For a graph H, define $ex(n, H) = \max\{e(G) \mid |G| = n, H \not\subset G\}$.

[Erdős-Stone-Simonovits] (1966)

For any $\epsilon > 0$, there is an n_0, such that for any $n \geq n_0$,

$$\left(1 - \frac{1}{\chi(H) - 1} - \epsilon\right) \frac{n^2}{2} \leq ex(n, H) \leq \left(1 - \frac{1}{\chi(H) - 1} + \epsilon\right) \frac{n^2}{2},$$

where $\chi(H)$ is the chromatic number of H.

[Kővári, Sós and Turán] (1954)

For any $s, t \in \mathbb{N}$, $t \leq s$, there is a constant c, such that

$$ex(n, K_{t,s}) \leq cn^{2-1/t}.$$

[Erdős and Spencer] (1974)

For any $s, t \in \mathbb{N}$, $t \leq s$, there is a constant c, such that

$$ex(n, K_{t,s}) \geq cn^{2-(s+t-2)/(st-1)}.$$

Construction of a $K_{2,2}$-free graph.

Let b be a non-degenerate symmetric bilinear form on $\mathbb{F}_q^3$, so that

$$x^\perp = \{y \in \mathbb{F}_q^3 \mid b(x,y) = 0\}.$$

Let G be the graph whose vertices are points of $\mathrm{PG}(2,q)$ and

$$\langle x \rangle \sim \langle y \rangle \Leftrightarrow y \in x^\perp.$$

G has $n \approx q^2$ vertices and $e \approx \frac{1}{2}nq \approx \frac{1}{2}n^{3/2}$ edges.

[Erdős-Rényi-Sós] (1966)

$$ex(n, K_{2,2}) = (1 + o(1))\frac{1}{2}n^{3/2}.$$

Construction of a $K_{3,3}$-free graph.

Let b be a non-deg. symmetric bilinear form on $\mathbb{F}_q^5$ defining $^\perp$.

Let C be cone in $\mathrm{PG}(4,q)$ with base $\mathrm{Q}^-(3,q)$ and vertex point z.

Let G be the graph whose vertices are points of $C \setminus z^\perp$ and

$$\langle x \rangle \sim \langle y \rangle \Leftrightarrow y \in x^\perp.$$

For a subset of vertices S, the neighbours of S are in $S^\perp = \langle S \rangle^\perp$.

G has $n \approx q^3$ vertices and $e \approx \frac{1}{2} n q^2 \approx \frac{1}{2} n^{5/3}$ edges.

[Brown] (1966) [Füredi] (1996)

$$ex(n, K_{3,3}) = (1 + o(1))\frac{1}{2} n^{5/3}.$$

We could generalise this construction with a set of cq^{m-1} points in $\mathrm{PG}(2t-3,q)$ with the property that every t span a $\mathrm{PG}(t-1,q)$ would give

$$ex(n, K_{t,t}) \geq c' n^{2-1/m}.$$

but applying the sphere-packing bound to the dual code gives

$$m \leq (2t-2)/(\lfloor t/2 \rfloor) \leq 4.$$

This can only give something better than the known lower bounds for $t = 5$ but as we shall see we can prove

$$ex(n, K_{5,5}) \geq cn^{2-(1/4)}$$

by other means.

Construction of a $K_{t,(t-1)!+1}$-free graph.

Let G be the graph whose vertices are elements of $\mathbb{F}_{q^{t-1}} \times \mathbb{F}_q^*$ and

$$(x, a) \sim (y, b) \Leftrightarrow (x + y)^{q^{t-2}+\ldots+q+1} = ab.$$

G has $n \approx q^t$ vertices and $e \approx \frac{1}{2} n q^{t-1} \approx \frac{1}{2} n^{2-1/t}$ edges.

[Alon-Rónyai-Szabó] (1999)

$$ex(n, K_{t,(t-1)!+1}) \geq cn^{2-1/t}.$$

The system $(x + y_i)^{q^{t-2}+\ldots+q+1} = c_i$ has at most $(t-1)!$ solutions, where $i = 1, \ldots, t-1$.

The probabilistic lower bound gives

$$ex(n, K_{t,t}) \geq cn^{2-2/(t+1)},$$

which in the case $t = 4$ we can improve to $ex(n, K_{4,4}) \geq cn^{5/3}$, since no $K_{3,3}$ implies no $K_{4,4}$.

[Bohman-Keevash] (2010)

$$ex(n, K_{t,t}) \geq c(\log n)^{1/(t^2-1)} n^{2-2/(t+1)}.$$

So for $t = 5$, we only have

$$c' n^{9/5} \geq ex(n, K_{5,5}) \geq c(\log n)^{1/24} n^{5/3}.$$

Construction of a $K_{4,7}$-free graph.

Let G be the graph whose vertices are elements of $\mathbb{F}_{q^3} \times \mathbb{F}_q^*$ and

$$(x, a) \sim (y, b) \Leftrightarrow (x + y)^{q^2+q+1} = ab.$$

G has $n \approx q^4$ vertices and $e \approx \frac{1}{2}nq^3 \approx \frac{1}{2}n^{7/4}$ edges.

In $\mathbb{F}_{q^3}^9$, define $b(u, v) = \sum_{i=1}^4 (u_i v_{9-i} + v_i u_{9-i}) - u_9 v_9$.

The vertices $u, v \in$

$$\{(1, x, x^q, x^{q^2}, x^{q+1}, x^{q^2+1}, x^{q^2+q}, x^{q^2+q+1}, a) \mid x \in \mathbb{F}_{q^3}, a \in \mathbb{F}_q^*\},$$

are adjacent iff $b(u, v) = 0$.

Let S be a subset of the vertices.

Lemma If $|S| \geq 4$ and $e_9 \notin \langle S \rangle$ then $\dim(\langle S \rangle) \geq 4$.

Lemma If $|S| = 5$ and $e_9 \notin \langle S \rangle$ and $\dim(\langle S \rangle) = 4$ then the number of vertices in $\langle S \rangle$ is at least q.

Now use the fact that $S^\perp = \langle S \rangle^\perp$ and that the graph contains no $K_{4,7}$ to conclude that the graph contains no $K_{5,5}$.

[Ball-Pepe] (2011)

$$ex(n, K_{5,5}) \geq cn^{7/4}.$$