: $\mathrm{Pin}(r,s) = \left\{ G : GG^{-1} = 1 \right\} \supset \mathrm{Spin}(r,s) = \left\{ G : G = G^{\#} \right\}$

on vectors: $\mathrm{O}(r,s) = \left\{ \underline{G} : \underline{G}(a) = G^{\#} a G^{-1} \right\} \supset \mathrm{SO}(r,s) \underset{2}{\cong} \mathrm{Spin}(r,s)$

Advantages over matrix representations:

- Coordinate-free
- Simple composition laws: $G_2 G_1 = G_3 \qquad \underline{G}_2 \underline{G}_1 = \underline{G}_3$
- Reducible to multiplication and reflection by vectors:
- Reflection in a hyperplane in $\mathbb{V}_{\mathrm{r,s}}$ with normal n_i: $\underline{G}_i(a) = -n_i a n_i^{-1}$

  $\Rightarrow$ *Cartan-Dieudonné Thm* (Lipschitz, 1880): $\underline{G} = \underline{G}_k \cdots \underline{G}_2 \underline{G}_1$

$\Rightarrow$ *Nearly all groups* [Doran et. al. (1993) "Lie Groups as Spin Groups"]

For example: All the classical groups!

In particular: Conformal group: $\mathrm{C}(r,s) \cong \mathrm{O}(r+1, s+1)$

Hence define: Conformal GA: $\mathbb{G}^{r+1,s+1}$

From

GEOMETRIC ALGEBRA

to

GEOMETRIC CALCULUS

What is a manifold? $\mathcal{M}^m$ of dimension m

— a set on which differential and integral calculus is well-defined!

— standard definition requires covering by charts of local coordinates.

- Calculus done indirectly by local mapping to $\mathcal{R}^m = \mathcal{R} \otimes \mathcal{R} \cdots \otimes \mathcal{R}$
- Proofs required to establish results independent of coordinates.

Geometric Calculus defines a manifold as any set isomorphic to a vector manifold

Vector manifold $\mathcal{M}^m = \{x\}$ is a set of vectors in GA that generates at each point x a tangent space with pseudoscalar $I_m(x)$

Advantages:

- Manifestly coordinate-free
- Calculus done directly with algebraic operations on points
- Geometry completely determined by derivatives of $I_m(x)$.

Remark: It is unnecessary to assume that $\mathcal{M}^m$ is embedded in a vector space, though embedding theorems can be proved.

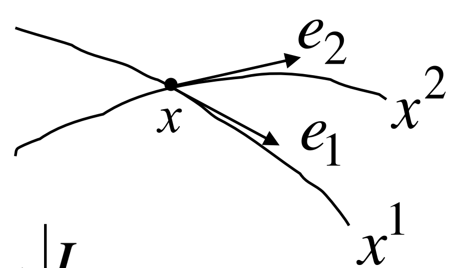
How GA facilitates use of coordinates on a vector manifold

Patch of $\mathcal{M}^m$ parametrized by coordinates: $x = x(x^1, x^2, \ldots x^m)$

Inverse mapping by coordinate functions: $x^\mu = x^\mu(x)$

Coordinate frame $\{e_\mu = e_\mu(x)\}$ defined by

$$\boxed{e_\mu = \partial_\mu x} = \frac{\partial x}{\partial x^\mu} = \lim \frac{\Delta x}{\Delta x^\mu}$$



With pseudoscalar: $e_{(m)} = e_1 \wedge e_2 \wedge \ldots \wedge e_m = |e_{(m)}| I_m$

Reciprocal frame $\{e^\mu\}$ implicitly defined by $e^\mu \cdot e_\nu = \delta^\mu_\nu$

with solution: $e^\mu = (e_1 \wedge \ldots \wedge (\quad)_\mu \wedge \ldots \wedge e_m) e_{(m)}^{-1}$

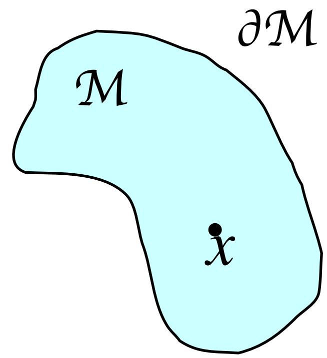
Vector derivative: $\partial = \partial_x = e^\mu \partial_\mu$ $\qquad \partial_\mu = e_\mu \cdot \partial = \dfrac{\partial}{\partial x^\mu}$

$$\Rightarrow \quad \boxed{e^\mu = \partial x^\mu}$$

Problem: How define vector derivative without coordinates?

Directed integrals in GA

$F = F(x)$ = multivector-valued function on $\mathcal{M} = \mathcal{M}^m$

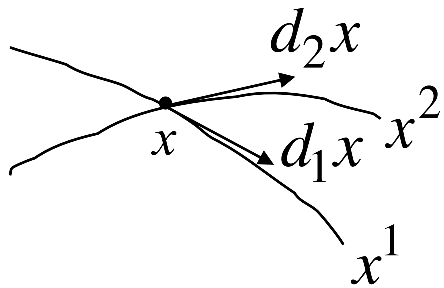


$d^m x = |d^m x| I_m(x)$ = directed measure on $\mathcal{M}$

In terms of coordinates:

$$d^m x = d_1 x \wedge d_2 x \wedge \ldots \wedge d_m x = e_1 \wedge e_2 \wedge \ldots \wedge e_m \, dx^1 dx^2 \ldots dx^m$$

where $d_\mu x = e_\mu(x)\, d^\mu x$ (no sum)



