The equivalence of linear codes implies semi-linear equivalence

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Abstract

We prove that if two linear codes are equivalent then they are semi-linearly equivalent. We also prove that if two additive MDS codes over a field are equivalent then they are additively equivalent.

1 Introduction

Let F be a finite set. A *code* $\mathcal{A}$ of *length* n is a subset of F^n. The Hamming distance $d(u, v)$ between $u, v \in F^n$ is the number of coordinates in which u and v differ. As in [8], we say that two codes $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if, after some permutation α of the coordinates of the elements of $\mathcal{A}$, there exist permutations σ_i of F such that for all $u = (u_1, \ldots, u_n) \in \mathcal{A}$,

$$(\sigma_1(u_1), \ldots, \sigma_n(u_n)) \in \mathcal{B}.$$

Let ψ denote the bijection from $\mathcal{A}$ to $\mathcal{B}$ induced by these permutations. Then it is clear that ψ preserves Hamming distance; that is

$$d(u, v) = d(\psi(u), \psi(v))$$

for all $u, v \in \mathcal{A}$.

If F is a finite field and $\mathcal{A}$ and $\mathcal{B}$ are k-dimensional subspaces of F^n then we say that $\mathcal{A}$ and $\mathcal{B}$ are *linear codes*. Two codes are $\mathcal{A}$ and $\mathcal{B}$ are *linearly equivalent* in the above definition of equivalence if, for all $i \in \{1, \ldots, n\}$,

$$\sigma_i(x) = \lambda_i x,$$

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for some non-zero $\lambda_i \in F$. In the case that F has non-trivial automorphisms (fields of non-prime order), for a fixed automorphism β of F, applying the permutation

$$\sigma_i(x) = \lambda_i x^\beta,$$

to the i-th coordinate of the codewords of $\mathcal{A}$, we get a linear code $\mathcal{B}$ which is equivalent to $\mathcal{A}$. This more general equivalence we will call *semi-linear equivalence*, following [3], [6] and [10]. It is also called *PΓL-equivalence* [2] and is referred to as simply *equivalence* in [4] and [7].

The main result of this note is the following theorem.

Theorem 1. *Two linear codes are equivalent if and only if they are semi-linearly equivalent.*

The reverse implication is trivial based on the above discussion, so the goal here is to prove that if $\mathcal{A}$ and $\mathcal{B}$ are equivalent linear codes then they are semi-linearly equivalent.

Theorem 1 can be compared to Theorem 1.5.10 in [3]. In that theorem they prove that if there is a Hamming distance preserving bijection of F^n which maps subspaces to subspaces then this map is semi-linear, i.e. it is additive and $\sigma(\lambda x) = \lambda^\beta \sigma(x)$, for all $\lambda \in F$, where β is an automorphism of F.

Theorem 1 has significant implications when classifying codes. For example, in a classification of codes up to equivalence, as in [1], [8] and [9], one will not find two linear codes in the same equivalence class. Perhaps of more interest, in a classification of linear codes up to semi-linear equivalence, as in [2] and [5], we can now be sure that two semi-linearly inequivalent codes are also inequivalent.

2   Equivalence implies semi-linear equivalence.

Let $\mathcal{A}$ and $\mathcal{B}$ be k-dimensional linear codes over a finite field $\mathbb{F}_q$ of length n which are equivalent. Since $\alpha(\mathcal{A})$ is also a linear code equivalent to $\mathcal{B}$, we can assume that α, the permutation of the coordinates of the elements of $\mathcal{A}$, is the identity by replacing $\mathcal{A}$ by $\alpha(\mathcal{A})$. Indeed, by re-ordering the coordinates in both codes, if necessary, we can find generator matrices A and B in standard form for $\mathcal{A}$ and $\mathcal{B}$ respectively. Recall that a generator matrix in standard form is a $k \times n$ matrix in which the initial $k \times k$ sub-matrix is the identity matrix.

As above, we denote by σ_i the permutation in the i-th coordinate which has the property that for all $u = (u_1, \ldots, u_n) \in \mathcal{A}$,

$$(\sigma_1(u_1), \ldots, \sigma_n(u_n)) \in \mathcal{B}.$$

Lemma 2. *If not all of the columns of A and B are of weight one then, for each $j \in \{1, \ldots, n\}$, the map*

$$\tau_j(x) = \sigma_j(x) - \sigma_j(0)$$

is additive.

Proof. Let $A = (\alpha_{jr})$ and let $B = (\beta_{jr})$.

