Larry Guth: "Polynomial Methods in Combinatorics". AMS, 2016, 273 pp

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One of the strengths that combinatorial problems have is that they are understandable to non-experts in the field. Although the solution to the problem in question may not always be straightforward, the statement of the problem is usually clear and accessible. One of the strengths that polynomials have is that they are well understood by mathematicians in general. Larry Guth manages to exploit both these strengths in this book and provide an accessible and enlightening drive through a selection of combinatorial problems for which polynomials have been used to great effect.

One of the many positives about this book is that before rushing to the current state-of-the-art of a particular problem, the author takes his time and often gives a historical account of how the solution to a problem arose. This leads us to consider attempts to solve the problem without using polynomials. Since these are problems mentioned in the book, this often means that attempts at solving the problem without polynomials were considerably less successful, thus highlighting the effectiveness of polynomial methods. In many cases the strongest results concerning a problem are only known to be attainable using polynomials.

Consider the joints problem in three-dimensional space. Here, we have a set L of lines and define a *joint* as a point in which three non co-planar lines of L meet. We are interested in knowing the maximum number of joints that a set of lines can define. The best known example is the following. From a set of t planes in general position (no three share a line), let L be the set of $\binom{t}{2}$ lines obtained by intersecting two of the planes. Any point that is the intersection of three planes is a joint, so there are $\binom{t}{3}$ joints. This example has roughly $\frac{2^{1/2}}{3}|L|^{3/2}$ joints. The joints problem was introduced in 1992 by Chazelle, Edelsbrunner, Guibasm Pollack, Seidel, Sharir and Snoeyink [4]. Using results from extremal graph theory they proved a bound of order $|L|^{7/4}$. Although the exponent was gradually improved, for example by Sharir in [9], it was not until the polynomial method proof of Guth and Katz [6] that the correct exponent was achieved. Guth and Katz proved that L defines at most $(6^{1/3}|L|)^{3/2}$ joints. As we have seen, up to the constant, this bound is tight. The example above is the best known example. The proof of Guth and Katz can be summarised as follows. Suppose that L defines a set S of j joints. The fact the polynomials of degree d form a subspace of dimension $\binom{d+3}{3}$ implies that there is a polynomial f of degree at most $(6j)^{1/3}$ which vanishes at all the joints. Let f be the minimum degree polynomial with the property that it vanishes at all the points of S. If every line of L contains more than $(6j)^{1/3}$ joints then f vanishes on all the lines of L. But then since at a joint f vanishes along three non co-planar lines, its derivatives must also vanish at the joint. This contradicts the minimality of the degree of f. Therefore there is some line that contains at most $(6j)^{1/3}$ joints. The bound then follows from a recursive argument.

There is some discussion in the book as to why polynomials are a useful tool in the resolution of combinatorial problems. The author suggests that the vanishing lemma is fundamental, that is that if a polynomial of degree d is zero at more than d points of a line, then it is zero on the whole line. Combine this with the fact the polynomials of degree d form a subspace, whose dimension can be easily computed, and we have a potent weapon of attack. This suggestion highlights something worth mentioning, that the problems considered are often geometric in nature. It could be argued that combinatorics covers a far larger umbrella of topics. Although graphs, for example, do appear, it is geometric graph theory that is considered. In other words, how a graph can be realised within in a geometry, in all cases the real plane.

Typically, the polynomial method is the following. Given a finite set S with some geometric property, we define a polynomial f (or a set of polynomials) and translate the geometric property of S to an algebraic property of f. Using results from algebraic geometry, or sometimes simply basic properties of polynomials, one derives additional algebraic properties that the polynomial f must have, which in turn translates to further properties of the set S. For example, imagine S is a set of points with a certain geometrical property and define a polynomial f which translates this geometric property to an algebraic one and whose degree is a function of $|S|$. If one can then prove that the degree of f is bounded then one obtains a bound on $|S|$.

As one might expect from a book about polynomials, there is a fair amount of algebraic geometry involved. However, those who do not have much knowledge of algebraic geometry should not be deterred in any way. On the contrary, where results from algebraic geometry are called upon to help in the resolution of a problem, proofs are given and are very well explained. From that point of view, the book may also serve as a way of getting a foot in the door to graduate level algebraic geometry. For example, the book includes a chapter on Bezout's theorem, which states that two polynomials p and q in two variables, which do not share a common factor, have at most $(\deg p)(\deg q)$ common zeros. The proof given here, relies on the dimension of subspaces of polynomials of a certain degree, together with the rank-nullity formula for linear maps. There are more elementary proofs known, but the author sticks to the theme throughout; that one of the useful properties of polynomials is that the polynomials of degree d form a subspace whose dimension can be easily computed.