$|d^m x| = |e_{(m)}| dx^1 dx^2 \ldots dx^m$ = *Volume element*

Directed Integral $\Big\}$ $\displaystyle\int_{\mathcal{M}} d^m x\, F = \int_{\partial\mathcal{M}} e_{(m)} F \, dx^1 dx^2 \ldots dx^m$

expressed as a standard multiple integral

$\boxed{\textbf{Fundamental Theorem} \text{ of Geometric Calculus}}$

For $\mathcal{M} = \mathcal{M}^m \subset \mathcal{V}^n$ = vector space,

∇ = vector derivative on $\mathcal{V}^n$

$$\int_{\mathcal{M}} (d^m x) \cdot \nabla F = \int_{\partial\mathcal{M}} d^{m-1}x\, F$$

$\partial = \partial_x = I_m^{-1}(I_m \cdot \nabla)$ = vector derivative on $\mathcal{M}$

$$\Rightarrow \qquad d^m x\, \partial = (d^m x)\cdot\partial = (d^m x)\cdot\nabla$$

✪ $$\boxed{\int_{\mathcal{M}} d^m x\, \partial F = \int_{\partial\mathcal{M}} d^{m-1}x\, F}$$

Inspires coordinate-free definition for the

tangential derivative: $\partial = \partial_x$ = derivative by x on $\mathcal{M}$

✪✪ $$\boxed{\partial F = \lim_{d\omega \to 0} \frac{1}{d\omega} \oint d\sigma F}$$

$$d\omega = d^m x$$

$$d\sigma = d^{m-1}x$$

Theory of differential forms generalized by GA

$L(d^k x, x)$ = multivector-valued k-form
= linear function of k-vector $d^k x$ at each point x.

e.g.: $L = d^k x$ = k-vector valued k-form

Exterior differential of k-form L:

$$dL \equiv \dot{L}(d^{k+1}x \cdot \dot{\partial}) = L(d^{k+1}x \cdot \dot{\partial}, \dot{x})$$

Fundamental Theorem:
(most general form)

$$\boxed{\int_{\mathcal{M}} dL = \oint_{\partial\mathcal{M}} L}$$

Special cases: $L = d^k x F(x)$ $\qquad L = \left\langle d^{m-1}x\, F \right\rangle$

$$dL = \left\langle d^m x\, \partial F \right\rangle = \left\langle d^m x\, \partial \wedge F \right\rangle$$

$$= (d^m x) \cdot (\partial \wedge F) \quad \text{if} \quad F = \langle F \rangle_{m-1}$$

Advantages over standard theory:

- Cauchy Theorem:
- Cauchy Integral Theorem

$$\partial F = 0 \quad \Leftrightarrow \quad \oint d^k x F = 0$$

Advantages of the **vector derivative**:

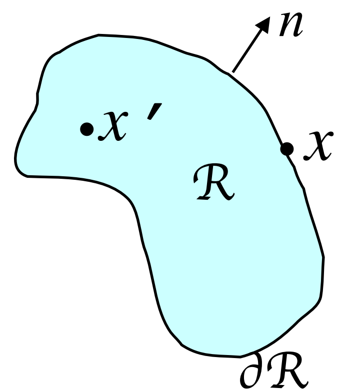
$$\nabla A = \underbrace{\lim}_{d\omega \to 0} \frac{1}{d\omega} \oint d\sigma A$$

- Applies to all dimensions
- Coordinate-free
- Simplifies Fund. Thm.
- Generalizes definition

Inverse operator given by generalized *Cauchy Integral Formula*

$$A(x') = \frac{(-1)^n}{\Omega_n I_n} \left\{ \int_R \frac{x - x'}{|x - x'|^n} d\omega \nabla A - \int_{\partial R} \frac{x - x'}{|x - x'|^n} d\sigma A \right\} = \nabla^{-1} s$$

$$\Omega_n = \frac{2\pi^{\frac{1}{2}n}}{\Gamma\left(\frac{n}{2}\right)} \qquad d\omega = I_n |d\omega| \qquad I_n^{-1} d\sigma = n |d\sigma|$$



or

$$A(x') = \frac{1}{\Omega_n} \left\{ -\int_R |d\omega| \frac{x - x'}{|x - x'|^n} \nabla A + \int_{\partial R} |d\sigma| \frac{x - x'}{|x - x'|^n} nA \right\}$$

Applies to Euclidean spaces of any dimension, including $n = 2$

Good for electrostatic and magnetostatic problems!

What's in a name? *Vector derivative* vs. *Dirac operator*

(vector $x = x^\mu \gamma_\mu$) $\boxed{\nabla = \partial_x}$ $D = \gamma^\mu \dfrac{\partial}{\partial x^\mu}$ (coordinates x^μ)

Geometric (multivector-valued) function: $A = A(x)$

Symbols: $\nabla A = \nabla \cdot A + \nabla \wedge A$ DA = divA + rot A (Riesz)

Names: del (grad) = div + curl Dirac op = Gauss + Maxwell

Main issue: How does ∇ (or D) relate to the

Fundamental Theorem of Calculus vs. Stokes Theorem ?