We have that for each $a = (a_1, \ldots, a_k) \in \mathbb{F}_q^k$, there is a $b = (b_1, \ldots, b_k) \in \mathbb{F}_q^k$, such that

$$(\sigma_1(a_1), \ldots, \sigma_k(a_k), \sigma_{k+1}(\sum_{j=1}^{k} \alpha_{j,k+1} a_j), \ldots) = (b_1, \ldots, b_k, \sum_{j=1}^{k} \beta_{j,k+1} b_j, \ldots). \tag{1}$$

Hence, for $j \in \{1, \ldots, k\}$,

$$b_j = \sigma_j(a_j)$$

and

$$\sum_{j=1}^{k} \beta_{jr} \sigma_j(a_j) = \sigma_r(\sum_{j=1}^{k} \alpha_{jr} a_j), \tag{2}$$

for $r \in \{k+1, \ldots, n\}$.

Setting $a_i = 0$ for $i \neq j$ we deduce that

$$\beta_{jr} \sigma_j(a_j) + \sum_{i \neq j} \beta_{ir} \sigma_i(0) = \sigma_r(\alpha_{jr} a_j). \tag{3}$$

Summing over j, and substituting in (2)

$$\sum_{j=1}^{k} \sigma_r(\alpha_{jr} a_j) - \sum_{j=1}^{k} \sum_{i \neq j} \beta_{ir} \sigma_i(0) = \sigma_r(\sum_{j=1}^{k} \alpha_{jr} a_j)$$

which implies

$$\sum_{j=1}^{k} \sigma_r(\alpha_{jr} a_j) - (k-1) \sum_{i=1}^{k} \beta_{ir} \sigma_i(0) = \sigma_r(\sum_{j=1}^{k} \alpha_{jr} a_j).$$

Again from (2)

$$\sum_{i=1}^{k} \beta_{ir} \sigma_i(0) = \sigma_r(0). \tag{4}$$

Thus,

$$\sum_{j=1}^{k} \sigma_r(\alpha_{jr} a_j) - (k-1)\sigma_r(0) = \sigma_r(\sum_{j=1}^{k} \alpha_{jr} a_j). \tag{5}$$

Define

$$\tau_r(x) = \sigma_r(x) - \sigma_r(0).$$

Then (5) becomes

$$\sum_{j=1}^{k} \tau_r(\alpha_{jr} a_j) = \tau_r(\sum_{j=1}^{k} \alpha_{jr} a_j). \tag{6}$$

Firstly, assume that the r-th column of A is not of weight one. Without loss of generality $\alpha_{1r}, \alpha_{2r} \neq 0$, so substituting in (6), $a_1 = \alpha_{1r}^{-1} x$ and $a_2 = \alpha_{2r}^{-1} y$, $a_j = 0$ for $j \in \{3, \ldots, k\}$, we have that

$$\tau_r(x + y) = \tau_r(x) + \tau_r(y).$$

We conclude that τ_r is an additive permutation of $\mathbb{F}_q$.

Subtracting (4) from (3), we have that

$$\beta_{jr}(\sigma_j(a_j) - \sigma_j(0)) = \tau_r(\alpha_{1r} a_1)$$

This implies that

$$\tau_j(x) = \sigma_j(x) - \sigma_j(0)$$

is additive too, for $j \in \{1, \ldots, k\}$.

If the r-th column of A is of weight one then, from (2),

$$\sigma_r(\alpha_{ir} a_i) = \sum_{j=1}^{k} \beta_{jr} \sigma_j(a_j)$$

for some $i \in \{1, \ldots, k\}$. Since this holds for all $(a_1, \ldots, a_k) \in \mathbb{F}_q^k$, we conclude that $\beta_{jr} = 0$ for $j \neq i$ and hence,

$$\sigma_r(\alpha_{ir} a_i) = \beta_{ir} \sigma_i(a_i).$$

This implies $\sigma_r(0) = \beta_{ir} \sigma_i(0)$ and so

$$\tau_r(\alpha_{ir} a_i) = \beta_{ir} \tau_i(a_i).$$

Since τ_i is additive, this implies τ_r is additive. $\qquad\square$

The following lemma implies that we can assume that σ_i is additive, by replacing σ_i by τ_i, for each $i \in \{1, \ldots, n\}$.

Lemma 3. *If not all of the columns of A and B are of weight one then, for all $u = (u_1, \ldots, u_n) \in \mathcal{A}$,*

$$(\tau_1(u_1), \ldots, \tau_n(u_n)) \in \mathcal{B}.$$

Proof. By Lemma 2 and (1), for each $a = (a_1, \ldots, a_k) \in \mathbb{F}_q^k$, there is a $b = (b_1, \ldots, b_k) \in \mathbb{F}_q^k$, such that

$$\left(\tau_1(a_1), \ldots, \tau_k(a_k), \tau_{k+1}\left(\sum_{j=1}^{k} \alpha_{j,k+1} a_j\right), \ldots\right) + v = (b_1, \ldots, b_k)B,$$

where $v = (v_1, \ldots, v_n) \in \mathbb{F}_q^n$ is defined by $v_j = \sigma_j(0)$.