The polynomial method in combinatorics can be traced back to Brouwer and Schrijver's proof of Jamison's bound [3]. This bounds the number of points required to block all the hyperplanes of an affine space over a finite field. Although only briefly mentioned in the book, along with Noga Alon's "Combinatorial Nullstellensatz" [1] these articles, [3] and [1], had already shown that polynomials have a role to play in solving combinatorial problems. Most of the content in the book relating the polynomial method is far more recent.

The starting point of the book is Dvir's proof of the finite field Kakeya conjecture [5] from 2009. The Kakeya conjecture is a conjecture from harmonic analysis whose finite field analogue was proposed by Wolff [12] in 1999. Suppose that we have a set L of lines in the n-dimensional affine space over the finite field with q elements. Let S be the set of points incident with a line of L. If L has the property that for each parallel class P of lines, L contains at least one of the lines of P, then there is a $c = c(n)$ such that $|S| \geqslant c(n)q^n$. In other words the set of lines L covers a proportion of all the points of the space. The proof is a straightforward application of the vanishing lemma and the dimension of the space of polynomials vanishing on S. The best known constant for the lower bound is 2^n, for $n \geqslant 4$, due to Saraf and Sudan [10], whereas the best known construction gives a lower bound of 2^{n-1}.

A recurring theme in the book is the Szemerédi-Trotter theorem [11] which bounds the number of incidences between a set S of points and a set L of lines in the real plane. The classical proof of Szemerédi-Trotter, included in the book under review, uses a lower bound on the crossing number of a graph drawn in the plane. This bound itself is proven using Euler's formula for planar graphs, along with some probabilistic arguments. To be explicit, denoting by $I(S, L)$ the number of incidences between a point of S and a line of L, then the Szemerédi-Trotter theorem asserts that

$$I(S, L) \leqslant c((|L||S|)^{2/3} + |S| + |L|),$$

for some constant c. The Szemerédi-Trotter theorem is one of the examples given to show how to apply polynomial partitioning to geometrical problems. More importantly, the following three-dimensional version of the Szemerédi-Trotter theorem is proven, which bounds the number of incidences between a set S of points and a set L of lines in the real three-dimensional space. If L has the property that there are at most b lines of L in any plane, for some $b \geqslant |L|^{1/2}$, then

$$I(S, L) \leqslant c(|L|^{3/4}|S|^{1/2} + b^{1/3}|L|^{1/3}|S|^{2/3} + |S| + |L|),$$

for some constant c.

Many of the results discussed and proven here concern the size of the set $P_r(L)$, which is the set of points incident with r lines of a set of lines L. The author devotes a chapter to proving the following theorem. Suppose that L is a set of lines in three-dimensional real or complex space. If L has the property that at most b of the lines of L are contained in a plane then

$$|P_2(L)| \leqslant b|L| + |L|^{3/2}.$$

The proof involves the analysis of flec-nodal points on algebraic surfaces, and again the algebraic geometry necessary for the proof is well-presented for the non-expert. The author notes that the bound is not valid for finite fields and that Kollar [8] has succeeded in proving asymptotically good bounds for general fields, although he doesn't go into details here.

There is only a passing mention of the Green-Tao theorem from [7] which could have been included here too, although the full proof of which would have been impossible to include due to its length. However, it might have been good to have included applications of Cayley-Bacharach theorem on cubic curves. This is another example of how the dimension of the subspace of polynomials leads to geometric results. The Cayley-Bacharach theorem states that for a set of nine points which is the intersection of two cubic curves, any cubic curve passing through eight of the points passes through the ninth. Now, suppose that S is a set of points in the real plane, not all collinear, and let L be the set of lines incident with exactly two points of S. Green and Tao prove that if $|L| < c|S|$ for some constant c, then most of the points of S lie on a (possibly degenerate) cubic curve. The author does mention that this result is at the cutting edge of research in the field.

There are many exercises included in the text. In many cases these involve generalising theorems which have been proved in the text. There are no solutions given to the exercises but there are hints for the more challenging ones.

The book would make a very good text for a master's course, probably concentrating mainly on the first half of the book and possibly combining it with Alon and Spencer's book on the probabilistic method [2]. Although the subject matter is essentially polynomials, there is a wonderful mix of combinatorial, probabilistic, linear algebraic and algebraic geometric techniques, which would interest a wide audience.

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