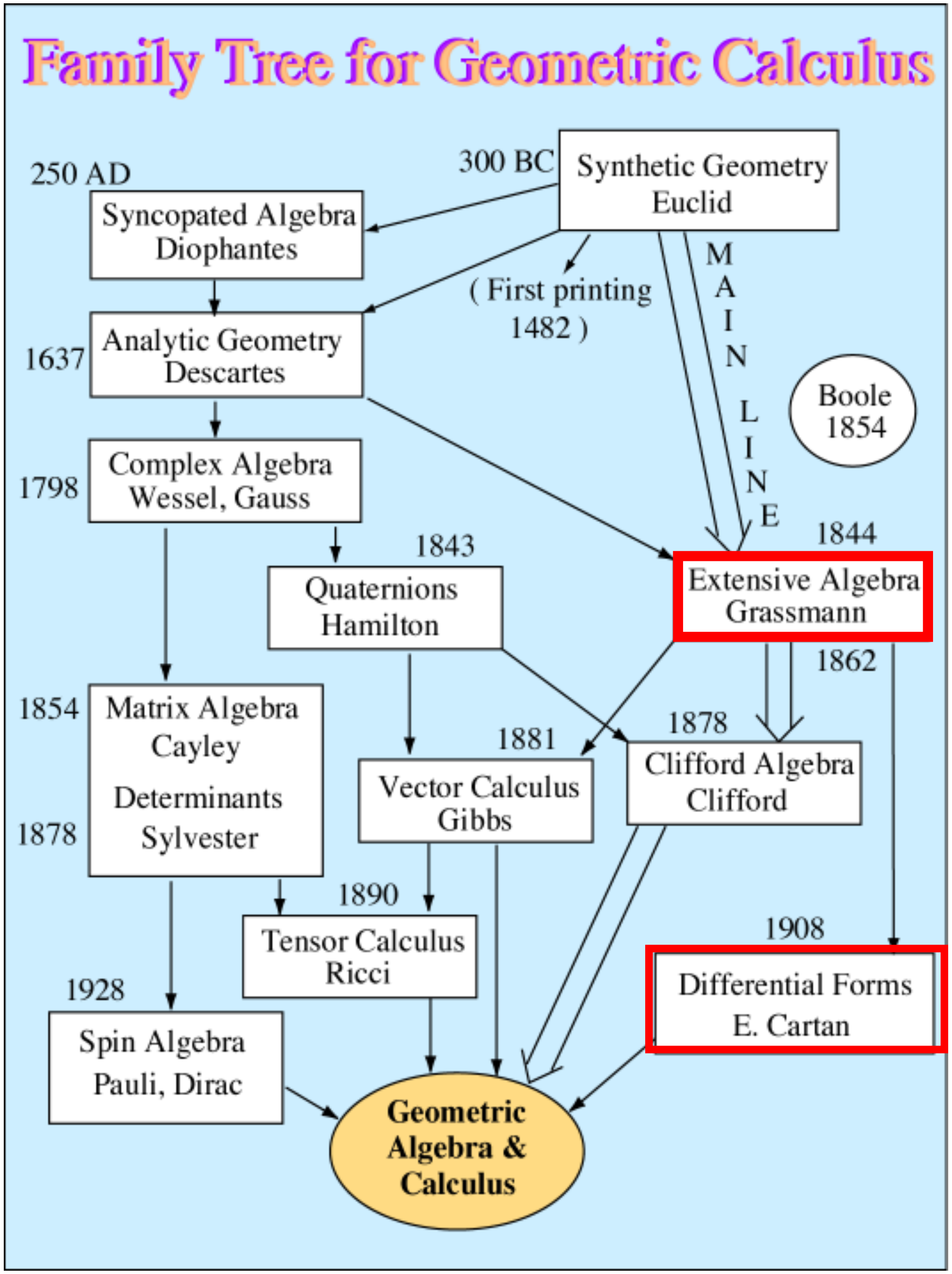
<u>Clifford analysis</u>: Applies differential forms to CA

<u>Geometric Calculus</u>: Develops differential forms within GA

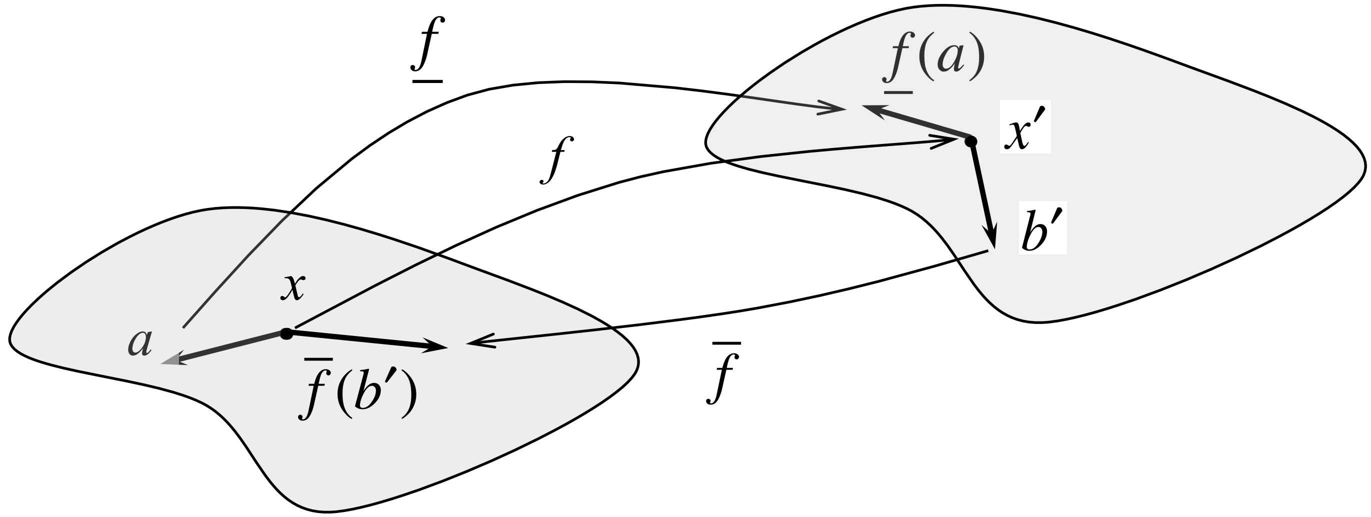
[Reference: *Differential Forms in Geometric Calculus* (1993)]

