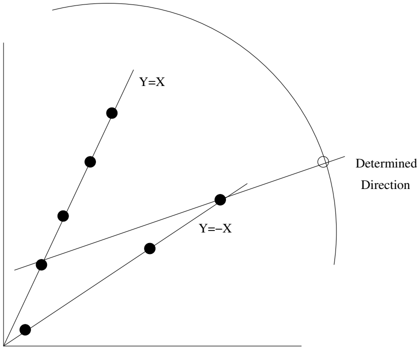
How few directions can a function over a finite field f determine ?
How small can the set $D(f)$ be ?

$$D(f) = \{\frac{f(y) - f(x)}{y - x} \mid x, y \in \mathbb{F}_q,\ x \neq y\}$$

ex. if f is linear then $|D(f)| = 1$.

ex. if f is linear over $\mathbb{F}_s \leq \mathbb{F}_q$ then
$q/s + 1 \leq |D(f)| \leq (q-1)/(s-1)$.

ex. if $f(x) = x^{(q+1)/2}$ and q is odd then $|D(f)| = (q+3)/2$.



Let p be a prime.

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If $|D(f)| > (p+3)/2$ then $|D(f)| > 2(p-1)/3$.

If $-c \notin D(f)$ then $x \mapsto f(x) + cx$ is a permutation.

Let $I(f)$ be maximum such $\sum_{x \in \mathbb{F}_p} (f(x) + xY)^k \equiv 0$ for all $k = 1, \ldots, I(f) - 1$.

Then $I(f) \geq p - |D(f)| + 1$.

[Gács] Consider $x^i f(x)^j$ as elements of $\mathbb{F}_p(x)/(x^p - x)$.

Note that the above implies that $x^i f(x)^j$ has degree $\leq p - 2$ for all $1 \leq i + 1 \leq I(f) - 1$.

Consider linear maps

$$\phi(A_1, \ldots, A_s) \mapsto \sum_{i=0}^{s} A_i(x) f(x)^i,$$

where the degree of $A_i(x)$ satisfies $\deg A_i \leq s - i$.

If $g, h \in \mathrm{Im}(\phi)$ then $\deg(gh) \neq p - 1$.

If $s < I(f)/2$ then only half the degrees can occur amongst the polynomials in $\mathrm{Im}(\phi)$.

[Ball and Gács] (2008) If $I(f) > (p-1)/t + t - 1$ for some $t \in \mathbb{N}$ then every line meets the graph of f in at most $t-1$ points or at least $(p-1)/t + 1$ points.

This implies that if $|D(f)| < p - 2\sqrt{p-1} + 15/4$ then the graph of f has additional properties.

[Conjecture] If $I(f) > (p-1)/t + t - 1$ for some $t \in \mathbb{N}$ then the graph of f is contained in an algebraic curve of degree $t-1$.

[Rédei] (1970) True for $t = 2$.

[Gács] (2003) True for $t = 3$.

Let q be a prime power.

[Ball, Blokhuis, Brouwer, Storme, Szőnyi] (1999), [Ball] (2003)

If $|D(f)| \leq (q+1)/2$ and s is maximal with the property that every line meets the graph of f is a multiple of s points then

$\mathbb{F}_s \leq \mathbb{F}_q$,

$q/s + 1 \leq |D(f)| \leq (q-1)/(s-1)$,

and for $s > 2$ the function f is linear over $\mathbb{F}_s$.