Since τ_i is additive, $\tau_i(0) = 0$. Thus, with $a = (0, \ldots, 0)$ the above implies there is a $(b'_1, \ldots, b'_k)$ such that

$$v = (b'_1, \ldots, b'_k)B.$$

Thus

$$(\tau_1(a_1), \ldots, \tau_k(a_k), \tau_{k+1}(\sum_{j=1}^{k} \alpha_{j,k+1}a_j), \ldots) = ((b_1, \ldots, b_k) - (b'_1, \ldots, b'_k))B,$$

and hence for each $a = (a_1, \ldots, a_k) \in \mathbb{F}_q^k$, there is a $c = (c_1, \ldots, c_k) \in \mathbb{F}_q^k$, such that

$$(\tau_1(a_1), \ldots, \tau_k(a_k), \tau_{k+1}(\sum_{j=1}^{k} \alpha_{j,k+1}a_j), \ldots) = (c_1, \ldots, c_k)B.$$

$\square$

We are now in a position to prove the main theorem. Note that an additive permutation of $\mathbb{F}_{p^h}$ is of the form

$$x \mapsto \sum_{i=0}^{h-1} c_i x^{p^i}.$$

This is easily verified since if we consider $\mathbb{F}_{p^h}$ as $\mathbb{F}_p^h$ then an additive map is given by an $h \times h$ matrix over $\mathbb{F}_p$. Thus, there are p^{h^2} in total, which coincides with the number of functions which can be defined as above.

Proof. (of Theorem 1)

Suppose all of the columns of A and B are of weight one. Then, after a suitable permutation of the coordinates, $\mathcal{A}$ is linearly equivalent to a code $\mathcal{A}'$ whose codewords are

$$(\underbrace{a_1, \ldots, a_1}_{m_1}, \underbrace{a_2, \ldots, a_2}_{m_2}, \ldots, \underbrace{a_k, \ldots, a_k}_{m_k}). \tag{7}$$

Thus, since $\mathcal{A}'$ and $\mathcal{B}$ are equivalent, there are permutations θ_i and a function f from $\{1, \ldots, n\}$ to $\{1, \ldots, k\}$, such that for all $(b_1, \ldots, b_k) \in \mathbb{F}_q^k$,

$$(\theta_1(b_{f(1)}\beta_{f(1),1}), \theta_2(b_{f(2)}\beta_{f(2),2}), \ldots, \theta_n(b_{f(n)}\beta_{f(n),n}))$$

is equal to (7). We can scale the columns of B and obtain a code $\mathcal{B}'$ which is linearly equivalent to a code $\mathcal{B}$ and has codewords

$$(\theta_1(b_{f(1)}), \theta_2(b_{f(2)}), \ldots, \theta_n(b_{f(n)})).$$

Now, comparing with (7) we have that, for example, if $m_1 \geqslant 2$,

$$\theta_1(b_{f(1)}) = \theta_2(b_{f(2)})$$

for all $(b_1, \ldots, b_k) \in \mathbb{F}_q^k$. This implies $f(1) = f(2)$, and hence $\theta_1 = \theta_2$, etc. Thus, we conclude that $\mathcal{B}'$ has codewords

$$(\underbrace{b_{g(1)}, \ldots, b_{g(1)}}_{m_1}, \underbrace{b_{g(2)}, \ldots, b_{g(2)}}_{m_2}, \ldots, \underbrace{b_{g(k)}, \ldots, b_{g(k)}}_{m_k}) \tag{8}$$

and is linearly equivalent to $\mathcal{A}'$. Hence, $\mathcal{A}$ is linearly equivalent to $\mathcal{B}$.

Suppose not all of the columns of A and B are of weight one.

By Lemma 3, we can assume for $j \in \{1, \ldots, n\}$,

$$\sigma_j(x) = \sum_i c_{ji} x^{p^i}.$$

As in the previous proofs, let $B = (\beta_{jr})$ and let $A = (\alpha_{jr})$.