Def: $$\boxed{\nabla A \equiv \underbrace{\lim}_{d\omega \to 0} \frac{1}{d\omega} \oint d\sigma A}$$

(Mitrea) $\nabla A \cong \dfrac{\partial A}{\partial \omega}$ Areolar derivative (Pompieu, 1910)
Volumetric deriv. (Théodorescu, 1931)


Family Tree for Geometric Calculus
250 AD
300 BC
Synthetic Geometry
Euclid
Syncopated Algebra
Diophantes
(First printing
1482)
MAIN LINE
1637
Analytic Geometry
Descartes
Boole
1854
1798
Complex Algebra
Wessel, Gauss
1843
1844
Quaternions
Hamilton
Extensive Algebra
Grassmann
1862
1854
Matrix Algebra
Cayley
Determinants
Sylvester
1878
1881
1878
Vector Calculus
Gibbs
Clifford Algebra
Clifford
1890
Tensor Calculus
Ricci
1908
Differential Forms
E. Cartan
1928
Spin Algebra
Pauli, Dirac
Geometric
Algebra &
Calculus

Mappings **<u>of</u>** & Transformations **<u>on</u>** Vector Manifolds



diffeomorphism: $f: x \;\rightarrow\; x' = f(x)$ $\qquad x = f^{-1}(x')$

<u>Induced transformations</u> <u>of vector fields</u> (active)

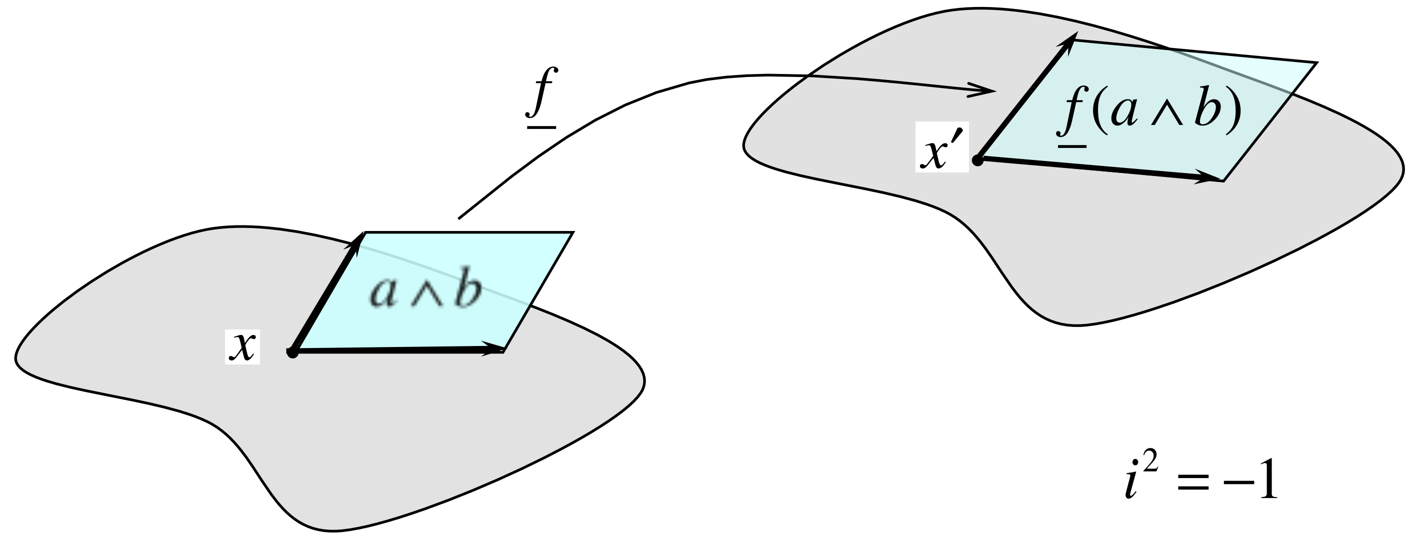
differential: $\underline{f}: a = a(x) \;\rightarrow\; a' = \boxed{\underline{f}(a) \equiv a\cdot\nabla f}$ $\qquad a = \underline{f}^{-1}(a')$

adjoint: $\overline{f}: b' = b'(x') \;\rightarrow\; b = \boxed{\overline{f}(b') \equiv \dot{\nabla}\dot{f}\cdot b' = \partial_x f(x)\cdot b'}$

Tensor fields: $T(a,b')$ *covariant:* $\overline{f}(b')$, *contravariant:* $\underline{f}(a)$

Theorem: $\boxed{\overline{f^{-1}} = \overline{f}^{-1}}$: $b(x) \;\rightarrow\; b'(x') = \overline{f^{-1}}\left[b\left(f(x')\right)\right]$

outermorphism: $\underline{f} : a \wedge b \;\rightarrow\; \boxed{\underline{f}(a \wedge b) = \underline{f}(a) \wedge \underline{f}(b)}$



$i^2 = -1$

Jacobian: $\underline{f} : i \;\rightarrow\; \underline{f}\, i = J_f\, i' \;\Rightarrow\; \boxed{J_f = \det \underline{f} = -i' \underline{f}\, i}$

