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[Alon and Tarsi] (1989)
True for q not prime.

Let $B = \{e_1, \ldots, e_k\}$ be a basis of $\mathbb{F}_q^k$ and let f be the endomorphism which has matrix A with respect to B.

Define linear maps α_i from $\mathbb{F}_q^k$ to $\mathbb{F}_q$ by

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Define a function $p(x)$ from $\mathbb{F}_q^k$ to $\mathbb{F}_q$ by

$$p(x) = \prod_{i=1}^{k} \alpha_i(x).$$

Assume that $p(x) = 0$, whenever $\prod_{i=1}^{k} x_i \neq 0$, where $x = (x_1, \ldots, x_k)$ are the coordinates of x with respect to B.

By Alon's Nullstellensatz, $p = \sum (X_i^{q-1} - 1) h_i(X)$, for some polynomials h_i of degree at most $k - q + 1$.

With respect to the dual basis $\{\alpha_1, \dots, \alpha_k\}$, the monomials $X_i = \sum c_{ij} \alpha_j$, for some c_{ij}.

Thus

$$p = \prod_{i=1}^{k} \alpha_i = \sum ((\sum c_{ij} \alpha_j)^{q-1} - 1) h_i,$$

which gives a contradiction for q non-prime, since $(q-1)! = 0$.