Substituting in (2), we have that for all $(a_1, \ldots, a_k) \in \mathbb{F}_q^k$ and $r \in \{k+1, \ldots, n\}$,

$$\sum_{j=1}^{k} \sum_{i=0}^{h-1} \beta_{jr} c_{ji} a_j^{p^i} = \sum_i c_{ri} (\sum_{j=1}^{k} \alpha_{jr} a_j)^{p^i}.$$

Therefore, for all $j \in \{1, \ldots, k\}$, $r \in \{k+1, \ldots, n\}$ and $i \in \{0, \ldots, h-1\}$,

$$\beta_{jr} c_{ji} = c_{ri} \alpha_{jr}^{p^i}. \tag{9}$$

Let $t \in \{0, \ldots, h-1\}$ be minimal such that $c_{1t} \neq 0$. We aim to show that we can multiply columns and rows of B by non-zero elements of $\mathbb{F}_q$ in such a way that the matrix B will transform into A^{p^t}. If both α_{jr} and β_{jr} are zero then they will be unaffected by these multiplications and so for this particular entry the previous statement holds. Thus, we assume that α_{jr} and β_{jr} are not both zero for any particular $j \in \{1, \ldots, k\}$ and $r \in \{k+1, \ldots, n\}$.

If $\beta_{jr} = 0$ then, by the previous paragraph, we can assume $\alpha_{jr} \neq 0$. Hence, $c_{ri} = 0$ for all $i \in \{0, \ldots, h-1\}$ which implies $\sigma_r = 0$, a contradiction.

If $\alpha_{jr} = 0$ then, by the previous paragraph, we can assume $\beta_{jr} \neq 0$. Hence, $c_{ji} = 0$ for all $i \in \{0, \ldots, h-1\}$ which implies $\sigma_j = 0$, a contradiction.

Thus, we can assume c_{1t}, α_{1r} and β_{1r} are non zero. Thus, (9) implies $c_{rt} \neq 0$ for all $r \in \{k+1, \ldots, n\}$, and so

$$\frac{\beta_{jr} c_{jt}}{\beta_{1r} c_{1t}} = \frac{\alpha_{jr}^{p^t}}{\alpha_{1r}^{p^t}}.$$

If $c_{jt} = 0$ for some j then $\alpha_{jr} = 0$, which we have already ruled out.

Thus, we can divide the r-th column of B by β_{1r} and multiply the j-th row of this generator matrix by

$$\frac{c_{jt}}{c_{1t}}$$

(which does not change the code) and we get a generator matrix whose j-th coordinate of the r-th column is

$$\frac{\alpha_{jr}^{p^t}}{\alpha_{1r}^{p^t}}$$

Now, multiplying the r-th column of the generator matrix by $\alpha_{1r}^{p^t}$, we get a generator matrix of a code linearly equivalent to $\mathcal{B}$ whose r-th column is the r-th column of A^{p^t}.

Therefore, we conclude that $\mathcal{A}$ and $\mathcal{B}$ are semi-linearly equivalent. $\qquad\square$

3   Additive MDS codes

The motivation for this article stems from the discussion after Theorem 3.4 in [2]. In that discussion the following question is asked. "If two additive codes $\mathcal{A}$ and $\mathcal{B}$ are equivalent then are they necessarily PΓL-equivalent? Equivalently, if two additive codes $\mathcal{A}$ and $\mathcal{B}$ are PΓL-inequivalent then is it true that they are inequivalent?" They also pose the same question for linear codes.'

We have proved in Theorem 1 that it is true for linear codes. Here, we answer the question in the affirmative for additive MDS codes. Recall that an MDS code is a code attaining the Singleton bound. That is, if C is a MDS code of length n over F of minimum distance d then $|C| = |F|^{n-d+1}$.

An additive code over $\mathbb{F}_q$ is linear over the prime subfield $\mathbb{F}_p$, where $q = p^h$. Two additive codes $\mathcal{A}$ and $\mathcal{B}$ are *additively equivalent* if, after some permutation of the coordinates of the elements of $\mathcal{A}$, there exist permutations

$$\sigma_j = \sum_{i=0}^{h-1} c_{ij} x^{p^i}.$$

such that for all $u = (u_1, \ldots, u_n) \in \mathcal{A}$,

$$(\sigma_1(u_1), \ldots, \sigma_n(u_n)) \in \mathcal{B}.$$

An additive MDS code of size q^k is the row space over $\mathbb{F}_p$ of a $kh \times n$ matrix, whose elements are from $\mathbb{F}_q$. We can consider the elements of $\mathbb{F}_q$ as row vectors of $\mathbb{F}_p^h$, so that the generator matrix becomes a $kh \times nh$ matrix.