Chain rule: (induced mapping of differential operators)

$$\overline{f} : \nabla' \;\rightarrow\; \boxed{\nabla = \overline{f}\, \nabla'} \qquad \text{or} \qquad \partial_x = \overline{f}(\partial_{x'})$$

$$\Rightarrow\; a \cdot \nabla = a \cdot \overline{f}(\nabla') = \underline{f}(a) \cdot \nabla' = a' \cdot \nabla'$$

$$\left.\begin{array}{l} x = x(\tau) \\ \dot{x} = \dfrac{dx}{d\tau} \end{array}\right\} \qquad \Rightarrow\; \boxed{\frac{d}{d\tau} = \dot{x} \cdot \nabla} = \dot{x} \cdot \overline{f}(\nabla') = \underline{f}(\dot{x}) \cdot \nabla' = \dot{x}' \cdot \nabla'$$

Derivation of the gauge tensor

New!!

Brilliant!!

Displacement Gauge Principle: The equations of physics must be invariant under arbitrary *field displacements*.

An arbitrary diffeomorphism of spacetime onto itself

$$f : x \quad \rightarrow \quad x' = f(x) \qquad\qquad x = f^{-1}(x')$$

induces a substitution *field displacement*: $F(x) \quad \rightarrow \quad F'(x) \equiv F(x') = F[f(x)]$

For the gradient of a scalar: $\nabla\varphi'(x) = \nabla\varphi[f(x)] = \overline{f}[\nabla'\varphi(x')]$

To make this invariant, *define a gauge tensor* $\overline{h}$ so that

$$\overline{h}\big[\nabla\varphi(x)\big] \quad \rightarrow \quad \overline{h}'\big[\nabla\varphi'(x)\big] = \overline{h}\big[\nabla'\varphi(x')\big] = \overline{h}\,\overline{f}^{-1}\,\nabla\varphi'(x)$$

$$\Rightarrow \quad \overline{h} \quad \rightarrow \boxed{\overline{h}' = \overline{h}\,\overline{f}^{-1}}$$

$\Rightarrow$ *Position gauge invariant vector derivative*: $\boxed{\overline{\nabla} \equiv \overline{h}\,\nabla} \quad \rightarrow \quad \overline{h}\,\nabla' = \overline{h}'\,\nabla$

Regard this as a NEW general approach to Differential Geometry!!

Summary: **Gauge Theory Gravity Principles** for **Differential Geometry**

I. *Rotation Gauge Principle*: The equations of physics must be *covariant* under *local Lorentz rotations*.

Physical significance: This can be regarded as a precise gauge theory formulation of *Einstein's Equivalence Principle.*

Physical implication: $\Rightarrow$ Existence of a ***geometric connexion*** (field)

II. *Displacement Gauge Principle*: The equations of physics must be *invariant* under arbitrary smooth remappings of events in spacetime.

Physical interpretation: This can be regarded as a precise gauge theory formulation of *Einstein's General Relativity Principle* as a *symmetry group* of mappings on spacetime.

– It cleanly separates *coordinate dependence* of spacetime maps from *physical dependence* of metrical relations.

Physical implication: $\Rightarrow$ Existence of a ***gauge tensor*** (field)

– which can be identified as a ***gravitational potential***,
– essentially equivalent to Einstein's metric tensor.

Where Topology meets Geometry!

Geometric Calculus needs to be extended
to treat singularities on/of manifolds:

Boundaries, holes and intersections
versus
Singular fields on manifolds

Crucial questions and examples come from physics!

Electromagnetic Field Singularities

Spacetime point: $x = x^\mu \gamma_\mu$ Coordinates: $x^\mu = x \cdot \gamma^\mu$

Derivative: $\partial = \partial_x = \gamma^\mu \partial_\mu$ $\partial_\mu = \dfrac{\partial}{\partial x^\mu} = \gamma^\mu \cdot \partial$

EM field: $F = F(x) = \tfrac{1}{2} F^{\nu\mu} \gamma_\mu \wedge \gamma_\nu$

Charge current: $J = J(x) = J^\mu \gamma_\mu$

Maxwell's Eqn: $\boxed{\partial F = J}$ $\partial F = \underset{\text{Div}}{\partial \cdot F} + \underset{\text{Curl}}{\partial \wedge F}$

$$\Rightarrow \quad \partial \cdot F = J \qquad \partial \wedge F = 0$$

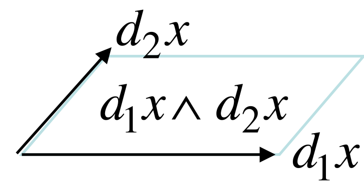
Potential: $F = \partial \wedge A = \partial A$ if $\partial \cdot A = 0$

$$\Rightarrow \quad \boxed{\partial^2 A = J}$$

Charge conservation: $\partial^2 F = \partial J = \partial \cdot J + \partial \wedge J$

$$\Rightarrow \quad \boxed{\partial \cdot J = 0} \qquad \partial^2 F = \partial \wedge J$$

Alternative formulations for E & M



EM 2-form: $\omega = \omega(d_1x\wedge d_2x) = (d_1x\wedge d_2x)\cdot F$

EM Tensor: $F^{\mu\nu} = (\gamma^{\mu}\wedge\gamma^{\nu})\cdot F = \gamma^{\nu}\cdot F\cdot\gamma^{\mu} = \omega(\gamma^{\mu}\wedge\gamma^{\nu})$

Current 1-form: $\alpha = J\cdot dx$ 3-form: $*\alpha = d^3x\cdot(Ji)$

Dual form: $*\omega = (d_1x\wedge d_2x)\cdot(Fi)$

Exterior differential: $d\omega = d^3x\cdot(\nabla\wedge F)$ $d^3x = d_1x\wedge d_2x\wedge d_3x$

Dual differential: $d*\omega = d^3x\cdot(\nabla\wedge(Fi)) = d^3x\cdot((\nabla\cdot F)i)$

	<u>STA</u>	<u>Tensor</u>	<u>Differential form</u>
Maxwell's Equations:	$\nabla\cdot F = J$	$\partial_{\mu}F^{\mu\nu} = J^{\nu}$	$d*\omega = *\alpha$
	$\nabla\wedge F = 0$	$\partial_{[\alpha}F_{\mu\nu]} = 0$	$d\omega = 0$
	invariant	covariant	

$$\boxed{\nabla F = J} \quad\Rightarrow\quad F = \nabla^{-1}J$$

Universal Electrodynamics for Material Media

Field $F = \mathbf{E} + i\mathbf{B}$ $\qquad$ $G = \mathbf{D} + i\mathbf{H}$ Field density or excitation (Sommerfeld)

Field Equations

$$\boxed{\begin{array}{lcl} \nabla \wedge F = 0 & \Rightarrow & F = \nabla \wedge A \\ \nabla \cdot G = J & \Rightarrow & \nabla \cdot J = 0 \end{array}}$$

Dual form: $\nabla \wedge (Gi) = Ji \quad \Rightarrow \quad \nabla \wedge (Ji) = 0$ $\qquad$ metric independent!

$$\boxed{\begin{array}{ll} \text{Maxwell field:} & M = F + Gi \\ \text{Field Equation:} & \nabla \wedge M = Ji \end{array}}$$

$\boxed{\text{Gravitation??}}$ $\qquad \nabla \cdot M = J_0$ $\qquad$ metric dependent!

Constitutive relations: $G = \chi(F)$

For the electron: $G = \rho^{-1} F$ $\qquad$ To be explained!

Cartan's Differential Forms in Geometric Calculus

Tangent vectors for coordinates x^μ: $d_\mu x = e_\mu dx^\mu$ (no sum on μ)

Volume elements: $d^k x = d_1 x \wedge d_2 x \wedge \ldots \wedge d_k x$

$$d^4 x = d_1 x \wedge d_2 x \wedge d_3 x \wedge d_4 x = \left|d^4 x\right| i \qquad i^2 = -1$$

Differential k-form: $\bar{K} = d^k x \cdot K$ for k-vector field: $K = \langle K \rangle_k = K(x)$

Exterior product: $\bar{A} \wedge \bar{K} = d^{k+1} x \cdot (A \wedge K)$ 1-form: $\bar{A} = dx \cdot A = A \cdot dx$

Exterior differential: $d\bar{K} = d^{k+1} x \cdot (\nabla \wedge K)$

Stokes Theorem: $\int_\Sigma d\bar{K} = \oint_{\partial\Sigma} \bar{K}$

Closed k-form: $\oint_{\partial\Sigma} \bar{K} = 0$ for all k-cycles

Exact k-form: $\bar{K} = d\bar{J} \Rightarrow d\bar{K} = 0 \quad \Leftrightarrow \quad K = \nabla \wedge J \Rightarrow \nabla \wedge K = 0$

$$dd\bar{J} = 0 \quad \Leftrightarrow \quad \nabla \wedge \nabla \wedge J = 0$$

D. Hestenes, Differential Forms in Geometric Calculus. In F. Brackx *et al.* (eds),
Clifford Algebras and their Applications in Mathematical Physics (1993)

Differential Forms in Physics

R. M. Kiehn: Cartan's Corner: http://www.cartan.pair.com/

Topological Electrodynamics	metric independence	Topological Thermodynamics

Vector Potential $\bar{A} = A \cdot dx$ $\qquad \mathcal{A} = \oint \bar{A}$ Action Integral

Field intensity $\bar{G} = (Gi) \cdot d^2x = (d^2x \wedge G) \cdot i$ Topological defects

$$d\bar{G} = \bar{J} = (Ji) \cdot d^3x = (d^3x \wedge J) \cdot i$$

Pfaff sequence:

1-form:	$\bar{A}$	Topological Action	
2-form:	$d\bar{A} = \bar{F}$	Topological Vorticity	$\bar{G}$
3-form:	$\bar{A} \wedge d\bar{A}$	Topological Torsion	$\bar{A} \wedge \bar{G}$: Topological spin
4-form:	$d\bar{A} \wedge d\bar{A}$	Topological Parity	$d(\bar{A} \wedge \bar{G}) = \bar{F} \wedge \bar{G} - \bar{A} \wedge \bar{J}$

Faraday's Law: $\oint \bar{F} = 0$ $\qquad$ Gauss-Ampere Law: $\oint \bar{G} = \int \bar{J}$

Recall the definition of free space in Maxwell Theory

Maxwell's equation for a homogeneous, isotropic medium

ε = permitivity (dielectric constant)

μ = (magnetic) permeability

$$\mathrm{G} = \mathbf{E} + \frac{i}{\sqrt{\mu\varepsilon}}\mathbf{B}$$

$$(\sqrt{\mu\varepsilon}\,\partial_t - \nabla\,)\mathrm{G} = 0 \qquad \text{Maxwell's Equation}$$

$$(\sqrt{\mu\varepsilon}\,\partial_t + \nabla\,)\times(\sqrt{\mu\varepsilon}\,\partial_t - \nabla\,)\mathrm{G} = 0$$

$$= (\,\mu\varepsilon\partial_t^2 - \nabla^2)\mathrm{G} = 0$$

$$(c^{-2}\partial_t^2 - \nabla^2)\mathrm{G} = 0 \qquad \text{Wave Equation}$$

$$\boxed{c = 1/\sqrt{\mu\varepsilon} \;=\; \text{velocity of light in the medium = free space}}$$