It is fairly easy to show that we can use Gaussian elimination and additive permutations to obtain a generator matrix for an additive MDS code, additively equivalent to the original code, whose generator matrix is of the form

$$
\begin{pmatrix}
I_h & O_h & O_h & O_h & \dots & \dots & O_h & O_h & \alpha_{r,1} & \alpha_{r+1,1} & \dots \\
O_h & I_h & O_h & O_h & \dots & \dots & O_h & O_h & \alpha_{r,2} & \alpha_{r+1,2} & \dots \\
O_h & O_h & I_h & O_h & \dots & \dots & O_h & O_h & \alpha_{r,3} & \alpha_{r+1,3} & \dots \\
O_h & O_h & O_h & I_h & \ddots & \dots & O_h & O_h & \alpha_{r,4} & \alpha_{r+1,4} & \dots \\
\vdots & \vdots & \dots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \\
\vdots & \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \vdots & \\
O_h & O_h & O_h & O_h & \dots & \dots & I_h & O_h & \alpha_{r,k-1} & \alpha_{r+1,k-1} & \dots \\
O_h & O_h & O_h & O_h & \dots & \dots & O_h & I_h & \alpha_{r,k} & \alpha_{r+1,k} & \dots
\end{pmatrix},
$$

where I_h is the $h \times h$ identity matrix, O_h is the $h \times h$ zero matrix and $\alpha_{i,j}$ are $h \times h$ non-singular matrices. See [2], for more details on this.

Suppose that the additive MDS codes $\mathcal{A}$ and $\mathcal{B}$ are equivalent. We can mimic the proof of Lemma 2 and Lemma 3, replacing a_i and b_i by row vectors in $\mathbb{F}_p^h$. The two proofs follows in the same way as for linear codes. The conclusion is then that σ_j can be taken to be additive and we conclude that $\mathcal{A}$ and $\mathcal{B}$ are additively equivalent. Thus, we have the following theorem.

Theorem 4. *Two additive MDS codes over a field are equivalent if and only if they are additively equivalent.*

Example 5. *In [2, Table 3], the second row shows that there are 3 additive MDS $(8, 9^3, 6)_9$ codes, i.e. MDS codes of length 8 and minimum distance 6. Two of these codes, C_1 and C_2 are linear MDS codes and have generator matrices*

$$
G_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & e & e^6 & e^6 & e \\ 0 & 1 & 0 & e & 1 & e & e^6 & e^6 \\ 0 & 0 & 1 & e^6 & e & 1 & e & e^6 \end{pmatrix}, \quad
G_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & e^5 & e^7 & e^7 & e^5 \\ 0 & 1 & 0 & e & 1 & e^5 & e^7 & e^7 \\ 0 & 0 & 1 & e^7 & e^5 & 1 & e^5 & e^7 \end{pmatrix}
$$

respectively. Note that $e \in \mathbb{F}_9$ is a primitive element such that $e^2 = e + 1$. It is explained in [2] that C_2 is a truncated Reed-Solomon code, while C_1 is not, implying that they are not semi-linearly equivalent. Additionally, one may show that C_1 and C_2 are not semi-linearly equivalent by confirming that the columns of G_2 are contained in the conic

$$
x_1 x_2 + e^3 x_1 x_3 + x_2 x_3,
$$

while the columns of G_1 are not contained in any conic.

The last of the three additive MDS $(8, 9^3, 6)_9$ codes, C_3, is not semi-linearly equivalent to a linear

code. It has generator matrix

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 2 & 2 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 1 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 2 & 2 & 2 & 0 & 0 & 2
\end{pmatrix}.$$

We now know, by Theorem 1, that C_1 and C_2 are not equivalent and, by Theorem 4, these codes are also not equivalent to C_3.

MDS codes have been classified (up to equivalence) over alphabets of size at most 8, see [1], [8] and [9]. If and when MDS codes over alphabets of size 9 are classified, there will be at least three equivalence classes of codes of length 8, two of which will contain linear codes and one of which will contain an additive code. We conjecture that there will be no further equivalence class.

4 Comments

It is unclear to us if Theorem 4 will apply to all additive codes. The principal problem in proving such a statement is that one cannot assume a generator matrix in standard form. It would be interesting to have a counterexample. In the proof of Theorem 4, we view additive MDS codes over $\mathbb{F}_q$, $q = p^h$, as linear codes over the vector space $\mathbb{F}_p^h$, i.e. $1 \times h$ matrices over $\mathbb{F}_p$. It is possible that one can extend Theorem 4 to codes over general matrices. Indeed it would be interesting to study MDS codes over matrices. The smallest alphabet which is not equivalent to a linear or additive code over $\mathbb{F}_q$ would be over 2×2 matrices over $\mathbb{F}_2$; so the alphabet would have size 16.

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