D'Alembertian: $\square^{\,2} = c^{-2}\partial_t^2 - \nabla^2$ Wave operator

Invariant under Lorentz transformations

$\Longrightarrow$ *Theory of relativity* $\qquad$ *But* $\sqrt{\dfrac{\mu}{\varepsilon}} = \rho(x) = ??$

Electron as singularity in the physical vacuum

Electromagnetic vacuum defined by: $\varepsilon\mu = \frac{1}{c^2} = \varepsilon_0\mu_0$ (Maxwell)

Vacuum impedence undefined: $Z(x) = \sqrt{\frac{\mu}{\varepsilon}} = \frac{1}{\rho(x)}\sqrt{\frac{\mu_0}{\varepsilon_0}}$ (E. J. Post)

Blinder function: $\boxed{\rho = \rho(x) = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{\varepsilon_0}{\mu_0}} = e^{-\lambda_e/r}}$

Point charge path & velocity: $z = z(\tau), \quad v = \dot{z} = \frac{1}{c}\frac{dz}{d\tau}$

Retarded distance: $r = (x - z(\tau)) \cdot v$ with $(x - z(\tau))^2 = 0$

Classical electron radius $\lambda_e = \frac{e^2}{m_e c^2}$

Vector potential: in Maxwell Thry $\boxed{\frac{e}{c}A_e = \frac{e^2}{c\lambda_e}\rho v = \rho m_e c v}$ **Momentum density** in Dirac Theory

Suggests unification of Maxwell & Dirac by reinterpreting: $\boxed{\rho = \psi\tilde{\psi}}$

THE END

Or a beginning

for reform of the mathematics curriculum

Where did mathematics come from?

Google: **V. I. Arnold** On Teaching Mathematics (Paris, 1997)

"Mathematics is a part of physics.

Physics is an experimental science, a part of natural science.

Mathematics is the part of physics where experiments are cheap."

"In the middle of the 20th century it was attempted to divide physics and mathematics.

The consequences turned out to be catastrophic.

Whole generations of mathematicians grew up without knowing half of their science and, of course in total ignorance of other sciences."

Current state: Physics is **no longer** a required minor for math students!!

Conclusion: Physics should be **fully integrated** into the math curriculum!!

Essential reforms of the Mathematics Curriculum

Linear Algebra [Ref. *Design of Linear Algebra*]

Begin with GA (universal number system)

Extend linear vector functions to whole GA
— *Outermorphisms*

Use coordinate–free methods

Treat reflections and rotations early

Subsume matrix algebra to GA

Conformal Geometric Algebra

Real and complex analysis,
multivariable and many-dimensional calculus
— unified, coordinate-free treatment with GC

Geometric Calculus and Differential geometry

Lie Groups & Transformations

Programming and Computing

More on history of mathematics
and origins of Geometric Algebra

Landmark Inventions in Mathematical Physics

☆ **Geometry** (-230 Euclid) the foundation for measurement

☆ **Analytic Geometry** (1637 Descartes)
first integration of algebra and geometry

☆ **Differential and Integral Calculus** (~ 1670 Newton & Leibniz)
- Newtonian Mechanics 1687
  Perfected ~1780+ by Euler, Lagrange, Laplace

19th Century

☆ **Complex variable theory** (~1820+ Gauss, Cauchy, Riemann)
- Celestial mechanics and chaos theory (1887 Poincaré)
- Quantum mechanics (1926 Schrödinger)

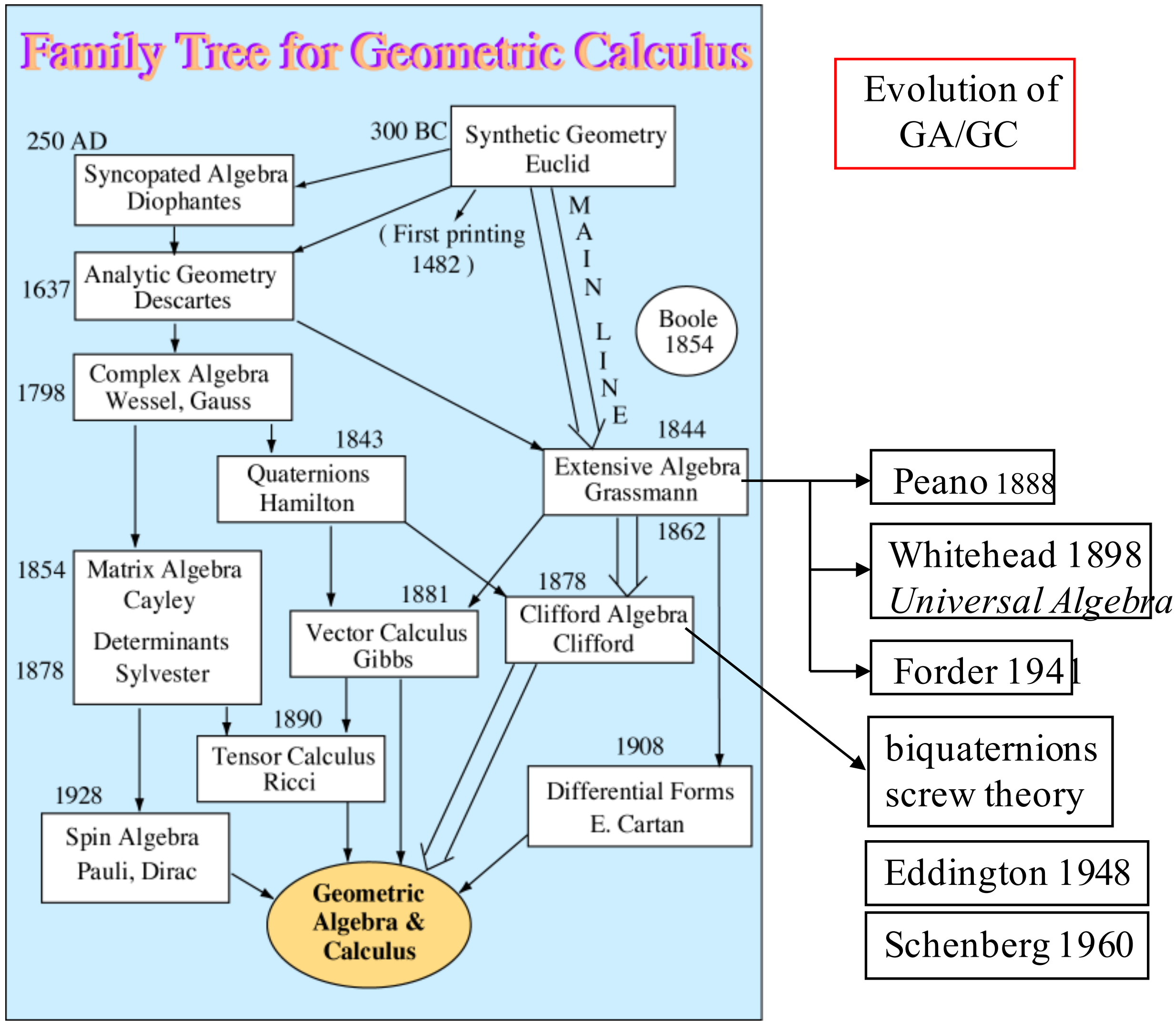
☆ **Vector calculus** (1881 Gibbs)
- Electrodynamics (1884 Heaviside)

☆ **Tensor calculus** (1890 Ricci)
- General Relativity (1955 Einstein)

☆ **Matrix algebra** (1854+ Cayley)
- Quantum mechanics (1925 Heisenberg, Born & Jordan)

☆ **Group Theory** (~1880 Klein, Lie)
- Quantum mechanics (1939 Weyl, Wigner)
- Particle physics (1964 Gell-Mann, etc.)

Evolution of GA/GC



The Figure shows only *major strands* in the history of GA

Real history is much more complex and nonlinear,
with many intriguing branches and loops

I welcome suggestions to improve my simplified account

The most important **historical loop** in the Figure is

The branches of Grassmann's influence
through Clifford and Cartan

- Essentially separated from Clifford algebra,
- Grassmann's geometric concepts evolved
  through differential forms to be formalized
  in the mid 20th century by Bourbaki
  (a step backward from Grassmann)

- The two were then combined to fulfill
  Grassmann's vision for a truly
  universal geometric algebra

Contributions of ***Marcel Reisz*** to Clifford (geometric) algebra

Mainly in his lecture notes *Clifford Numbers and Spinors* (1958)
- Origins mysterious – one paper on Dirac equation in GR (1953) in Swedish conference proceedings
  - Main research on analysis and Cauchy problem in Rel.
- Known through reference in my *Space Time Algebra* (1966)
- Lounesto arranged publication (Kluwer 1993) with notes

Immediate impact on me (Nov 1958)
- I was prepared in differential forms, Dirac theory & QED
- Catalyzed insights to integrate them geometrically
- Supplied algebraic techniques that I combined with Feynman's
- Suggested elimination of matrices by identifying
  - spinors with elements of minimal ideals

Launched me on a research program to
- develop unified, coordinate-free methods for physics
- discover geometric meaning for complex numbers in QM

Multiple discoveries and **isolated results** in the historical record

Multiple discovery of the generalized Cauchy Integral formula
— discussed in my 1985 lecture

Another example, Maxwell's equation: $\boxed{\nabla F = J}$

Silberstein (1924), Lanczos (1929) – complex quaternions
Juvet (1930), Riesz (1953, 58), . . .
— who deserves the credit?

Priority vs. Impact

Impact and influence as tests of historical significance:

- Is the work systemic or isolated?
- Does it generate more results from the author?
- Does it stimulate work by others?

Isolated results — impact depends on access besides intrinsic value

Quaternions — favorite example

Classical geometry & screw theory — branches of math isolated

Invariant theory — marginalized

Outline and References

<http:\\modelingnts.la.asu.edu> <http://www.mrao.cam.ac.uk>

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- NFCM (Kluwer, 2nd Ed.1999)
- *New Foundations for Mathematical Physics* (Web)

  1. Synopsis of GA 2. Geometric Calculus

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- NFCM (chapter 9 in 2nd Ed.)
- *Electrodynamics* (W. E. Baylis, Birkhäuser, 1999)

III. Spacetime Physics (invariant formulation)

- Spacetime Physics with Geometric Algebra (Web & AJP)
- Doran, Lasenby, Gull, Somaroo & Challinor,
  Spacetime Algebra and Electron Physics (Web)

Lasenby & Doran, *Geometric Algebra for Physicists*
(Cambridge: The University Press, Fall 2002).